BLOWUP OF SOLUTIONS OF A NONLINEAR WAVE EQUATION

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We establish a blowup result to an initial boundary value problem for the nonlinear wave equation $u_{tt} - M(\|B^{1/2}u\|^2)Bu + ku_t = |u|^{p-2}u$, $x \in \Omega$, $t > 0$.

1. Introduction

We consider the initial boundary value problem (IBVP) for the nonlinear wave equation

$$\begin{aligned} &u_{tt} - Au + ku_t = |u|^{p-2}u, && x \in \Omega,\ t > 0, \\ &u(x,t) = 0, && x \in \partial\Omega,\ t \geq 0, \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), && x \in \Omega, \end{aligned} \tag{1.1}$$

where

$$\begin{gathered} Au = M\left(\|B^{1/2}u\|^2\right)e^{-\Phi(x)} \operatorname{div}\left(e^{\Phi(x)}\nabla u\right), \\ \|B^{1/2}u\|^2 = \int_\Omega e^{\Phi(x)}|\nabla u|^2 dx, \end{gathered} \tag{1.2}$$

$p > 2$ is a constant, k is a positive constant, $M : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, $\Phi \in L^\infty(\Omega)$, and $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary Γ so that the divergence theorem can be applied.

When $M \equiv 1$ and $\Phi \equiv 0$, for the case $k = 0$, it is well known that the source term $|u|^{p-2}u$ is responsible for finite blowup (global nonexistence) of solutions with negative initial energy (see [1, 9]). The interaction

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between the damping term and the source has been first considered by Levine [11, 12]. For $k > 0$, the author showed that solutions, with negative initial energy, blow up in finite time. In [5], Georgiev and Todorova extended Levine's result to the case of nonlinear damping of the form $|u_t|^m u_t$. This result was generalized to an abstract setup by Levine and Serrin [14], Levine et al. [13], and Vitillaro [18]. In [16], Messaoudi extended the result of Levine to the situation where $\Phi \neq 0$.

When $\Phi \equiv 0$ and M is not a constant function, the equation without the damping and source terms is often called the wave equation of Kirchhoff type which has been introduced by Kirchhoff [10] in order to study the nonlinear vibrations of an elastic string. The existence of local and global solutions in Sobolev and Gevrey classes was investigated by many authors (see [2, 3, 4, 6, 7, 8, 15, 17]).

In the present paper, we investigate the blowup of solutions of the initial boundary value problem (1.1). We show that, for suitably chosen initial data, any strong solution blows up in finite time. Our work is based on the results of [14].

2. Main result

In order to state our main result, we introduce the weighted space

$$L^s(\Omega,\Phi) := \left\{ v : \Omega \longrightarrow \mathbb{R} / \int_\Omega e^{\Phi(x)} |u_0|^s dx < \infty \right\},$$
$$E(0) = \frac{1}{2}\int_\Omega e^{\Phi(x)} u_1^2 dx + \frac{1}{2}\bar{M}\left(\|B^{1/2}u_0\|^2\right) - \frac{1}{p}\int_\Omega e^{\Phi(x)} |u_0|^p dx. \tag{2.1}$$

We also make the following hypothesis:

$$M \in C(\mathbb{R}_+,\mathbb{R}_+), \quad \bar{M}(s) = \int_0^s M(k)dk, \tag{2.2}$$

such that

$$r\bar{M}(s) \geq sM(s), \quad \forall s \geq 0,\ 1 < r < \frac{p}{2}. \tag{2.3}$$

Theorem 2.1. *Let $p > 2$ and assume that (2.2) and (2.3) hold. Then, for any initial data satisfying $E(0) < 0$, the solution of (1.1) blows up in finite time.*

Proof. Except for the operator Au, this problem is similar to [14, problem (4.1)–(4.3)] for $l = 2$. So the proof goes exactly like the one of [14, Theorem 5]. It remains only to show that Au and $F(u) = |u|^{p-2}u$ satisfy conditions (1s) and (2s) in [14, page 346]. To do this, we set

$$V = \Upsilon = L^2(\Omega,\Phi), \qquad W = L^p(\Omega,\Phi),$$
$$D = H_0^1(\Omega,\Phi) = \{u \in H_0^1(\Omega,\Phi)/u,\ \nabla u \in \Upsilon\}. \tag{2.4}$$

It is clear that A and F are Frechet derivatives of the C^1 real-valued potentials given by

$$\mathcal{A}u = \frac{1}{2}\bar{M}\left(\|B^{1/2}u\|^2\right), \qquad \mathcal{F}(u) = \frac{1}{p}\|u\|_W^p. \tag{2.5}$$

Now we have, by virtue of (2.3),

$$\begin{aligned}\langle Au,u\rangle_V &= \int_\Omega e^{\Phi(x)}uM\left(\|B^{1/2}u\|^2\right)e^{-\Phi(x)}\operatorname{div}\left(e^{\Phi(x)}\nabla u\right)\\ &= M\left(\|B^{1/2}u\|^2\right)\int_\Omega u\operatorname{div}\left(e^{\Phi(x)}\nabla u\right)\\ &= \left(\|B^{1/2}u\|^2\right)\|B^{1/2}u\|^2 \le r\bar{M}\left(\|B^{1/2}u\|^2\right) \le 2r\mathcal{A}u,\\ \langle F(u),u\rangle_V - 2r\mathcal{F}(u) &= \left(1-\frac{2r}{p}\right)\|u\|_W^p = (p-2r)\mathcal{F}(u).\end{aligned} \tag{2.6}$$

Therefore, conditions (1s) and (2s) in [14, page 346] are satisfied. This completes the proof. □

Remark 2.2. Conditions (3s) and (1d)–(3d) of [14] are automatically satisfied since $Pu_t = u_t$ and $Q(t,u_t) = ku_t$ are linear. See the proof of [14, Theorem 5].

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