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Applied Mathematics Letters 12 (1999) 23–28

Applied Mathematics Letters

Formation of Singularities in Solutions of a Wave Equation

M. A. SALIM
Department of Mathematics
King Fahd University of Petroleum and Minerals
Dharan 31261, Saudi Arabia
messaoud@kfupm.edu.sa

(Received and accepted August 1998)

Communicated by M. Slemrod

Abstract—We prove a blow up result for the equation $w_{tt}(x,t) = a(x)\varphi(w_x(x,t))w_{xx}(x,t)$, which can be taken as a model for a transverse motion of a string with nonconstant density. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords—Wave equation, Smooth solutions, Singularities, Nonlinear response, Global solutions.

1. INTRODUCTION

The aim of this paper is to study the existence and nonexistence of classical solutions to the one-dimensional nonlinear equation of the form

$$w_{tt}(x,t) = a(x)\varphi(w_x(x,t))w_{xx}(x,t), \tag{1}$$

where $x \in I$ (bounded or nonbounded interval), $t \geq 0$. This equation can be regarded as a model for a transverse motion of a nonhomogeneous vibrating string (the density is a function of x). By assuming that

$$a(x) > 0, \quad \forall x \in I, \qquad \varphi(\xi) > 0, \quad \forall \xi \in \mathbb{R}, \tag{2}$$

equation (1) is strictly hyperbolic.

Generally, classical solutions of problems associated with (1) develop singularities in finite time, if the elastic response function φ is 'genuinely' nonlinear. Many authors studied 'initial' boundary value problems associated to (1), with $a(x) \equiv 1$, and proved results concerning existence and formation of singularities. Lax [1] and MacCamy and Mizel [2] showed that classical solutions break down in finite time even for smooth and small initial data. In his work, Lax assumed that φ' does not change sign, whereas MacCamy and Mizel allowed φ' to change sign. They also showed, under appropriate conditions on φ, that intervals of x can exist, in which the solution must exist for all time t even though it breaks down for values x outside these intervals.

The author would like to express his deep thanks to KFUPM for its sincere support.

0893-9659/99/\$ - see front matter Typeset by AMS-TEX
PII: S0893-9659(99)00027-0

In the dissipative case, the situation is different. For initial data small and smooth enough, the effect of the damping term dominates the nonlinear elastic response and global solutions can be obtained (see [3]). However, for large initial data the nonlinearity in the elastic response takes over and classical solutions may develop singularities in finite time. These results have been established by several authors (see [4–6]).

It is interesting to mention that nonlinear hyperbolic systems, of which equation (1) with $a \equiv 1$ is a special case, have attracted the attention of many authors, and several results concerning global existence and blow up have been established (see [6–10]).

This work will be divided into two parts. In the first part we state, without proof, a local existence theorem. In the second part, we state and prove our main blow up result.

2. LOCAL EXISTENCE

In this section, we state a local existence theorem. The proof is omitted. It can be easily established by either using a classical energy argument [11], or applying the nonlinear semigroup theory presented in [12]. We set

$$u(x,t) := w_t(x,t), \qquad v(x,t) := w_x(x,t)$$

and substitute in (1) to get the system

$$\begin{aligned} u_t(x,t) &= a(x)\varphi(v(x,t))v_x(x,t), \\ v_t(x,t) &= u_x(x,t), \qquad x \in \mathbb{R}, \quad t \geq 0. \end{aligned} \tag{3}$$

We consider (3) together with the initial data

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x \in \mathbb{R}. \tag{4}$$

In order to state the local existence result, we make the following hypotheses.

(H1)

$$a \in W^{1,+\infty}(\mathbb{R}), \qquad \varphi \in C^2(\mathbb{R}).$$

(H2)

$$a(x) \geq \alpha, \qquad \varphi(\xi) \geq \alpha, \qquad x, \xi \in \mathbb{R}, \qquad \alpha > 0.$$

PROPOSITION. *Assume that (H1), (H2) hold and let u_0, v_0 in $H^2(\mathbb{R})$ be given. Then the initial value problem (3),(4) has a unique local solution (u,v) on a maximal time interval $[0,T)$ such that*

$$u, v \in C\left([0,T); H^2(\mathbb{R})\right) \cap C^1\left([0,T); H^1(R)\right). \tag{5}$$

REMARK 2.1. The Sobolev embedding theorem implies that u, v are C^1 functions on $\mathbb{R} \times [0,T)$. Hence (u,v) is a classical solution.

REMARK 2.2. If φ is a C^{k+1} function and $u_0, v_0 \in H^k(\mathbb{R})$, then $u(\cdot,t), v(\cdot,t) \in H^k(\mathbb{R})$, $k \geq 1$.

REMARK 2.3. A similar result holds if φ were depending on both u and v.

3. FORMATION OF SINGULARITIES

In this section, we state and prove our main result. We first start with establishing uniform upper bounds on the solution (u,v) in terms of the initial data.

LEMMA. *Assume (H1) and (H2) hold. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that given any u_0, v_0 in $H^2(\mathbb{R})$ satisfying*

$$|u_0(x)| < \delta, \quad |v_0(x)| < \delta, \qquad \forall x \in \mathbb{R}, \tag{6}$$

the solution (3) obeys

$$u(x,t)| < \epsilon, \quad |v(x,t)| < \epsilon, \quad \forall x \in \mathbb{R}, \quad t \in [0,T). \tag{7}$$

PROOF. We define the quantities

$$\begin{aligned} r(x,t) &:= \frac{u(x,t)}{\sqrt{a(x)}} + A(v(x,t)), \\ s(x,t) &:= \frac{u(x,t)}{\sqrt{a(x)}} - A(v(x,t)), \end{aligned} \tag{8}$$

and the differential operators

$$\begin{aligned} \partial_t^- &:= \frac{1}{\rho(x,t)} \frac{\partial}{\partial t} - \frac{\partial}{\partial x}, \\ \partial_t^+ &:= \frac{1}{\rho(x,t)} \frac{\partial}{\partial t} + \frac{\partial}{\partial x}, \end{aligned} \tag{9}$$

where

$$A(z) = \int_0^z \sqrt{\varphi(\xi)}\, d\xi, \qquad \rho(x,t) = \sqrt{a(x)\varphi(v(x,t))}. \tag{10}$$

Straightforward computations yield

$$\partial_t^- r = \partial_t^+ s = -\frac{a'}{4a}(r+s). \tag{11}$$

We then define the nonegative functions

$$R(t) := \max_x |r(x,t)|, \quad S(t) := \max_x |s(x,t)|, \qquad t \in [0,T). \tag{12}$$

These maxima are attained since r and s die at infinity; consequently for each $t \in [0,T)$, there exist $\hat{x}, \check{x} \in \mathbb{R}$ such that

$$R(t) = |r(\hat{x},t)|, \qquad S(t) = |s(\check{x},t)|. \tag{13}$$

Also by the definitions of R and S, we have

$$\begin{aligned} R(t-h) &\geq |r(\hat{x} + h\rho(\hat{x},t), t-h)|, \\ S(t-h) &\geq |s(\check{x} - h\rho(\check{x},t), t-h)|, \qquad 0 < h < t. \end{aligned} \tag{14}$$

We subtract (14) from (13), divide by h, and let h go to zero to arrive at

$$\begin{aligned} \dot{R}(t) &\leq \frac{|a'(\hat{x})|}{4\sqrt{a(\hat{x})}} \sqrt{\varphi(v(\hat{x},t))}(R(t)) + S(t)) \\ \dot{S}(t) &\leq \frac{|a'(\check{x})|}{4\sqrt{a(\check{x})}} \sqrt{\varphi(v(\check{x},t))}(R(t)) + S(t)), \end{aligned} \tag{15}$$

for almost every $t \in [0,T)$. By setting

$$M := \max_{|\xi| \leq \epsilon} \sqrt{\varphi(\xi)} \qquad (\epsilon \text{ given in the lemma})$$

and using (H1), (H2), and (15), we get

$$\frac{d}{dt}(R(t)+S(t)) \le kM(R(t)+S(t)), \qquad k>0, \tag{16}$$

for almost every $t \in [0,T)$, provided that

$$|v(x,t)| < \epsilon. \tag{17}$$

Therefore, (16) and Gronwall's inequality yield

$$(R(t)+S(t)) \le (R(0)+S(0))e^{kMT}, \tag{18}$$

for any $t \in [0,T)$, provided that (12) holds. We then choose $\delta > 0$ small enough so that

$$\frac{(R(0)+S(0))e^{kMT}}{2\alpha} < \frac{\epsilon}{2}. \tag{19}$$

Thus, we conclude that if v satisfies (12), it satisfies, in fact,

$$|v(x,t)| < \frac{\epsilon}{2},$$

by (8), (10), (18), and (H2). Therefore, by continuity, (7) is established.

REMARK 3.1. We can obtain the same result if $\varphi(\xi) \ge \alpha > 0$ holds only on a neighborhood of zero.

THEOREM. *Assume that (H1) and (H2) hold. Assume further that*

$$a'' \in L^\infty(\mathbb{R}), \qquad \varphi'(0) < 0. \tag{20}$$

Then we can choose initial data u_0, v_0 in $H^2(\mathbb{R})$ such that the solution (3) blows up in finite time.

PROOF. We take a t-partial derivative of (11) to get

$$\partial_t^- r_t = -\frac{1}{2}a^{1/2}\varphi^{-1/2}\varphi' v_t r_t - \frac{a'}{4a}(r_t+s_t). \tag{21}$$

By using

$$u_t = \frac{\sqrt{a}}{2}(r_t+s_t), \qquad v_t = \frac{r_t - s_t}{2\sqrt{\varphi}} \tag{22}$$

and substituting in (21), we obtain

$$\partial_t^- r_t = -\frac{a^{1/2}\varphi'}{4\varphi}r_t^2 + \frac{a^{1/2}\varphi'}{4\varphi}r_t s_t - \frac{a'}{4a}(r_t+s_t). \tag{23}$$

We then set

$$W(x,t) := \gamma(x,t)r_t(x,t), \qquad \gamma(x,t) := e^{(a(x)/4)\varphi(v(x,t))} \tag{24}$$

and substitute in (23) to arrive at

$$\partial_t^- W = -\frac{a^{1/2}\varphi'}{4\gamma\varphi}W^2 - \frac{a^{1/2}\varphi' a'}{16}W - \frac{a'}{4a}W - \frac{a'\gamma}{4a}s_t. \tag{25}$$

To handle the last term in (25), we first note that

$$s_t = -a^{1/2}\varphi\partial_t^- v$$

and then introduce the function

$$f(x,t) := \int_0^{v(x,t)} e^{(a(x)/4)\varphi(\xi)}\varphi(\xi)\, d\xi.$$

Direct calculations yield

$$\begin{aligned}
-\frac{a'\gamma}{4a}s_t &= \frac{a'a^{-1/2}\gamma}{4}\varphi\partial_t^- v \\
&= \frac{1}{4}\partial_t^-(a'a^{-1/2}f) - \frac{1}{16}a'^2a^{-1/2}\int_0^v e^{(a(x)/4)\varphi(\xi)}\varphi^2(\xi)\, d\xi \\
&\quad - \frac{1}{4}\left(a''a^{-1/2} - \frac{1}{2}a'^2a^{-3/2}\right).
\end{aligned} \tag{26}$$

By substituting in (25), we have

$$\partial_t^- W = -\frac{a^{1/2}\varphi'}{4\gamma\varphi}W^2 + AW + B + \partial_t^-\left(\frac{a'a^{-1/2}}{4}f\right), \tag{27}$$

where A and B are functions depending continuously and only on a, a', a'', *and* v. By setting

$$F := W - \frac{a'a^{-1/2}}{4}f$$

and substituting in (27), we get

$$\partial_t^- F = -\frac{a^{1/2}\varphi'}{4\gamma\varphi}\left(F + \frac{a'a^{1/2}}{4}f\right)^2 + \left(F + \frac{a'a^{-1/2}}{4}f\right) + B. \tag{28}$$

By applying Young's inequality to the first and the second term, we obtain

$$\partial_t^- F \geq -\frac{a^{1/2}\varphi'}{8\gamma\varphi}F^2 + C(x,t) \tag{29}$$

where C is a function depending on f, f^2, φ, φ', γ, a, a', and a$''$. We choose $\delta > 0$ so small that

$$\max_{x,t}\left(\frac{-a^{1/2}\varphi'(v(x,t))}{8\gamma(x,t)\varphi(v(x,t))}\right) \geq \beta$$

and

$$\max_{x,t}|C(x,t)| \leq C. \tag{30}$$

Therefore, (29) yields

$$\partial_t^- F \geq \beta F^2 - C. \tag{31}$$

By choosing initial data small in L^∞ norm so that (30) is satisfied with derivatives large enough, the quadratic term in (31) blows up in finite time.

REMARK 3.2. The same result can be obtained if $\varphi'(0) < 0$ is replaced by $\varphi'(0) > 0$.

REMARK 3.3. The calculations show that the larger the derivatives are, the shorter the time of existence of the solution is.