A GLOBAL NONEXISTENCE RESULT FOR THE NONLINEARLY DAMPED MULTI-DIMENSIONAL BOUSSINESQ EQUATION

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الخلاصــة

سوف نناقش في هذا البحث مسألة ابتدائية حدية غير خطية متعددة الأبعاد تتعلق بمعادلة بوسيناسك ، ونثبت نتيجة حدوث شذوذ في الحلول الضعيفة. وسوف نطور في هذا البحث أعمالاً سابقة تتعلق بهذه المعادلة سواء في البعد الواحد أو الأبعاد المتعددة.

ABSTRACT

In this paper we consider a multi-dimensional nonlinear initial-boundary value problem related to the Boussinesq equation and prove a global nonexistence result. This work improves an earlier one by Gmira and Guedda [1].

Keywords: Nonlinear damping, Boussinesq equation, source, nonexistence.

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1. INTRODUCTION

The Boussinesq equation

$$u_{tt} + \alpha u_{xxxx} - u_{xx} = \beta \left(u^2\right)_{xx}, \quad x \in \mathbb{R}, \quad t > 0 \tag{1.1}$$

where $\alpha, \beta > 0$, was first derived by Boussinesq [2] in 1872 and since then very many mathematicians have studied it and used it to model real world problems such as the propagation of long waves on shallow water and oscillations of nonlinear elastic beams. Varlamov [3] considered the damped equation of the form

$$u_{tt} - 2bu_{txx} + \alpha u_{xxxx} - u_{xx} = \beta \left(u^2\right)_{xx}, \quad x \in (0, \pi), \quad t > 0, \tag{1.2}$$

for small initial data and constructed, for the case $\alpha > b^2$, the solution in the form of a Fourier series. He also showed that, on $[0, T), T < \infty$, the solution of (1.1) is obtained by letting b go to zero. In 2001, Varlamov [4] improved his earlier result by considering the three-dimensional version of (1.2) in the unit ball and used the eigenfunctions of the Laplace operator to construct solutions. He examined the problem, for homogeneous boundary conditions and small initial data, and obtained global mild solutions in appropriate Sobolev spaces. He also addressed the issue of the uniqueness and the long-time behavior of the solution. Lai and Wu [5] considered the following more generalized equation

$$u_{tt} - au_{ttxx} - 2bu_{txx} + cu_{xxxx} - u_{xx} = -p^2 u + \beta \left(u^2\right)_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.3}$$

where $a, b, c > 0$, $p \neq 0$, and β is a real number. They used the Fourier transform and the perturbation theory to establish the well-posedness of global solutions to small initial data for the Cauchy problem. The same techniques have been applied by Lai *et al.* [6] to establish a global existence and an exponential decay results for an initial-boundary value problem related to (1.3).

For the nonexistence, we mention the result of Levine and Sleeman [7], in which the authors considered an initial boundary value problem related to the eqaution

$$u_{tt} = 3u_{xxxx} + u_{xx} - 12 \left(u^2\right)_{xx} \tag{1.4}$$

and showed that, under appropriate conditions for the initial data, no positive weak or classical solution can exist for all time. Recently Bayrack and Can [8] studied the behavior of a one-dimensional riser vibrating due to effects of waves and current involving linear dissipation. Precisely, they looked into the following problem

$$\begin{cases} u_{tt} + \alpha u_t + 2\beta u_{xxxx} - 2\left[(ax+b)\, u_x\right]_x + \frac{\beta}{3}\left(u_x^3\right)_{xxx} \\ -\left[(ax+b)\, u_x^3\right]_x - \beta\left(u_{xx}^2 u_x\right)_x = f(u)\,, \ (x,t) \in (0,1) \times (0,T) \\ u(x,0) = u_0(x)\,, \ u_t(x,0) = u_1(x)\,, \ x \in (0,1) \\ u(0,t) = u(1,t) = 0, \ u_{xx}(0,t) = u_{xx}(1,t) = 0, \ t \in (0,T)\,. \end{cases} \tag{1.5}$$

and proved that, under suitable conditions on f and the initial data, all solutions of (1.5) blow up in finite time in the L^2 space. To establish their result, the authors used the standard concavity method due to [9]. Gmira and Guedda [1] extended the result of [8] to the multi-dimensional version of the problem (1.5). So, they considered

$$\begin{cases} u_{tt}+\rho(x)\,u_t+\beta\triangle^2 u-div\,(g(x)\nabla u)+\Gamma\triangle\left(|\nabla u|^2\triangle u\right) \\ \quad -div\left(h(x)\,|\nabla u|^{p-2}\nabla u\right)-\Gamma div\left((\triangle u)^2\nabla u\right)=f(u) \end{cases} \tag{1.6}$$

and established a nonexistence result, under suitable conditions on u_0, u_1, f, by using the "modified" concavity method introduced in [10]. The use of the latter method by Gmira and Guedda allowed them to remove the condition of cooperative initial data $\left(\int_\Omega u_0u_1dx>0\right)$ imposed by Bayrack and Can [8]. However, some conditions can be further weakened.

In this paper we are concerned with the following nonlinearly damped problem

$$\begin{cases} u_{tt}+\rho(x)\,|u_t|^{m-2}\,u_t+\beta\triangle^2 u-div\,(g(x)\nabla u)+\Gamma\triangle\left(|\nabla u|^2\triangle u\right) \\ -div\left(h(x)\,|\nabla u|^{p-2}\nabla u\right)-\Gamma div\left((\triangle u)^2\nabla u\right)=|u|^{l-2}\,u, \quad x\in\Omega, \quad t>0 \\ u(x,0)=u_0(x)\,, \ \ u_t(x,0)=u_1(x)\,,\ x\in\Omega, \\ u(x,t)=\frac{\partial u}{\partial\eta}(x,t)=0,\ x\in\partial\Omega, \quad t>0, \end{cases} \tag{1.7}$$

where $\Omega\subset\mathbb{R}^n$, $n\geq 1$, is a bounded domain with sufficiently smooth boundary, η is the unit outer normal on $\partial\Omega$, $\rho\geq 0$, is a smooth bounded function given on Ω, g, $h\in C^1\left(\overline{\Omega},\mathbb{R}^+\right)$, p, l, $m\geq 1$, and β and Γ are nonnegative constants. In addition to allowing the damping to be nonlinear, we establish a blow up result under weaker conditions, than those required in [1], on the initial data as well as the constants p, l, and m. To achieve our goal we exploit the method of Georgiev and Todorova [11] (see also [12]). This work is divided into three sections. In Section two we state and demonstrate our main result. In Section three, the linear damping ($m=2$) case is treated.

2. MAIN RESULT

In order to state and prove our result, we introduce the energy functional

$$\begin{aligned} E(t)=\frac{1}{2}\int_\Omega u_t^2dx+\frac{\beta}{2}\int_\Omega(\triangle u)^2dx+\frac{1}{2}\int_\Omega g\,|\nabla u|^2dx \\ +\frac{\Gamma}{2}\int_\Omega(\triangle u)^2|\nabla u|^2dx+\frac{1}{p}\int_\Omega h\,|\nabla u|^p-\frac{1}{l}\int_\Omega|u|^l dx. \end{aligned} \tag{2.1}$$

Theorem 1. *Assume that $m, p\geq 1$, and $l>\max\{4,m,p\}$. Assume further that*

$$E(0)<0. \tag{2.2}$$

Then any classical solution of (1.7) blows up in finite time.

Remark 2.1. In [1], the authors require that $l > 2(4+\gamma) > p, \gamma > 0$, which is obviously stronger than our requirements on both l and p. (See (2.4) of [1]). Moreover in [1], l may depend on $||\rho||_\infty$ since γ does (See (2.6) of [1] again).

Remark 2.2. The result can be established for weak solution by means of density.

Proof. A multiplication of Equation (1.7) by u_t and integration over Ω yields

$$E^{'}(t) = -\int_\Omega \rho(x)\left|u_t\right|^m dx \leq 0. \tag{2.3}$$

By setting $H(t) = -E(t)$, we get from (2.1) and (2.2),

$$0 < H(0) \leq H(t) \leq \frac{1}{l}\int_\Omega |u|^l\, dx, \qquad \forall\, t \geq 0. \tag{2.4}$$

We then define

$$L(t) = H^{1-\sigma}(t) + \varepsilon \int_\Omega u u_t dx \tag{2.5}$$

for ε small to be chosen later and

$$0 < \sigma \leq \min\left(\frac{l-m}{l(m-1)},\ \frac{l-2}{2l}\right). \tag{2.6}$$

By taking a derivative of (2.5) we obtain

$$L^{'}(t) = (1-\sigma)\, H^{-\sigma}(t)\, H^{'}(t) + \varepsilon\int_\Omega u_t^2 dx + \varepsilon\int_\Omega u u_{tt} dx. \tag{2.7}$$

By using (1.7), Equation (2.7) becomes

$$\begin{aligned} L^{'}(t) = {} & (1-\sigma)\, H^{-\sigma}(t)\, H^{'}(t) + \varepsilon\int_\Omega u_t^2 dx \\ & - \beta\varepsilon\int_\Omega (\triangle u)^2\, dx - \varepsilon\int_\Omega \rho\left|u_t\right|^{m-2} u_t u dx \\ & - \varepsilon\int_\Omega g\left|\nabla u\right|^2 dx - 2\Gamma\varepsilon\int_\Omega \left|\nabla u\right|^2 (\triangle u)^2\, dx \\ & - \varepsilon\int_\Omega h\left|\nabla u\right|^p dx + \varepsilon\int_\Omega |u|^l\, dx. \end{aligned} \tag{2.8}$$

We then exploit Young's inequality to get

$$\begin{aligned} \int_\Omega \rho\left|u_t\right|^{m-1} u dx &\leq \frac{\lambda^m}{m}\int_\Omega |u|^m\, dx + \frac{m-1}{m}\lambda^{-m/(m-1)}\int_\Omega \left|\rho^{1/(m-1)} u_t\right|^m dx \\ &\leq \frac{\lambda^m}{m}\int_\Omega |u|^m\, dx + b\frac{m-1}{m}\lambda^{-m/(m-1)}\int_\Omega \rho\left|u_t\right|^m dx. \end{aligned}$$

where $b = \|\rho\|_\infty^{1/(m-1)}$. This yields, by substitution in (2.8),

$$\begin{aligned}
L'(t) \geq{} & (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon \int_\Omega u_t^2 dx \\
& - \beta\varepsilon \int_\Omega (\triangle u)^2 dx - \varepsilon \frac{\lambda^m}{m} \int_\Omega |u|^m dx \\
& - \varepsilon b \frac{m-1}{m} \lambda^{-m/(m-1)} \int_\Omega \rho |u_t|^m dx \\
& - \varepsilon \int_\Omega g |\nabla u|^2 dx - 2\Gamma\varepsilon \int_\Omega |\nabla u|^2 (\triangle u)^2 dx \\
& - \varepsilon \int_\Omega h |\nabla u|^p dx + \varepsilon \int_\Omega |u|^l dx.
\end{aligned} \tag{2.9}$$

Therefore, choosing λ so that

$$\lambda^{-m/(m-1)} = M H^{-\sigma}(t),$$

for large M to be specified later, and substituting in (2.9), we arrive at

$$\begin{aligned}
L'(t) \geq{} & (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon \int_\Omega u_t^2 dx \\
& - \beta\varepsilon \int_\Omega (\triangle u)^2 dx - \varepsilon \frac{M^{-(m-1)}}{m} H^{\sigma(m-1)}(t) \int_\Omega |u|^m dx \\
& - \varepsilon b \frac{m-1}{m} M H^{-\sigma}(t) H'(t) - \varepsilon \int_\Omega g |\nabla u|^2 dx \\
& - 2\Gamma\varepsilon \int_\Omega |\nabla u|^2 (\triangle u)^2 dx - \varepsilon \int_\Omega h |\nabla u|^p dx + \varepsilon \int_\Omega |u|^l dx.
\end{aligned}$$

That is

$$\begin{aligned}
L'(t) \geq{} & \left[(1-\sigma) - \varepsilon b \frac{m-1}{m} M\right] H^{-\sigma}(t) H'(t) + \varepsilon \int_\Omega u_t^2 dx \\
& - \beta\varepsilon \int_\Omega (\triangle u)^2 dx - \varepsilon \frac{M^{-(m-1)}}{m} H^{\sigma(m-1)}(t) \int_\Omega |u|^m dx \\
& - \varepsilon \int_\Omega g |\nabla u|^2 dx - 2\Gamma\varepsilon \int_\Omega |\nabla u|^2 (\triangle u)^2 dx \\
& - \varepsilon \int_\Omega h |\nabla u|^p dx + \varepsilon \int_\Omega |u|^l dx.
\end{aligned} \tag{2.10}$$

We then use the embedding $L^l(\Omega) \hookrightarrow L^m(\Omega)$ to get

$$\int_\Omega |u|^m\, dx \le C\left(\int_\Omega |u|^l\, dx\right)^{m/l},$$

where C is a positive constant depending on Ω only. So we have, from (2.4),

$$H^{\sigma(m-1)}(t)\int_\Omega |u|^m\, dx \le \frac{C}{l}\left(\int_\Omega |u|^l\, dx\right)^{\sigma(m-1)+(m/l)}.$$

By using (2.6) and the inequality

$$z^\nu \le z+1 \le \left(1+\frac{1}{a}\right)(z+a), \quad \forall z>0,\ 0<\nu\le 1,\ a\ge 0, \tag{2.11}$$

we have the following

$$\begin{aligned}\left(\int_\Omega |u|^l\, dx\right)^{\sigma(m-1)+(m/l)} &\le d\left(\int_\Omega |u|^l\, dx + H(0)\right) \\ &\le d\left(\int_\Omega |u|^l\, dx + H(t)\right),\ \forall\, t\ge 0,\end{aligned} \tag{2.12}$$

where $d = 1 + 1/H(0)$. Inserting the estimate (2.12) into (2.10) we get

$$\begin{aligned}L'(t) \ge{}& \left[(1-\sigma) - \varepsilon b\frac{m-1}{m}M\right] H^{-\sigma}(t)\, H'(t) + \varepsilon\int_\Omega u_t^2 dx \\ &- \beta\varepsilon\int_\Omega (\triangle u)^2\, dx - \varepsilon C d\frac{M^{-(m-1)}}{lm}\left(\int_\Omega |u|^l\, dx + H(t)\right) \\ &- \varepsilon\int_\Omega g\,|\nabla u|^2\, dx - 2\Gamma\varepsilon\int_\Omega |\nabla u|^2 (\triangle u)^2\, dx \\ &- \varepsilon\int_\Omega h\,|\nabla u|^p\, dx + \varepsilon\int_\Omega |u|^l\, dx.\end{aligned}$$

By using (2.1) and $H(t) = -E(t)$, we can write, for some positive constant K,

$$\begin{aligned}L'(t) \ge{}& \left[(1-\sigma) - \varepsilon b\frac{m-1}{m}M\right] H^{-\sigma}(t)\, H'(t) + \left(\frac{K}{2}+\varepsilon\right)\int_\Omega u_t^2 dx \\ &+ \beta\left(\frac{K}{2}-\varepsilon\right)\int_\Omega (\triangle u)^2\, dx + \left(K - \varepsilon C d\frac{M^{-(m-1)}}{lm}\right) H(t) \\ &+ \left(\varepsilon - \frac{K}{l} - \varepsilon C d\frac{M^{-(m-1)}}{lm}\right)\int_\Omega |u|^l\, dx + \left(\frac{K}{2}-\varepsilon\right)\int_\Omega g\,|\nabla u|^2\, dx \\ &+ \Gamma\left(\frac{K}{2}-2\varepsilon\right)\int_\Omega |\nabla u|^2 (\triangle u)^2\, dx + \left(\frac{K}{p}-\varepsilon\right)\int_\Omega h\,|\nabla u|^p\, dx.\end{aligned} \tag{2.13}$$

At this point we choose $K = r\varepsilon$, for $r = \max\{p, 4\}$; hence (2.13) becomes

$$\begin{aligned} L'(t) &\geq \left[(1-\sigma) - \varepsilon b \frac{m-1}{m} M\right] H^{-\sigma}(t) H'(t) + \varepsilon \left(\frac{r}{2}+1\right) \int_\Omega u_t^2 dx \\ &+ \beta\varepsilon \left(\frac{r}{2}-1\right) \int_\Omega (\triangle u)^2 dx + \varepsilon \left(r - Cd\frac{M^{-(m-1)}}{lm}\right) H(t) \\ &+ \varepsilon \left(1 - \frac{r}{l} - Cd\frac{M^{-(m-1)}}{lm}\right) \int_\Omega |u|^l dx + \varepsilon \left(\frac{r}{2}-1\right) \int_\Omega g |\nabla u|^2 dx \\ &+ \Gamma\varepsilon \left(\frac{r}{2}-2\right) \int_\Omega |\nabla u|^2 (\triangle u)^2 dx + \varepsilon \left(\frac{r}{p}-1\right) \int_\Omega h |\nabla u|^p dx \\ &\geq \left[(1-\sigma) - \varepsilon b \frac{m-1}{m} M\right] H^{-\sigma}(t) H'(t) + \varepsilon \left(\frac{r}{2}+1\right) \int_\Omega u_t^2 dx \\ &+ \varepsilon \left(r - Cd\frac{M^{-(m-1)}}{lm}\right) H(t) + \varepsilon \left(1 - \frac{r}{l} - Cd\frac{M^{-(m-1)}}{lm}\right) \int_\Omega |u|^l dx. \end{aligned} \tag{2.14}$$

We then choose M large enough so that

$$a_1 = r - Cd\frac{M^{-(m-1)}}{lm} > 0, \quad a_2 = 1 - \frac{r}{l} - Cd\frac{M^{-(m-1)}}{lm} > 0.$$

Therefore (2.14) yields

$$\begin{aligned} L'(t) &\geq \left((1-\sigma) - \varepsilon b \frac{m-1}{m} M\right) H^{-\sigma}(t) H'(t) \\ &+ \gamma\varepsilon \left[H(t) + \int_\Omega u_t^2 dx + \int_\Omega |u|^l dx\right], \end{aligned} \tag{2.15}$$

where

$$\gamma = \max\{a_1, a_2, \frac{r}{2}+1\}.$$

Once M is fixed (hence γ), we choose ε sufficiently small that

$$(1-\sigma) - \varepsilon b \frac{m-1}{m} M \geq 0$$

and

$$L(0) = H^{1-\sigma}(0) + \varepsilon \int_\Omega u_0 u_1 dx > 0.$$

Therefore we have, from (2.15),

$$L^{'}(t) \geq \gamma\varepsilon \left[H(t) + \int_{\Omega} u_t^2 dx + \int_{\Omega} |u|^l \, dx\right], \tag{2.16}$$

and

$$L(t) \geq L(0) > 0, \;\; \forall\, t \geq 0.$$

Next, it is clear that

$$L^{\frac{1}{1-\sigma}}(t) \leq 2^{\frac{1}{1-\sigma}} \left\{H(t) + \varepsilon^{\frac{1}{1-\sigma}} \left(\int_{\Omega} u_t u dx\right)^{\frac{1}{1-\sigma}}\right\}.$$

By the Cauchy–Schwarz inequality and the embedding of the $L^p(\Omega)$ spaces we have

$$\begin{aligned}\left|\int_{\Omega} u_t u dx\right| &\leq \left(\int_{\Omega} u^2 dx\right)^{1/2} \left(\int_{\Omega} u_t^2 dx\right)^{1/2} \\ &\leq C \left(\int_{\Omega} |u|^l \, dx\right)^{1/l} \left(\int_{\Omega} u_t^2 dx\right)^{1/2},\end{aligned}$$

which implies

$$\left|\int_{\Omega} u_t u dx\right|^{\frac{1}{1-\sigma}} \leq C \left(\int_{\Omega} |u|^l \, dx\right)^{\frac{1}{(1-\sigma)l}} \left(\int_{\Omega} u_t^2 dx\right)^{\frac{1}{2(1-\sigma)}}.$$

Also Young's inequality gives

$$\left|\int_{\Omega} u_t u dx\right|^{\frac{1}{1-\sigma}} \leq C \left[\left(\int_{\Omega} |u|^l \, dx\right)^{\frac{\mu}{(1-\sigma)l}} + \left(\int_{\Omega} u_t^2 dx\right)^{\frac{\theta}{2(1-\sigma)}}\right]$$

for $1/\mu + 1/\theta = 1$. We take $\theta = 2(1-\sigma)$, (hence $\mu = \frac{2(1-\sigma)}{(1-2\sigma)}$) to get

$$\left|\int_{\Omega} u_t u dx\right|^{\frac{1}{1-\sigma}} \leq C \left[\left(\int_{\Omega} |u|^l \, dx\right)^{\frac{2}{(1-2\sigma)l}} + \int_{\Omega} u_t^2 dx\right].$$

Again by using (2.6) and (2.11) we deduce, as in (2.12),

$$\left(\int_{\Omega} |u|^l \, dx\right)^{\frac{2}{(1-2\sigma)l}} \leq d \left(\int_{\Omega} |u|^l \, dx + H(t)\right).$$

Therefore,

$$\left|\int_{\Omega} u_t u dx\right|^{\frac{1}{1-\sigma}} \leq C \left[H(t) + \int_{\Omega} |u|^l \, dx + \int_{\Omega} u_t^2 dx\right], \;\; \forall t \geq 0;$$

consequently

$$L^{\frac{1}{1-\sigma}}(t) \leq C_1 \left[H(t) + \int_{\Omega} |u|^l \, dx + \int_{\Omega} u_t^2 dx \right] \tag{2.17}$$

where C_1 is positive constant. A combination of (2.16) and (2.17) , thus, yields

$$L^{'}(t) \geq \xi L^{\frac{1}{1-\sigma}}(t), \quad \forall t \geq 0, \tag{2.18}$$

where $\xi = \gamma\varepsilon / C_1$. Integration of (2.18) over $(0, t)$ gives

$$L^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{L^{\frac{-\sigma}{1-\sigma}}(0) - \frac{\xi\sigma}{(1-\sigma)} t};$$

hence $L(t)$ blow up in a time

$$T^* \leq \frac{1-\sigma}{\xi \sigma L^{\frac{\sigma}{1-\sigma}}(0)}. \tag{2.19}$$

This completes the proof. □

3. THE LINEAR DAMPING CASE

The next result improves the one given in Remark 3.3 of [1]. In fact we will show that the blow up for solutions of (1.7), when the damping is linear ($m = 2$), takes places if $\int_{\Omega} u_0 u_1 dx > -\frac{1}{2} \int_{\Omega} \rho u_0^2 dx$ instead of $\int_{\Omega} u_0 u_1 dx > 0$.

Theorem 2. *Assume that* $p \geq 1$ *and* $l > \max\{4, p\}$. *Assume further that*

$$E(0) \leq 0, \quad \int_{\Omega} u_0 u_1 dx > -\frac{1}{2} \int_{\Omega} \rho u_0^2 dx. \tag{3.1}$$

Then the solution of (1.7), for $m = 2$, *blows up in finite time.*

Proof. Let

$$L(t) = \int_{\Omega} u_t u dx + \frac{1}{2} \int_{\Omega} \rho(x) u^2 dx. \tag{3.2}$$

By taking a derivative of (3.2) and using (1.7) we obtain

$$\begin{aligned} L^{'}(t) = & \int_{\Omega} u_t^2 dx - \beta \int_{\Omega} (\triangle u)^2 dx \\ & - \int_{\Omega} g |\nabla u|^2 dx - 2\Gamma \int_{\Omega} |\nabla u|^2 (\triangle u)^2 dx \\ & - \int_{\Omega} h |\nabla u|^p dx + \int_{\Omega} |u|^l dx. \end{aligned} \tag{3.3}$$

For some positive constant K, (3.3) takes the form

$$L^{'}(t) = \left(\frac{K}{2}+1\right)\int_\Omega u_t^2 dx + \beta\left(\frac{K}{2}-1\right)\int_\Omega (\triangle u)^2\, dx + KH(t)$$

$$+\left(\frac{K}{2}-1\right)\int_\Omega g\left|\nabla u\right|^2 dx + \Gamma\left(\frac{K}{2}-2\right)\int_\Omega \left|\nabla u\right|^2 (\triangle u)^2\, dx$$

$$+\left(\frac{K}{p}-1\right)\int_\Omega h\left|\nabla u\right|^p dx + \left(1-\frac{K}{l}\right)\int_\Omega \left|u\right|^l dx. \tag{3.4}$$

We choose K so that $l > K > \max\{4, p\}$, then we have from (3.4),

$$L^{'}(t) \geq \left(1-\frac{K}{l}\right)\left[\int_\Omega (\triangle u)^2\, dx + \int_\Omega u_t^2 dx + \int_\Omega \left|u\right|^l dx + H(t)\right]. \tag{3.5}$$

Therefore

$$L(t) \geq L(0) = \int_\Omega u_0 u_1 dx + \frac{1}{2}\int_\Omega \rho u_0^2 dx > 0,\ \ \forall\ t \geq 0.$$

Next, it is clear that

$$L^{\frac{2l}{l+2}}(t) \leq C\left\{\left(\int_\Omega u_t u dx\right)^{\frac{2l}{l+2}} + \left(\int_\Omega \rho u^2 dx\right)^{\frac{2l}{l+2}}\right\}. \tag{3.6}$$

By the Cauchy–Schwarz inequality we have

$$\left|\int_\Omega u_t u dx\right| \leq \left(\int_\Omega u^2 dx\right)^{1/2}\left(\int_\Omega u_t^2 dx\right)^{1/2}$$

$$\leq C\left(\int_\Omega \left|u\right|^l dx\right)^{1/l}\left(\int_\Omega u_t^2 dx\right)^{1/2},$$

which implies

$$\left|\int_\Omega u_t u dx\right|^{\frac{2l}{l+2}} \leq C\left(\int_\Omega \left|u\right|^l dx\right)^{\frac{2}{l+2}}\left(\int_\Omega u_t^2 dx\right)^{\frac{l}{l+2}}.$$

With the help of Young's inequality, we get

$$\left|\int_\Omega u_t u dx\right|^{\frac{2l}{l+2}} \leq C\left[\left(\int_\Omega \left|u\right|^l dx\right)^{\frac{2\mu}{l+2}} + \left(\int_\Omega u_t^2 dx\right)^{\frac{\theta l}{l+2}}\right],$$

for $1/\mu + 1/\theta = 1$. Choosing $\theta = \frac{l+2}{l}$, (hence $\mu = \frac{l+2}{2}$), we obtain

$$\left|\int_\Omega u_t u dx\right|^{\frac{2l}{l+2}} \leq C\left[\int_\Omega \left|u\right|^l dx + \int_\Omega u_t^2 dx\right]. \tag{3.7}$$

Similarly we also have

$$\int_\Omega \rho u^2 dx \leq \left(\int_\Omega (\rho u)^2\, dx\right)^{1/2} \left(\int_\Omega u^2 dx\right)^{1/2},$$

which gives

$$\left(\int_\Omega \rho u^2 dx\right)^{\frac{2l}{l+2}} \leq C \left(\int_\Omega (\rho u)^l\, dx\right)^{\frac{2}{l+2}} \left(\int_\Omega u^2 dx\right)^{\frac{l}{l+2}}$$

$$\leq C \left[\left(\int_\Omega |\rho u|^l\, dx\right)^{\frac{2\mu}{l+2}} + \left(\int_\Omega u^2 dx\right)^{\frac{\theta l}{l+2}}\right].$$

With the same choice of θ and μ as in above and the use of the boundary conditions, we easily deduce

$$\left(\int_\Omega \rho u^2 dx\right)^{\frac{2l}{l+2}} \leq C \left[\int_\Omega |\rho u|^l\, dx + \int_\Omega u^2 dx\right]$$

$$\leq C \left[\|\rho\|_\infty^l \int_\Omega |u|^l\, dx + \int_\Omega u^2 dx\right]$$

$$\leq C_1 \left[\int_\Omega |u|^l\, dx + \int_\Omega u^2 dx\right]$$

$$\leq C_2 \left[\int_\Omega |u|^l\, dx + \int_\Omega (\triangle u)^2\, dx\right]. \tag{3.8}$$

Combining (3.7) and (3.8) , we have

$$L^{\frac{2l}{l+2}}(t) \leq C \left[\int_\Omega (\triangle u)^2\, dx + \int_\Omega u_t^2 dx + \int_\Omega |u|^l\, dx\right] \tag{3.9}$$

$$\leq C \left[\int_\Omega (\triangle u)^2\, dx + \int_\Omega u_t^2 dx + \int_\Omega |u|^l\, dx + H(t)\right].$$

A combination of (3.5) and (3.9) leads to

$$L'(t) \geq \frac{1}{C}\left(1 - \frac{K}{l}\right) L^{\frac{2l}{l+2}}(t). \tag{3.10}$$

A simple integration of (3.10) yields

$$L^{(l+2)/(l-2)}(t) \geq \frac{1}{L^{-(l+2)/(l-2)}(0) - at}, \tag{3.11}$$

where $a = (1/C)[2l/(l-2)][1-(K/l)]$. Therefore (3.11) shows that L blows up in finite time. □

Remark 3.1. The above result remains valid if $|u|^{l-2}u$ if replaced by $f(u)$ provided that

$$uf(u) - KF(u) \geq \delta|u|^l, \quad \delta > 0 \qquad F(u) = \int_0^u f(s)ds.$$

Again this is a weaker requirement than (2.4) of [1].

Remark 3.2. We do not require that $u_1 \neq 0$ as in Theorem 2.1 of [1].

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