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Formation of Singularities in Heat Propagation Guided by Second Sound

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In the classical theory of heat propagation, the flux is usually given by Newton's laws. As a result, we get a parabolic equation: namely the heat equation (équation de chaleur). An other approach in the modern theory, is to assume that the heat flux satisfies Cattaneo's law and instead we obtain a hyperbolic system describing the propagation of the hear. Such a phenomenon, that occurs in some materials, is called propagation guided by Second Sound.

In the absence of deformation, heat propagation in a unidimensional case is given by the following equation of balance of energy

$$e_t + q_x = 0, \tag{0.1}$$

where e is the internal energy and q is the heat flux (e and q are functions of x and t and subscripts indicate partial derivatives).

We assume that e depends on the absolute temperature and q satisfies the Cattaneo[1] law; i.e.,

$$e = e_0(\theta) \tag{0.2}$$

$$\tau(\theta) q_t + q = -\kappa(\theta)\theta_x, \tag{0.3}$$

where τ and κ are positive functions. By substituting in (0.1), we get

$$e_0'(\theta)\theta_t = -q_x \tag{0.4}$$

where

$$e_0'(\theta) > 0 \tag{0.5}$$

is assumed to hold and consequently the system (0.4), (0.5) is hyperbolic.

Global existence and decay of classical solutions, for smooth and small initial data, have been established by Coleman *et al.* [2]. In their paper, the authors used a classical energy argument to prove their res[illegible] As they

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pointed out, Kato's method based on semigroup theory [5] is also applicable to their initial-value problem.

In the case of large initial data, one expects that classical solutions blow up in finite time due to the formation of shock waves. This phenomenon is often found in hyperbolic and coupled hyperbolic–parabolic systems (See, e.g., [3, 4, 6–8]).

It is also interesting to note that a global existence and decay to equilibrium state result, to the one-dimensional nonlinear thermoelasticity—where the heat flux is given by (0.3)—was proved by Tarabek [9] in 1989.

In this work, we prove a blow up result to the above system. Our argument to accomplish this will be very close to the one used by Slemrod [8] and Hrusa and Messaoudi [4].

This paper will be divided into two sections. In the first one, we state a local existence theorem. In the second section, our main result will be presented.

1. Local Existence

We consider the following Cauchy problem

$$\theta_t(x,t) = -c(\theta(x,t))\, q_x(x,t) \tag{1.1}$$

$$q_t(x,t) = -\sigma(\theta(x,t))\, \theta_x(x,t) - \lambda(\theta(x,t))\, q(x,t), \tag{1.2}$$

$$\theta(x,0) = \theta_0(x), \qquad q(x,0) = q_0(x), \tag{1.3}$$

$$x \in \mathbb{R}, \quad t \geqslant 0.$$

Proposition. *Assume that c, σ, and λ are C^3 positive functions, with*

$$\lambda(y) \leqslant \gamma, \quad \forall y \in \mathbb{R}, \tag{1.4}$$

and let θ_0 and $q_0 \in H^2(\mathbb{R})$ be given. Then the initial value problem (1.1)–(1.3) *has a unique local solution (θ, q), on a maximal time interval* $[0, T)$, *satisfying*

$$\theta, g \in C([0, T); H^2(\mathbb{R})) \cap C^1([0, T); H^1(\mathbb{R})). \tag{1.5}$$

For a proof of this result, we can use a classical energy argument [2], as well as the nonlinear semigroup theory presented by Hughes *et al.* [5].

Remark 1.1. The Sobolev embedding theorem implies that θ and q are in $C^1(\mathbb{R} \times [0, T))$.

Remark 1.2. If c, σ, λ a[illegible]$^{k+1}$ functions and $\theta_0, q_0 \in H^k(\mathbb{R})$. Then the solution $(\theta, q) \in [H^k(\mathbb{R})]^2$.

By choosing initial data small enough (in L^∞ norm), we are guaranteed to have

$$a := \inf \frac{\rho' \rho^{-1/2}}{2\alpha} > 0. \tag{2.28}$$

We set

$$M := \max \left| \frac{\lambda}{2} + \frac{\lambda'}{2\alpha} (r+s) - \frac{\rho' \rho^{-1/2}}{2\alpha} f \right| \tag{2.29}$$

$$m := \max \left| -\frac{\rho' \rho^{-1/2}}{2\alpha} f^2 - \left[\frac{\lambda}{2} + \frac{\lambda'}{2\alpha} (r+s) \right] f \right.$$

$$\left. + \frac{\lambda}{2} (r+s) \cdot \int_0^\infty (\lambda' \rho^{3/2})(\xi)\, d\xi \right|. \tag{2.30}$$

(We note that these maxima exist since they depend only on θ and q.) Thus, we have

$$\partial_t F \geqslant aF^2 - M\,|F| - m. \tag{2.31}$$

We then use

$$M\,|F| \leqslant \frac{a}{2} F^2 + \frac{1}{2a} M^2 \tag{2.32}$$

to obtain

$$\partial_t F \geqslant \frac{a}{2} (F^2 - B^2) \tag{2.33}$$

where

$$B = \frac{M^2}{a^2} + \frac{2m}{a}. \tag{2.34}$$

From Lemma 3.1 of [8], it suffices to choose θ_0 and q_0 small enough in the L^∞ norm and with positive derivatives such that $q_0'(x) + \alpha(\theta_0(x))\,\theta_0'(x)$ is large enough to make F blow up in a time $T < L$.

Remark 2.3. A similar result can also be obtained for certain initial-boundary value problem.

2. Formation of Singularities

This section is devoted to the statement and a proof to our main result. We first begin with a lemma which gives a pointwise upper bound on the solution in terms of the initial data.

Lemma. *Assume that c, σ, and λ are as in the proposition and let θ_0, $q_0 \in H^2(\mathbb{R})$ be given. Then any solution (θ, q) to problems* (1.1)–(1.3) *satisfies*

$$\max_{(x,t)} \{|\theta(x,t)| + |q(x,t)|\} \leqslant \Gamma \max_x \{|\theta_0(x)| + |q_0(x)|\}, \tag{2.1}$$

where Γ is a positive constant independent of θ and q.

Proof. We first introduce the quantities

$$\begin{aligned} r(x,t) &= q(x,t) + A(\theta(x,t)) \\ s(x,t) &= q(x,t) - A(\theta(x,t)), \end{aligned} \tag{2.2}$$

where

$$\alpha := \sqrt{\frac{\sigma}{c}}, \qquad A(\theta) = \int_0^\theta \alpha(\xi)\, d\xi \tag{2.3}$$

and the differential operators

$$\partial_t := \frac{\partial}{\partial t} + \rho(\theta) \frac{\partial}{\partial x}, \qquad D_t := \frac{\partial}{\partial t} - \rho(\theta) \frac{\partial}{\partial x}. \tag{2.4}$$

where

$$\rho := \sqrt{\sigma c} \tag{2.5}$$

and simple and straightforward computations lead to

$$\partial_t r = D_t s = -\lambda(\theta) q = -\lambda(\theta) \left(\frac{s+r}{2}\right). \tag{2.6}$$

We then define

$$R(t) := \max_x |r(x,t)|, \qquad S(t) := \max_x |s(x,t)|. \tag{2.7}$$

(The maxima in (2.7) are attained because R and S die at infinity.)

The last term in (2.22) can be handled as follows

$$\frac{\rho^{-3/2}}{2}\frac{\lambda'}{2\alpha}(r+s)\,\partial_t(r-s)$$

$$=\frac{\lambda'\rho^{-3/2}}{2\alpha}(r-A)\,\partial_t(r-s) \tag{2.22}$$

$$=\frac{\lambda'\rho^{-3/2}}{2\alpha}r\partial_t(r-s)-\frac{\lambda'\rho^{-3/2}}{2\alpha}A\partial_t(r-s)$$

$$=\partial_t\left[r\int_0^{\theta}(\lambda'\rho^{-3/2})(\xi)\,d\xi\right]-\partial_t r\cdot\int_0^{\theta}(\lambda'\rho^{-3/2})(\xi)\,d\xi$$

$$-\partial_t\int_0^{\theta}(\lambda' A\rho^{-3/2})(\xi)\,d\xi. \tag{2.23}$$

We note that

$$\partial_t\theta=\frac{\partial_t(r-s)}{2}.$$

By combining (2.22), (2.23), we get

$$\partial_t\Phi=-\frac{\rho'\rho^{-1/2}}{2\alpha}\Phi^2-\left[\frac{\lambda}{2}+\frac{\lambda'}{2\alpha}(r+s)\right]\Phi$$

$$+\frac{\lambda}{2}(r+s)\cdot\int_0^{\theta}(\lambda'\rho^{-3/2})(\xi)\,d\xi+\partial_t f \tag{2.24}$$

where

$$f:=r\int_0^{\theta}(\lambda'\rho^{-3/2})(\xi)-\int_0^{\theta}\left[\rho^{-3/2}\left(\frac{\lambda\alpha}{2}+\lambda' A\right)\right](\xi)\,d\xi. \tag{2.25}$$

Now, we are ready to conclude our proof. For this purpose, we set

$$F:=\Phi-f. \tag{2.26}$$

Hence (2.24) gives

$$\partial_t F=-\frac{\rho'\rho^{-1/2}}{2\alpha}F^2+\frac{\lambda}{2}(r+s)\cdot\int_0^{\theta}(\lambda'\rho^{-3/2})(\xi)\,d\xi$$

$$-\left[\frac{\lambda}{2}+\frac{\lambda'}{2\alpha}(r+s)-\frac{\rho'\rho^{-1/2}}{2\alpha}f\right]F$$

$$-\frac{\rho'\rho^{-1/2}}{2\alpha}f^2-\left[\frac{\lambda}{2}+\frac{\lambda'}{2\alpha}(r+s)\right]f. \tag{2.27}$$

For any $t \in (0, T)$, we can choose x_1 and x_2 so that

$$R(t) = |r(x_1, t)|, \qquad |s(x_2, t)|. \tag{2.8}$$

Therefore, for any $h \in (0, t)$, we have

$$R(t-h) \geqslant |r(x_1 - h\rho(\theta(x_1, t)), t-h)|, \tag{2.9}$$

$$S(t-h) \geqslant |s(x_2 + h\rho(\theta(x_2, t)), t-h)|. \tag{2.10}$$

By substracting (2.9), (2.10) from (2.8), dividing the resulting inequalities by h, and letting h go to zero, we obtain

$$R(t) \leqslant |\partial_t r(x_1, t)| \leqslant \frac{\gamma}{2}\,[R(t) + S(t)] \tag{2.11}$$

$$S(t) \leqslant |D_t s(x_2, t)| \leqslant \frac{\gamma}{2}\,[R(t) + S(t)], \tag{2.12}$$

for almost every $t \in [0, T)$; hence (2.11), (2.12) yield

$$\frac{d}{dt}\,[R(t) + S(t)] \leqslant \gamma[R(t) + S(t)], \tag{2.13}$$

for almost every $t \in [0, T)$.

A straightforward integration—using Gronwall's inequality—leads to

$$[R(t) + S(t)] \leqslant [R(0) + S(0)]\,e^{\gamma t}, \quad \forall t \in [0, T]. \tag{2.14}$$

Therefore (2.1) follows.

Remark 2.1. σ, c, and λ need not be positive on $\mathbb{R}$. It suffices that (1.4) holds near equilibrium. In this case, we choose the initial data small enough and make a slight, but not crucial modification in the proof.

THEOREM. *Let σ, c, and λ be as in the proposition. Assume further that*

$$\rho'(0) < 0. \tag{2.15}$$

Then, for any $L > 0$, there exist initial data θ_0 and $q_0 \in H^2(\mathbb{R})$ for which the solution (θ, q) blows up in finite $T < L$.

Remark 2.2. An analogous result can be obtained if (2.15) is replaced by $\rho'(0) > 0$.

Proof. We first take an x-partial derivative of (2.6) to get

$$\begin{aligned}(\partial_t r)_x &= r_{tx} + \rho r_{xx} + \rho' \theta_x r_x \\ &= -\frac{\lambda}{2}(r_x + s_x) - \frac{\lambda}{2}(r+s)\theta_x. \end{aligned} \tag{2.16}$$

We then use

$$\theta_x = \frac{r_x - s_x}{2\alpha} \tag{2.17}$$

to obtain

$$\begin{aligned}&\partial_t r_x + \rho' r_x \frac{r_x - s_x}{2\alpha} \\ &\quad = -\frac{\lambda}{2}(r_x + s_x) - \frac{\lambda}{2}(r+s)\frac{r_x - s_x}{2\alpha}. \end{aligned} \tag{2.18}$$

Straigthforward calculations then give

$$s_x = -\frac{1}{2\rho}\,\partial_t(r-s) \tag{2.19}$$

and substitution in (2.18) yields

$$\begin{aligned}\partial_t r_x = &-\rho'(r_x)^2 - \frac{\rho'}{4\alpha\rho}\, r\partial_t(r-s) - \left[\frac{\lambda}{2} + \frac{\lambda'}{2\alpha}(r+s)\right] r_x \\ &+ \left[\frac{\lambda}{2} - \frac{\lambda'}{2\alpha}(r+s)\right] \frac{\partial_t(r-s)}{2\rho}. \end{aligned} \tag{2.20}$$

We now introduce

$$\Phi := \rho^{-1/2} r_x \tag{2.21}$$

and substitute in (2.20) to obtain

$$\begin{aligned}\partial_t \Phi = &-\frac{\rho' \rho^{-1/2}}{2\alpha}\Phi^2 - \left[\frac{\lambda}{2} + \frac{\lambda'}{2\alpha}(r+s)\right]\Phi \\ &+ \frac{\rho^{-3/2}}{2}\left[\frac{\lambda}{2} - \frac{\lambda'}{2\alpha}(r+s)\right]\partial_t(r-s). \end{aligned} \tag{2.22}$$

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and substitute in (2.20) to obtain

$$\begin{aligned}\partial_t \Phi = &-\frac{\rho' \rho^{-1/2}}{2\alpha}\,\Phi^2 - \left[\frac{\lambda}{2} + \frac{\lambda'}{2\alpha}(r+s)\right]\Phi \\ &+ \frac{\rho^{-3/2}}{2}\left[\frac{\lambda}{2} - \frac{\lambda'}{2\alpha}(r+s)\right]\partial_t(r-s). \end{aligned} \tag{2.22}$$

The last term in (2.22) can be handled as follows

$$\frac{\rho^{-3/2}}{2}\frac{\lambda'}{2\alpha}(r+s)\,\partial_t(r-s)$$

$$=\frac{\lambda'\rho^{-3/2}}{2\alpha}(r-A)\,\partial_t(r-s) \tag{2.22}$$

$$=\frac{\lambda'\rho^{-3/2}}{2\alpha}\,r\partial_t(r-s)-\frac{\lambda'\rho^{-3/2}}{2\alpha}\,A\partial_t(r-s)$$

$$=\partial_t\left[r\int_0^{\theta}(\lambda'\rho^{-3/2})(\xi)\,d\xi\right]-\partial_t r\cdot\int_0^{\theta}(\lambda'\rho^{-3/2})(\xi)\,d\xi$$

$$-\partial_t\int_0^{\theta}(\lambda' A\rho^{-3/2})(\xi)\,d\xi. \tag{2.23}$$

We note that

$$\partial_t\theta=\frac{\partial_t(r-s)}{2}.$$

By combining (2.22), (2.23), we get

$$\partial_t\Phi=-\frac{\rho'\rho^{-1/2}}{2\alpha}\Phi^2-\left[\frac{\lambda}{2}+\frac{\lambda'}{2\alpha}(r+s)\right]\Phi$$

$$+\frac{\lambda}{2}(r+s)\cdot\int_0^{\theta}(\lambda'\rho^{-3/2})(\xi)\,d\xi+\partial_t f \tag{2.24}$$

where

$$f:=r\int_0^{\theta}(\lambda'\rho^{-3/2})(\xi)-\int_0^{\theta}\left[\rho^{-3/2}\left(\frac{\lambda\alpha}{2}+\lambda' A\right)\right](\xi)\,d\xi. \tag{2.25}$$

Now, we are ready to conclude our proof. For this purpose, we set

$$F:=\Phi-f. \tag{2.26}$$

Hence (2.24) gives

$$\partial_t F=-\frac{\rho'\rho^{-1/2}}{2\alpha}F^2+\frac{\lambda}{2}(r+s)\cdot\int_0^{\theta}(\lambda'\rho^{-3/2})(\xi)\,d\xi$$

$$-\left[\frac{\lambda}{2}+\frac{\lambda'}{2\alpha}(r+s)-\frac{\rho'\rho^{-1/2}}{2\alpha}f\right]F$$

$$-\frac{\rho'\rho^{-1/2}}{2\alpha}f^2-\left[\frac{\lambda}{2}+\frac{\lambda'}{2\alpha}(r+s)\right]f. \tag{2.27}$$

By choosing initial data small enough (in L^∞ norm), we are guaranteed to have

$$a := \inf \frac{\rho' \rho^{-1/2}}{2\alpha} > 0. \tag{2.28}$$

We set

$$M := \max \left| \frac{\lambda}{2} + \frac{\lambda'}{2\alpha} (r+s) - \frac{\rho' \rho^{-1/2}}{2\alpha} f \right| \tag{2.29}$$

$$m := \max \left| -\frac{\rho' \rho^{-1/2}}{2\alpha} f^2 - \left[\frac{\lambda}{2} + \frac{\lambda'}{2\alpha} (r+s) \right] f \right.$$

$$\left. + \frac{\lambda}{2} (r+s) \cdot \int_0^\infty (\lambda' \rho^{3/2})(\xi) \, d\xi \right|. \tag{2.30}$$

(We note that these maxima exist since they depend only on θ and q.) Thus, we have

$$\partial_t F \geqslant aF^2 - M|F| - m. \tag{2.31}$$

We then use

$$M|F| \leqslant \frac{a}{2} F^2 + \frac{1}{2a} M^2 \tag{2.32}$$

to obtain

$$\partial_t F \geqslant \frac{a}{2} (F^2 - B^2) \tag{2.33}$$

where

$$B = \frac{M^2}{a^2} + \frac{2m}{a}. \tag{2.34}$$

From Lemma 3.1 of [8], it suffices to choose θ_0 and q_0 small enough in the L^∞ norm and with positive derivatives such that $q_0'(x) + \alpha(\theta_0(x))\, \theta_0'(x)$ is large enough to make F blow up in a time $T < L$.

Remark 2.3. A similar result can also be obtained for certain initial-boundary value problem.

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