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Interpolation outside the Interval of Convergence

by

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Final Report

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1. Abstract

1.1

النتيجة الأنيقة المنسوبة لأردوس وتوران تنص على أن متتابعة لاغرانج التي تتفق مع دالة ف متصلة في أصفار كثيرة حدود متعامدة خلال الفترة [ج ، د] تؤول إلى ف في معيار ل٢. لقد طورنا هذه النتيجة الى متتابعة ذات طبيعة مماثلة ولكنها تتفق في عدد محدد من نقاط محددة سلفا وواقعة خارج الفترة (ج ، د). ويتم هذا العمل من خلال تحديد عدد محدود من النقاط x_i ، i= ١،٢،...، ك ، خارج [ج ، د]، ولكل منها تعدد م. سوف نستخدم و(س) = $\prod_{\text{ن}=1}^{\text{م}} (\text{س} - \text{س}_{\text{ن}})^{\text{م}_{\text{ن}}}$ لبناء دالة صفرية متعامدة ومستوفية ثم نحدد اصفارها البسيطة من خلال منهج قولب. لاغارنج المستوفاة وهرمت المستوفاة كذلك المعتمدة على الاصفار البسيطة تذيل في حاصل الضرب و(س) والدوال الصفرية المتعامدة المستوفية. يتضح أن سلسلة كثيرة الحدود الناتجة تؤول الى ف وكذلك تتفق معها في النقاط الثابتة في معنى هرمت. كذلك قمنا بالتحقيق في سلوك هذه المتتابعات من خلال النظر في عناصر مختلفة. كذلك تم تصميم طريقة حساب كثيرة الحدود هذه كما وضعت دراسة مقارنة لدالة الرونغ في الفترة (-٥،٥).

1.2 Abstract in English

An elegant result due to Erdos and Turan states that the sequence of Lagrange interpolants to a continuous function f at the zeros of orthogonal polynomials over an interval $[c,d]$ converges to f in L_2 norm. We have extended this result in the sense that the sequences of similar nature interpolate f at a finite number of pre-assigned points lying outside (c,d). This work is carried out by fixing a finite number of nodes x_i, $i = 1,2,\ldots,k$, outside $[c,d]$, each with multiplicity m_i. We used $W(x) = \prod_{i=1}^{k}(x - x_i)^{n_i}$ to construct orthogonal 0-interpolants (*OZI*) and then determined their simple zeros by the approach of Golub. The Lagrange interpolants or Hermite interpolants based on the simple zeros are appended with the product of $W(x)$ and the *OZI*. It is shown that the sequence of resulting polynomials converged to f as well as interpolated it at the fixed nodes in the sense of Hermite. We have investigated the behavior of these sequences by considering different parameters. A method of computing these polynomials is also devised and a comparative study is presented for Runge function on $[-5,5]$.

2. Introduction

Interpolating polynomials play a vital role in the numerical solution of several scientific and engineering problems. Nevertheless, they require a carefully designed algorithm to avoid excessive accumulation of round-off errors.

Another drawback, the interpolating polynomials suffer from, is that they lack the convergence property in general. This phenomenon appears particularly in case of several functions having a singularity, e.g., the Runge function $f(x)=\frac{1}{1+x^2}$ defined on $[-5,5]$. For this example, the sequence of interpolating polynomials does not converge to f when constructed at the uniformly distributed points on $[-5,5]$, e.g. see [8] and 21]. The convergence characteristic of interpolating polynomials, however, changes if the nodes are the zeros of orthogonal polynomials. Erdös and Turan noticed this particular phenomenon in 1937 [15]. They constructed a sequence of polynomial interpolants to a given continuous function f in the sense of Lagrange where the nodes were the zeros of orthogonal polynomials over an interval $[a,b]$ and showed that such a sequence converges to f in L_2- norm over $[a,b]$.

2.1. Theme of the project

In some applications, we may require the members of an approximating sequence to interpolate a given function f at a finite number of pre-assigned points. These points may lie inside the interval of convergence or in its outer part where f is defined. It is known that the sequences of best least squares or uniform polynomial approximants to an $f \in C[c,d]$ converges to f. Some modifications of these sequences exist in the literature in which additional interpolating nodes within $[c,d]$ are incorporated in each member of the sequence and yet its convergence to f is preserved on $[c,d]$, e.g. see, [1], [3], [13] and [17]. On the other hand, if we take up the case of Erdos-Turan type of polynomials as discussed above, we find that these polynomials interpolate f at the zeros of orthogonal polynomials over $[c,d]$ and that the sequence of such polynomials converges to f in the least squares sense over $[c,d]$ [15]. These zeros by default lie within (c,d) [24]. A natural question that may be asked is as follows: Can we modify these polynomials by inducting additional nodes, simple or multiple, outside the interval (c,d) in such a way that each polynomial interpolates f at these nodes without affecting the convergence of the sequence to f over $[c,d]$? Our project by and large addresses this aspect of approximating polynomials. More precisely, we consider the following problem and determine its solution along with its computational procedure:

Problem A: Let $f:[a,b]\to\Re$. Consider an interval $[c,d]\subset[a,b]$ with K a finite subset of $[a,b]\setminus(c,d)$. Construct a sequence of polynomials p_n such that

1) p_n interpolates f at the zeros of orthogonal polynomials on $[c,d]$

2) $\{p_n\}$ is L_2-convergent over $[c,d]$.

3) Each p_n interpolates f in the sense of Lagrange or Hermite at the points of K.

We also extend our work by considering the points in K as multiple nodes. In computational part (cf Chapter 7) we shall see that the approximation by the suggested polynomials is improved outside the interval of convergence, i.e., $[c,d]$.

2.2. Literature survey

There is a wide range of literature on the theoretical and computational aspects of polynomials interpolating in the sense of Lagrange or Hermite and their applications in physical, engineering and business problems. Gander [16] considered several representations of interpolating polynomials, for example, Lagrange, Newton, Hermite etc, where each one is characterized by some basis functions. He investigate the transformations between the basis functions which map a specific representation to another and showed that for this purpose the LU- and the QR-decomposition of the Vandermonde matrix play a crucial role. The choice of nodes in the interpolation process plays an important role. A variety of problems and approaches related to these topics may be seen in every text on approximation theory. The article “A Chronology of Interpolation: From Ancient Astronomy to Modern Signal and Image Processing” [20] and the monograph "Orthogonal Polynomials: Computation and Approximation" [17] are worth seeing in this regard.

Although a significant literature exists on interpolation at orthogonal zeros where convergence has also been discussed, the problem of interpolation beyond the interval of convergence has not been discussed thoroughly to the best of our knowledge. A remarkable application of this phenomenon may be observed in Gauss-Radau and Gauss-Lobbato rules [22], [19] where end points of the interval of integration are included among the nodes of these quadrature rules. In various other directions on interpolation at the zeros of orthogonal polynomials and their convergence, a lot of work is carried out by the researchers. The convergence and boundedness of the extended Lagrange interpolating operator with additional nodes were studied in the space $L^p_{u,t}$ of Sobolev type in [7] by Capobianco and Russo**.** In [11], Demelin and Lubinski investigated mean convergence of Lagrange interpolation at the zeros of orthogonal polynomials $p_n(W^2, x)$ for Erdos weights $W^2 = e^{-2Q}$. and provided necessary and sufficient condition for mean convergence of Lagrange interpolants for these weights. Demelin *et al* further discussed the mean convergence of Lagrange interpolation for fast decaying even and smooth exponential weights on the line [12]. A lot of literature exists alone on the convergence of Lagrange interpolation as well. Criscuolo *et al* has discussed Point-wise simultaneous convergence of extended Lagrange interpolation with additional knots in [9]. Criscuolo *et al* also discussed convergence of extended Lagrange interpolation [10]. We also find a significant work on the side of applications and computational procedures related to interpolation polynomials. For example, in [24], Streltosy proposed the theory of function interpolation, based on the use of Chebyshev and Legendre orthogonal polynomials on a discrete point set. He provided effective method of solving Fredholm linear integral equations of the first and second kind and showed that Legendre polynomials are more preferable than Chebyshev polynomials of the first kind for solving Fredholm equations. In [14], Deun and Bultheel provide an interpolation algorithm for orthogonal rational functions.

In the proposed project, we plan to study Problem A via interpolating orthogonal polynomials. This type of polynomials appears in our earlier work on constrained least squares approximation [1], [3] and extension of Gauss quadrature rules [2], [4]. We plan to discuss this problem initially by including the end point of $[c,d]$ as interpolating nodes and then extend our result to the nodes lying outside $[c,d]$. In addition, we intend to explore the possibility of considering these nodes with multiple order. We shall also design some algorithm for the computation of our proposed interpolants. For this, we plan to use a modified form of 3-term recurrence relation [23] for the computation of interpolating orthogonal polynomials. In order to determine the zeros of these orthogonal polynomials, we shall compute eigenvalues of the Jacobi matrix related to abovementioned 3-term recurrence relation. This technique, originally proposed by Galoub and Welsch, is considered very effective and stable in the evaluation of zeros of orthogonal polynomials [18].

2.3. Organization of report

The distribution of material in this report is as follows:

We provided a review of some fundamentals of approximation theory that mainly deal with interpolation, orthogonal polynomials, L_2-approximation, and the Erdös-Turan Theorem in the next chapter. A review of the notions of 0-interpolants and orthogonal 0-interpolants is given in Chapter 4. Here, the sequences of orthogonal 0-interpolants and their convergence are also discussed. The material provided in rest of the chapters is a core of the report. The Erdos-Turan type result by considering additional interpolation only at the end points of $[c,d]$, the interval of convergence, is discussed in Chapter 5. We referred it to as Problem I. It is followed by Problem II where a finite number of simple additional nodes are considered outside the interval of convergence. In Chapter 6, we carried out an extension of Problems I and II for multiple nodes lying outside the interval (c,d) and established convergence result. Chapter 7 deals with the computational aspects for determining the required interpolating polynomial. We also explained the measure of various errors between the function and the polynomial. In Chapter 8, we devised algorithm for computing the polynomials. The computational procedure is applied to specific functions in the MATLAB environment and simulation results are given along with corresponding graphs. Some concluding remarks are given in Chapter 9.

3. Some Fundamentals Relevant to Erdös-Turan Theorem

We discuss some basic concepts of approximation theory in the context of Erdös-Turan Theorem in this chapter. These concepts or their modifications will be required in our main work. The material discussed here may be found in the standard texts and current literature on approximation theory and numerical analysis [4], [5], [6], [15], [18], [21], [23].

3.1 Nomenclature

We list some standard notations frequently used in current and later chapters:

$$\left.\begin{array}{lll}
\text{a) } \omega(x) & := & \text{non-negative weight function continuous on } [c,d] \\
\text{b) } \langle f,g\rangle_{[c,d],\omega} & := & \int_c^d f(x)g(x)\omega(x)dx \\
\text{c) } \|f\|_\omega & := & \sqrt{\langle f,f\rangle_\omega} \\
\text{d) } \pi_n & := & \text{Class of all polynomials of degree} \le n \\
\text{e) } C[c,d] & := & \text{Class of all continuous real valued functions on } [c,d] \\
\text{f) } L_n(.,U,g) & := & n\text{th degree Lagrange interpolant to a function } g \text{ at} \\
 & & \text{the set } U \text{ consisting of } (n+1) \text{ distinct points}
\end{array}\right\} \qquad (3.1)$$

3.2 Interpolation problem

The *polynomial interpolation problem* is to determine a polynomial p of minimal degree where p or its derivative(s) assumes the given values at a finite number of points. In fact, it deals with finite data: $\{(x_i, y_i): i = 0,1,\ldots,k\}$ where the second coordinates y_i's may be related to a real valued function f under consideration, i.e., either $y_i = f(x_i)$ or $y_i = f^{(j)}(x_i)$. We classify the interpolation problem into two types:

Type I: *Lagrange interpolation problem* when all the first coordinates x_i's in the data are distinct.

Type II: *Hermite interpolation problem* when at least one of the first coordinates x_i's in the data is repeated.

3.3 Lagrange interpolation problem (solution)

In this case, the first coordinates x_i's of $(k+1)$ data points are distinct. We set

$$A = \{x_0, x_1, \ldots, x_k\}. \tag{3.2}$$

The resultant polynomial p is of degree k and is denoted by $L_k(.,A,f)$ (cf (3.1)-f). Thus, we have

$$L_k(x_i, A, f) = f(x_i),\ i = 0,1,2,\ldots,k\ .$$

Note that $L_k(.,A,f) \in \pi_k$. It is uniquely determined by $k+1$ interpolation conditions. Some explicit representations of $L_k(.,A,f)$ are as follows:

a. Set

$$W_A(x) = \prod_{\alpha \in A}(x-\alpha) \tag{3.3}$$

then

$$L_k(x, A, f) = \sum_{\alpha \in A} f(\alpha) \frac{W_A(x)}{(x-\alpha)W_A'(\alpha)} \tag{3.4}$$

b. Set

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{k} \frac{(x-x_j)}{(x_i-x_j)},\ i = 0,1,2,\ldots,k, \tag{3.5}$$

then

$$L_k(x, A, f) = \sum_{i=0}^{k} f(x_i) l_i(x) . \tag{3.6}$$

The $k+1$ polynomials l_i's defined in (3.5) are known as the *fundamental polynomials* of the Lagrange interpolant at the nodes $x_0, x_1, x_2, \ldots, x_k$. It may be noted that $l_i \in \pi_k, i = 0,1,..,k$, and that $l_i(x_j) = \delta_{ij}$. From (3.3)-(3.6), we note that

$$l_i(x) = \frac{W_A(x)}{(x-x_i)W_A'(x_i)} \tag{3.7}$$

c. If we denote the divided difference[1] [8], [21] of order j by $f[x_0,x_1,...,x_j]$ then based on the Newton's interpolation formula [8], [21], we have another representation:

$$L_k(x,A,f)=f[x_0]+\sum_{j=1}^{k} f[x_0,x_1,...,x_j]\prod_{i=0}^{j-1}(x-x_i) \tag{3.8}$$

3.4 Hermite interpolation problem (solution)

Here we discuss the case when one or more x_i's in the data $\{(x_i,y_i): i=0,1,\ldots,k\}$ are not distinct. In particular, we consider a situation when an x_i appears m_i times with $m_i \geq 1$ in the data. In such a case, we need the interpolation polynomial p to satisfy $p^{(j)}(x_i)=f^{(j)}(x_i)$, $j=0,1,2,...,m_i-1$. In order to determine the desired interpolant, we identify each of the distinct x_i's and re-label them as follows: if r is the number of distinct x_i's, we set $U=\{u_1,u_2,\ldots,u_r\}$ where each u_j correspond to one of the distinct first coordinates repeating m_y-times in the data. The resulting polynomial p is of degree $S(r)-1$ with $S(r)=\sum_{j=1}^{k} m_j$, and will be denoted by $H_{S(r)-1,k}(.,U,f)$.

Thus, we have $H^{(j)}_{S(r)-1,k}(u_i,U,f)=f^{(j)}(u_i)$, $i=0,1,2,...,k$; $j=0,1,2,...,m_j-1$. Here we say that $H_{S(r)-1,k}$ interpolates f at u_i's in the sense of Hermite.

Remark 3.1. The Hermite interpolants may be computed by Newton's interpolation formula (cf (3.8)). Here, one has to take care of divided difference of repeated nodes[2] [21].

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[1] Note that for the points $x_0,x_1,...,x_k$, the 0-order divided difference $f[x_i]$ is defined as $f[x_i]=f(x_i)$, i = 0,1,…,k; and the m^{th}-order divided difference is by the recursive formula $f[x_0,x_1,...,x_m]=\dfrac{f[x_0,x_1,...,x_{m-1}]-f[x_1,x_2...,x_m]}{x_0-x_m}$, m = 1, 2, …j

[2] If $x_0=x_1=...=x_k=u$ then $f[x_0,x_1,...,x_k]=f[\underbrace{u,u,...,u}_{k+1\text{ times}}]=f^{(k)}(u)/k!$

3.5 Existence and uniqueness of best approximants [21]

i. Best approximant. Let $A \subset B$ and $(B, \|.\|)$ be a normed linear space[3]. Let $b \in B$. We say that $a^* \in A$ is a best approximant of b from A if $\|a^* - b\| \le \|a - b\|$, $\forall a \in A$.

ii. Existence of best approximant. Let A be a finite dimensional linear space in a normed linear space $(B, \|.\|)$. Then for every $b \in B$, there exists an element of A that is a best approximant of b from A.

iii. Uniqueness of best approximant. Let A be a convex set in a normed linear space $(B, \|.\|)$ whose norm is strictly convex. Then for every $b \in B$, there is at most one best approximant of b from A.

iv. Application of convexity and strict convexity. Every linear space is convex. Also, L_2-norm on $C[a,b]$ is strictly convex.

3.6 L_2-approximation problem

An *L_2-approximation problem* over π_n for an $f \in C[a,b]$ with respect to a weight function, say ω, may be posed as follows:

$$\min_{p \in \pi_n} \|f - p\|_\omega . \tag{3.9}$$

The problem (3.9) has a unique solution, say p_n^*, since π_n is a finite dimensional linear space and the norm $\|\ \|_\omega$ on $C[a,b]$ is strictly convex (*See ii and iii, section 3.5*).

3.7 Mean squared convergence

Let p_n^* be a solution of the problem given by (3.9). If $f \in C[a,b]$ then

$$\lim_{n \to \infty} \|p_n^* - f\|_\omega = 0. \tag{3.10}$$

This result is a consequence of the Weierstrass approximation theorem with the following heuristic argument:

(i) $\left\{\|p_n^* - f\|_\omega\right\}_{n=0}^{\infty}$ is a decreasing sequence of nonnegative numbers since $\pi_n \subset \pi_{n+1}$.

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[3] The norm $\|.\|$ is a real-valued function defined on B which satisfies the following properties:
(i) $\|x\| > 0$ unless $x = 0$; (ii) $\|\lambda x\| = |\lambda| \|x\|$ where λ is a scalar; (iii) $\|x + y\| \le \|x\| + \|y\|$.

(ii) We can have $p^* \in \pi_N$ for some N such that $\left\| p^* - f \right\|_\infty < \varepsilon$. Then

$\left\| p_N^* - f \right\|_\infty < \left\| p^* - f \right\|_\infty < \varepsilon$.

(iii) (3.10) is an outcome of (i) and (ii).

3.8 Orthogonal polynomials and their role in L_2-approximation

We say that two polynomials p and q are orthogonal with respect to a weight function ω over an interval $[a,b]$ if $\langle p,q \rangle_\omega := \int_a^b p(x)q(x)\omega(x)dx = 0$. Orthogonal polynomials are mutually linearly independent and they play a vital role in the computation of L_2-approximants p_n^*. In particular, if "$\varphi_0, \varphi_1, ..., \varphi_n$" is an orthogonal basis of π_n, then

$$p_n^*(x) = \sum_{i=0}^{n} \frac{\langle \varphi_i, f \rangle_\omega}{\langle \varphi_i, \varphi_i \rangle_\omega} \varphi_i(x) \tag{3.11}$$

3.9 3-Term recurrence relation for orthogonal polynomials[…]

An orthogonal basis "$\varphi_0, \varphi_1, ..., \varphi_n$" of π_n, as required in (3.11) may be determined by the following method (known as 3-Term recurrence relation):

$$\varphi_{i+1}(x) = (x - \alpha_i)\varphi_i(x) - \beta_i \varphi_{i-1}(x), \qquad i = 1, 2, \dots \tag{3.12}$$

with

$$\varphi_0(x) \equiv 1, \qquad \varphi_1(x) = x - \alpha_0; \;\; \alpha_0 = \langle \varphi_0, \varphi_0 \rangle_w \tag{3.13}$$

The recursion coefficients in (3.12) according to the Steiltjes procedure [17] are given by

$$\left. \begin{aligned} \alpha_i &= \frac{\langle x\varphi_i, \varphi_i \rangle_\omega}{\langle \varphi_i, \varphi_i \rangle_\omega}, \qquad i = 1, 2, \dots \\ \beta_i &= \frac{\langle \varphi_i, \varphi_i \rangle_\omega}{\langle \varphi_{i-1}, \varphi_{i-1} \rangle_\omega}, \qquad i = 1, 2, \dots \end{aligned} \right\}. \tag{3.14}$$

3.10 Orthogonal zeros and their evaluation

It may be noted that the orthogonal polynomial φ_n discussed above has n distinct zeros in the open interval (a,b) [21]. These zeros are in fact eigenvalues of the Jacobi matrix [18].

$$J_n = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & 0 & 0 & . & 0 \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & 0 & . & 0 \\ 0 & \sqrt{\beta_2} & \alpha_2 & \ddots & . & . \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ . & . & . & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ 0 & 0 & . & 0 & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix} \tag{3.15}$$

3.11 Erdös-Turan Theorem based on orthogonal zeros

Here, we discuss an L_2-convergence process related to the Lagrange interpolants (cf (Section 3.3)) at the zeros of orthogonal polynomials. This is in fact the underlying theme of our project. A remarkable result based on this type of intrpolants is due to Erdös and Turan [15] which may stated as

Theorem 3.1.[15] *Let $P_0, P_1, P_2, \ldots$ be a system of polynomials which are orthogonal on a finite closed interval $[c,d]$ with respect to a weight function ω. Suppose that $Z_{n+1} := \{z_1, z_2, \ldots, z_{n+1}\}$ is the set of $n+1$ zeros of P_{n+1}. Consider a real valued function f with $Z_{n+1} \subset \mathrm{Dom}(f)$ and let $L_n(.,Z_{n+1},f)$ be the Lagrange interpolant of degree $\leq n$ to f at the zeros of P_{n+1}. If $f \in C[c,d]$, then*

$$\lim_{n\to\infty} \int_c^d \left(L_n(x, Z_{n+1}, f) - f(x)\right)^2 dx = 0. \tag{3.16}$$

In Theorem 3.1, we note that the interpolating nodes being the zeros of orthogonal polynomial are distinct and located within the interval (c, d) [Powell]. Moreover, these nodes are readily available as eigenvalues of the matrix (3.15)

4. Orthogonal 0-Interpolants

In order to include additional nodes outside the interval of convergence in Erdos-Turan Theorem (cf Theorem 3.1), we introduce a system of orthogonal zero-interpolants. For formal explanation, first we shall define the notion of 0-interpolants and then describe the relevant constrained minimization problem. On the same lines, we shall discuss the notion of orthogonal Hermite zero-interpolants and its corresponding minimization problem. [6]

4.1. Zero-interpolants

We shall say that a real-valued function g is a *zero-interpolant* at a finite set $\{x_0, x_1, \ldots, x_k\}$ if $g(x_i) = 0$, $i = 0,1,...,k$, i.e., g interpolates the zero-function at the nodes x_i, $i = 0,1,\ldots,k$.

Notations: For a finite set A with $k+1$ elements, i.e., $|A| = k+1$, we shall use the following notations:

$$\pi_{n,k}(A) := \{p \in \pi_{n+k} : p \text{ is a zero-interpolant at } A\},\ n \geq 0. \tag{4.1}$$

$$C\left([c,d], A, k\right) := \{g \in C[c,d] : A \subset \mathrm{Dom}(g) \text{ and } g \text{ is a zero-interpolant at } A\} \tag{4.2}$$

Remark 4.1. *A may or may not be a subset of the interval* $[c,d]$ *in (4.2).*

Setting $W_{A,k}(x) := \prod_{\alpha \in A}(x - \alpha)$, we describe some properties of $\pi_{n,k}(A)$ in the following lemma:

Lemma 4.1 [1]. If $|A| = k+1$, then the class of algebraic polynomials $\pi_{n,k}(A)$ has the following properties:

(a) $\pi_{n,k}(A)$ is an $n+1$ dimensional subspace of π_{n+k+1}, which is generated by the polynomials $W_{A,k}, xW_{A,k}, \ldots, x^n W_{A,k}$.

(b) Every polynomial p in $\pi_{n,k}(A)$ is of the form

$$p = qW_{A,k} \tag{4.3}$$

for some $q \in \pi_n$.

(*c*) $\bigcup_{n=0}^{\infty} \pi_{n,k}(A)$ is uniformly dense in $C\left([c,d],A,k\right)$.

Proof. The parts (*a*) and (*b*) of the theorem follow from the structure of $\pi_{n,k}(A)$. The proof of (*c*) may be found in [1].

4.2. Orthogonal 0-interpolants at a finite set *A*

Note that $\pi_{n+1}\left(A,k\right)$ being an (*n* + 2)- dimensional space has a basis that consists of monic orthogonal polynomials with respect to weight function $\omega(x)$ over a given interval [*c,d*] [1]. These orthogonal polynomials which we shall denote by $\phi_{j,k}(x)$, *j* = 0,1,2,…, are determined by the 3-term recurrence relation by making some appropriate changes in (3.12)-(3.14) as follows:

i. Replace $\varphi_j(x)$ by $\phi_{j,k}(x)$, $j = 0,1,2,\ldots$
ii. Replace $\varphi_0(x) = 1$ by $\phi_{0,k}(x) = W_{A,k}(x)$.

Note that A is a part of the set of zeros of every polynomial in $\pi_{n,k}(A)$. Thus, the orthogonal polynomial $\phi_{j,k}$, $j = 0,1,2\ldots,n+1$, has a unique decomposition (cf (4.3))

$$\phi_{j,k}(x) = p_{j,k}(x)W_{A,k}(x) \tag{4.4}$$

for some $p_{j,k} \in \pi_j$.

Orthogonal Zero-Interpolants: We refer to the polynomials $\phi_{j,k}$, *j* = 0,1,2…, as orthogonal 0-interpolants at the set A with respect to ω.

Remark 4.2. *In the notation of* ((3.1)-*b*) *it follows that for* $i \neq j$ *, we have* $0 = \left\langle \phi_{i,k}, \phi_{j,k} \right\rangle_{[c,d],\omega} = \left\langle W_{A,k}\, p_{i,k}, W_{A,k}\, p_{j,k} \right\rangle_{[c,d],\omega} = \left\langle p_{i,k}, p_{j,k} \right\rangle_{[c,d],\omega W_{A,k}^2}$. *Therefore,* $p_{j,k}$, *j* = 0,1,2…,*n* +1, *are monic orthogonal polynomials with respect to the weight function* $\omega(x)W_A^2(x)$ *over [c,d], and thus, each* $p_{j,k}$ *has j real distinct zeros lying in the open interval (c,d) [1].*

4.3. Constrained L_2-approximation problem with simple nodes

Let $A=\{x_1,x_2,\ldots,x_{k+1}\}$ be a subset of real numbers. Let $f:[c,d]\to\Re$ be such that $\int_c^d f^2(x)w(x)dx<\infty$ and that $A\subset \mathrm{Dom}(f)$. Constrained L_2-approximation problem over π_N (where $N\geq n+k$ with $n\geq 0$) may be posed as follows:

$$\min_{\substack{p\in\pi_N\\ p(x_i)=f(x_i)\\ i=0,1..,k}} \|f-p\|_{[c,d],\omega}\ . \tag{4.5}$$

Solution. If n = 0, then $L_k(.,A,f)$, the Lagrange interpolating polynomial to f at the k + 1 distinct nodes $x_i, i=0,1,2,\ldots,k$, provides an optimal solution of the problem (4.5).

In order to solve (4.5) when n > 0, we convert it into an unconstrained minimization problem by modifying the given function f and merging the interpolating constrains in the feasible set [1]. Thus, an equivalent form of (4.5) is given by

$$\min_{p\in\pi_{n,k}(A)}\|f_L-p\|_\omega \tag{4.6}$$

where

$$f_L=f-L_k(.,A,f). \tag{4.7}$$

The solution, $\phi_n^*\in\pi_{n,k}(A)$, of the problem (4.6) is uniquely determined by

$$\phi_{n,k}^*(x)=\sum_{i=0}^{n}\frac{\langle\phi_{i,k},f_L\rangle_{[c,d],\omega}}{\langle\phi_{i,k},\phi_{i,k}\rangle_{[c,d],\omega}}\phi_{i,k}(x) \tag{4.8}$$

where $\phi_{0,k},\phi_{1,k},\ldots,\phi_{n,k}$ form an orthogonal basis of $\pi_{n,k}(A)$.

Convergence [1]. If $\phi_{n,k}^*\in\pi_{n,k}(A)$ is the solution of problem (4.6), then (cf (4.7)-(4.8))

$$\lim_{n\to\infty}\left\|f_L-\phi_{n,k}^*\right\|_\omega=0. \tag{4.9}$$

provided that $f\in C[a,b]$.

4.4. Orthogonal Hermite 0-interpolants at a finite set A

In Section 4.1, we discussed the notion of *zero-interpolant* at simple nodes. Now we extend this concept to multiple nodes. For this, we consider $\{u_1,u_2,\ldots,u_{k+1}\}$ where each distinct node u_i has multiplicity m_i . We shall say that a real-valued function g is a *Hermite zero-interpolant* at a finite set $\{u_1, u_2, \ldots, u_{k+1}\}$ if $g^{(j)}(u_i) = 0$, $i = 1,2,\ldots,k+1$; $j = 0,1,\ldots,m_i - 1$, i.e., g interpolates the zero-function and its first (m_i – 1) derivatives at each node u_i, i = 0,1,…,k. In order to define the orthogonal Hermite 0-interpolants, we introduce additional notations:

$$s(k) := \{m_i\}_{i=0}^{k}; \quad S(k) := \sum_{i=0}^{k} m_i \tag{4.10}$$

$$W_{s(k)}(x) := \prod_{j=0}^{k} (x - u_j)^{m_j} . \tag{4.11}$$

On the lines of Section 4.2, we consider (n +2)-dimensional subspace $\pi_{n+1,s(k)}(A)$ and its orthogonal basis, say $\{\phi_{j,s(k)}\}_{j=0}^{n+1}$ with respect to a weight function ω over the interval $[c,d]$. As noticed in Lemma 4.1, we briefly state some properties relevant to the orthogonal polynomials $\phi_{j,s(k)}$:

1. $\phi_{n+1,s(k)}(x) = p_{n+1,s(k)}(x) W_{s(k)}(x)$ for some $p_{n+1,s(k)}(x) \in \pi_{n+1}$
2. $Z_{n+1,s(k)} := \{z_1, z_2, \ldots, z_{n+1}\}$ will denote the set of (n +1) distinct zeros of the factor polynomial $p_{n+1,s(k)}$. These zeros lie in $[c,d]$.
3. Each orthogonal polynomial $\phi_{j,s(k)}$ has k +1 zeros u_i each with multiplicity $m_i, i = 1,2\ldots,k+1$. These are in addition to (j +1) distinct zeros of $p_{j+1,s(k)}$.

Orthogonal Hermite 0-Interpolants: The polynomials $\phi_{j,s(k)}$, j = 0,1,2…, will be referred to as orthogonal Hermite 0-interpolants at the set $\{u_1, u_2, \ldots, u_{k+1}\}$ with respect to ω where each node u_i has multiplicity m_i.

Remark 4.3. *We also notice that for* $i \neq j$,

$$0 = \left\langle \phi_{i,s(k)}, \phi_{j,s(k)} \right\rangle_{[c,d],\omega} = \left\langle p_{i,s(k)} W_{s(k)}, p_{j,s(k)} W_{s(k)} \right\rangle_{[c,d],\omega W_{s(k)}^2} = \left\langle p_{i,s(k)}, p_{j,s(k)} \right\rangle_{[c,d],\omega W_{s(k)}^2}.$$

Therefore, $p_{j,s(k)}$, *j = 0,1,2…,n+1, are monic orthogonal polynomials with respect to weight function* $\omega(x) W_{s(k)}^2(x)$ *over [c,d], and thus, each* $p_{j,s(k)}$ *has j real distinct zeros lying in the open interval (c,d) [3].*

4.5. Constrained L_2-approximation problem with multiple nodes

The constrained L_2-approximation problem as discussed in Section 4.3 can be reformulated for multiple nodes as follows:

Let $U = \{u_1, u_2, ..., u_{k+1}\}$ be a subset of real numbers where each node x_i has multiplicity m_i . Let $f : [c,d] \to \Re$ be such that $\int_c^d f^2(x) w(x) dx < \infty$ and $U \subset \mathrm{Dom}(f)$. Then a constrained L_2-approximation problem over π_N (where $N \geq n + S(k) - 1$ with $n \geq 0$) may be stated as follows:

$$\min_{\substack{p \in \pi_N \\ p^{(j)}(x_i) = f^{(j)}(x_i) \\ i=0,1..,k \\ j=0,1,\dots,m_i}} \|f - p\|_{[c,d]\omega} \tag{4.12}$$

Remark 4.4. Problem (4.12) can be resolved on the lines of the method given for problem (4.5). Here, we modify f as follows (cf (4.7)):

$$f_H(x) := f(x) - H_{S(k)-1,k}(x, U, f). \tag{4.13}$$

If $N = S(k) + n - 1$. Then an equivalent form of (4.6) can be expressed as

$$\min_{p \in \pi_{n,s(k)}(A)} \|f_H - p\|_\omega , \tag{4.14}$$

where $\pi_{n,s(k)}(A) = \langle W_{s(k)}, xW_{s(k)}, x^2 W_{s(k)}, \dots, x^n W_{s(k)} \rangle$ with $W_{s(k)}$ given by (4.11). The solution of (4.14) is similar to that of the problem (4.6) by considering the orthogonal basis of $\pi_{n,s(k)}(A)$, say $\{\phi_{j,s(k)}\}_{j=0}^n$, as discussed in Section 4.2.

Convergence [3]. If $\phi^*_{n,s(k)} \in \pi_{n,s(k)}(A)$ is a solution of problem (4.14), then

$$\lim_{n \to \infty} \left\| f_H - \phi^*_{n,s(k)} \right\|_\omega = 0. \tag{4.15}$$

provided that $f \in C^{(k^*)}[a,b]$ with $k^* := \left(\max_{1 \leq i \leq k+1} m_i \right) - 1$.

The details of the proof for (4.15) may be found in [3].

5. Extension of Erdös-Turan Theorem (Simple Nodes)

This chapter deals with extensions of Erdös-Turan Theorem (cf Theorem 3.1) where the underlying interval is $[c,d]$ and additional nodes lying outside the open interval (c,d) are simple. Here, we address the following problems:

Problem I: Let $f:[c,d]\to\Re$. Construct a sequence of polynomials P_n such that

1. P_n interpolates f at the zeros of orthogonal polynomials with respect to appropriate weight function on $[c,d]$.
2. $\{P_n\}$ is L_2-convergent to f over $[c,d]$.
3. Each P_n interpolates f in the sense of Lagrange at the end point(s) of interval $[c,d]$.

Problem II: Let $f:[a,b]\to\Re$. Consider an interval $[c,d]\subset[a,b]$ with A a finite subset of $[a,b]\setminus(c,d)$. Construct a sequence of polynomials R_n such that

1. R_n interpolates f at the zeros of orthogonal polynomials with respect to appropriate weight function on $[c,d]$.
2. $\{R_n\}$ is L_2-convergent to f over $[c,d]$.
3. Each R_n interpolates f in the sense of Lagrange at the points of A.

The solution of both problems depends on appropriate orthogonal 0-interpolants which replace the orthogonal system of polynomials considered in Erdös-Turan Theorem. Here, we modify the given function f to a 0-interpolant at the points c and d in case of Problem I and at the given set A in case of Problem II, and then construct a sequence of desired interpolants.

5.1. System of orthogonal 0-interpolants at A and fundamental polynomials

Recall that $\{\phi_{j,k}\}_{j=0}^{n+1}$ as described in Section 4.2 is an orthogonal basis of the space $\pi_{n+1,k}(A):=\langle W_{A,k},xW_{A,k},\ldots,x^{n+1}W_{A,k}\rangle$ with $W_{A,k}(x):=\prod_{\alpha\in A}(x-\alpha)$. We shall regard

$\phi_{j,k}$, j = 0,1,2,….. , as the system of orthogonal 0-interpolants at A. From (4.4) and Remark 4.2, we note that $\phi_{n+1,k}(x) = W_{A,k}(x)p_{n+1,k}(x)$ where $p_{n+1,k}$ has n + 1 distinct zeros in (c,d). We shall write these zeros in the set notation form as

$$Z_{n+1,k} := \{z_1, z_2, \ldots, z_{n+1}\}. \tag{5.1}$$

Next, we consider the polynomials $W_A l_i, i = 0,1,2,\ldots,n+1$, $l_i \in \pi_{n+1}$ are the fundamental polynomials (cf (3.5)) of the Lagrange interpolant based on the points of $Z_{n+1,k}$. We shall require following Lemma in sequel:

Lemma 5.1. The polynomials $W_A l_i, i = 1,2,\ldots,n+1$, are mutually orthogonal with respect to the weight function ω on the interval [c,d].

Proof. First, we note that

$$l_i(x) = \frac{p_{n+1}(x)}{(x - z_i)p'_{n+1}(z_i)}, i = 0,1,2,\ldots,n+1$$

Thus, for $i \neq j$, i, j = 1,2,…,n +1, we have after reshuffling of some factors

$$\langle W_A l_i, W_A l_i \rangle_{[c,d],\omega} = \frac{1}{p'_{n+1,k}(z_i)p'_{n+1,k}(z_j)} \int_c^d \phi_{n+1,k}(x) \frac{p_{n+1,k}(x)W_A(x)}{(x - z_i)(x - z_j)} \omega(x)dx . \tag{5.2}$$

If $i \neq j$, then the expression $\dfrac{p_{n+1,k}(x)W_A(x)}{(x - z_i)(x - z_j)}$ in (5.2) reduces to a polynomial of class $\pi_{n-1,k}(A)$. Noting that $\phi_{n+1,k}$ is orthogonal to $\pi_{n-1,k}(A)$ with respect to ω on the interval [c,d], the right side of (5.2) vanishes. This completes the proof.

5.2. Interpolants with additional nodes outside (c,d)

In order to modify Erdos-Turan type interpolants (cf Theorem 3.1) that may absorb a finite number of additional nodes lying outside (c,d), we proceed as follows:

i. Let A be a finite set of real numbers such that A and (c,d) are disjoint.

ii. Define a class of real valued functions $K_A[c,d]$ in which any function f has the following attributes:

a. $f \in C[c,d]$,

b. Dom (f) = [a,b] with $A \cup (c,d) \subset [a,b]$,

c. f is differentiable at the points of A

iii. Following the notation of (4.7), set

$$f_A(x) = f(x) - L_{|A|-1}(x, A, f) \tag{5.3}$$

Now we are in position to provide an explicit form of the desired interpolants.

Lemma 5.2. *For an* $f \in K_A[c,d]$, set

$$f_{W_A}(x) := \begin{cases} \dfrac{f_A(x)}{W_A(x)}, & x \notin A, \\ \lim\limits_{t \to x} \dfrac{f_A(t)}{W_A(t)}, & x \in A. \end{cases} \tag{5.4}$$

Then the polynomial $\mathbb{L}_{n,|A|}(x, A, f) \in \pi_{n+|A|-1}$ defined as

$$\mathbb{L}_{n,|A|}(x, A, f) := L_{|A|-1}(x, A, f) + W_A(x) L_n(x, Z_{n+1}, f_{W_A}) \tag{5.5}$$

interpolates f at the zeros of the orthogonal polynomial $\phi_{n+1,k}(x) = W_A(x) p_{n+1,k}(x)$ (cf Section 5.1).

Proof. Note that $A \cup Z_{n+1,A}$ is the set of all zeros of $\phi_{n+1,k}$ whereas $Z_{n+1,A}$ is the set of the zeros of $p_{n+1,k}$ (cf (5.1)). Moreover, A and Z_{n+1} are disjoint sets. If $v \in A$, then $L_{|A|-1}(v, A, f) = f(v)$ along with $W_A(v) = 0$. Thus, $\mathbb{L}_{n,|A|}(v, A, f) := f(v)$. On the other hand, if $v \in Z_{n+1,A}$, then $\mathbb{L}_{n,|A|}(v, A, f) = L_{|A|-1}(v, A, f) + W_A(v) f_{W_A}(v)$ (cf (5.5)). Using (5.3)-(5.4), we conclude that $\mathbb{L}_{n,|A|}(v, A, f) = f(v)$. Thus, the polynomial $\mathbb{L}_{n,|A|}(., A, f)$ interpolates f at the zeros of $\phi_{n+1,k}$.

5.3. Extension I: Inclusion of end points of [*c*,*d*] as additional nodes

As a first step towards extension of Erdös-Turan Theorem, we start with a simple situation where the end-points c and d appear as simple nodes in the modified interpolants (cf (5.5)). In such a case, $A = \{c, d\}$ and the corresponding interpolant can be written from (5.5) as

$$\mathbb{L}_{n,2}(x, A, f) := L_1(x, A, f) + W_A(x) L_n(x, Z_{n+1}, f_{W_A}) \tag{5.6}$$

Our first result is as follows:

Theorem 5.1. *Let $\phi_{j,k}$, j = 0,1,2,..... , be the system of orthogonal 0-interpolants at A={c,d} with respect to ω on the interval [c,d]. If* $f \in K_A[c,d]$ *then*

$$\lim_{n\to\infty}\left\|\mathbb{L}_{n,2}(.,A,f)-f\right\|_\omega = 0. \tag{5.7}$$

where $\mathbb{L}_{n,2}(.,A,f)$ *is the Lagrange interpolant to f at the zeros of* $\phi_{n+1,2}$ *as given in (5.6)*

Proof. Using (5.6) with a slight reshuffling and then considering f_{W_A} from (5.4), we get

$$\begin{aligned} f-\mathbb{L}_{n,2}(.,A,f) &= W_A\left[\left\{\left(f-L_1(.,A,f)\right)/W_A\right\}-L_n(.,Z_{n+1},f_{W_A})\right] \\ &= W_A\left[f_{W_A}-L_n(.,Z_{n+1},f_{W_A})\right] \end{aligned} \tag{5.8}$$

Since $f_{W_A} \in C[c,d]$, there exists the best uniform approximant $Q_n \in \pi_n$ to f_{W_A} from the class π_n and [21],

$$\lim_{n\to\infty}\left\|f_{W_A}-Q_n\right\|_{[c,d],\infty} = 0. \tag{5.9}$$

Thus,

$$\lim_{n\to\infty}\left\|f_{W_A}-Q_n\right\|_{[c,d],\omega} = 0. \tag{5.10}$$

By triangle inequality, (5.8) can be expressed as

$$\begin{aligned} &\left\|W_A\left[f_{W_A}-L_n(.,Z_{n+1},f_{W_A})\right]\right\|_{[c,d],\omega} \\ &\le \left\|W_A\left[f_{W_A}-Q_n\right]\right\|_{[c,d],\omega} + \left\|W_A\left[Q_n-L_n(.,Z_{n+1},f_{W_A})\right]\right\|_{[c,d],\omega}. \end{aligned} \tag{5.11}$$

Since $W_A(x)$ being a quadratic polynomial is bounded on [*c,d*], it follows from (5.10) that

$$\left\|W_A\left[f_{W_A}-Q_n\right]\right\|_{[c,d],\omega} \to 0 \text{ as } n\to\infty. \tag{5.12}$$

Note that $Q_n = L_n(.,Z_{n+1},Q_n)$ and that L_n is a linear operator. Therefore, the second term on the right side of (5.11) can be written as

$$\left\|W_A\left[Q_n-L_n(.,Z_{n+1},f_{W_A})\right]\right\|_{[c,d],\omega} = \left\|W_A L_n(.,Z_{n+1},Q_n-f_{W_A})\right\|_{[c,d],\omega}. \tag{5.13}$$

Expanding L_n in terms of its fundamental polynomials, we have

$$W_A(x)L_n(x,Z_{n+1},Q_n-f_{W_A}) = \sum_{i=1}^{n+1}(Q_n-f_{W_A})(z_i)W_A(x)l_i(x)$$

Hence,

$$\left\|W_A L_n(.,Z_{n+1},Q_n - f_{W_A})\right\|^2_{[c,d],\omega} = \left\|\sum_{i=1}^{n+1}(Q_n - f_{W_A})(z_i)W_A l_i\right\|^2_{[c,d],\omega}. \tag{5.14}$$

Next, using the orthogonality of $W_A\, l_i, i = 1,2,\ldots,n+1$ (cf Lemma 5.1), we have

$$\begin{aligned}&\left\|\sum_{i=1}^{n+1}(Q_n - f_{W_A})(z_i)W_A l_i\right\|^2_{[c,d],\omega}\\ &= \sum_{i=1}^{n+1}\left(Q_n(z_i) - f_{W_A}(z_i)\right)^2 \int_c^d [W_A(x) l_i(x)]^2 \omega(x)dx\end{aligned} \tag{5.15}$$

Since the points $z_i \in (c,d), i = 1,2,\ldots,n+1$ and $Q_n - f_{W_A}$ is continuous on [*c,d*], the right side of the above equation can be estimated as

$$\begin{aligned}&\sum_{i=1}^{n+1}\left(Q_n(z_i) - f_{W_A}(z_i)\right)^2 \int_c^d [W_A(x) l_i(x)]^2 \omega(x)dx\\ &\le \left\|Q_n - f_{W_A}\right\|^2_{[c,d],\infty} \sum_{i=1}^{n+1}\int_c^d [W_A(x) l_i(x)]^2 \omega(x)dx.\end{aligned} \tag{5.16}$$

Noting that $\sum_{i=1}^{n+1} l_i(x) = 1$ [8] and consequently, $\left[\sum_{i=1}^{n+1} l_i(x)\right]^2 = 1$, we have

$$\begin{aligned}\int_c^d [W_A(x)]^2 \omega(x)dx &= \int_c^d W_A^2(x)\left[\sum_{i=1}^{n+1} l_i(x)\right]^2 \omega(x)dx\\ &= \int_c^d \left[\sum_{i=1}^{n+1} W_A(x) l_i(x)\right]^2 \omega(x)dx\\ &= \int_c^d W_A^2(x)\sum_{i=1}^{n+1}[l_i(x)]^2 \omega(x)dx\\ &= \sum_{i=1}^{n+1}\int_c^d [W_A(x) l_i(x)]^2 \omega(x)dx\end{aligned} \tag{5.17}$$

where the second last expression is due to the orthogonality of the polynomials $W_A(x) l_i(x),\ i = 1,2,\ldots,n+1$ (cf Lemma 5.1). Combining (5.13)-(5.17), we arrive at

$$\left\|W_A\left[L_n(.,Z_{n+1},Q_n - f_{W_A})\right]\right\|_{[c,d],\omega} \le \left\|Q_n - f_{W_A}\right\|_{[c,d],\infty} \sqrt{\int_c^d [W_A(x)]^2 \omega(x)dx} \to 0 \text{ as } n \to \infty.$$

This along with (5.8), (5.11) and (5.12) shows that

$$\|f - \mathbb{L}_{n,2}(.,A,f)\|_{[c,d],\omega} \to 0 \text{ as } n \to \infty.$$

This completes the proof.

5.4. Extension II: Inclusion of a finite number of points outside (*c*,*d*)

In this section, we consider an extension of Erdös-Turan Theorem on a finite set of real numbers lying outside (*c*,*d*). More precisely, we consider $A = \{x_1, x_2, \ldots, x_{k+1}\}$ as a set of additional nodes where $x_i \notin (c,d)$. Based on the notations and terminology given in (5.1)-(5.5). we state

Theorem 5.2. *Let* $A = \{x_1, x_2, \ldots, x_{k+1}\}$ with all $x_i \notin (c,d)$. *Let* $\phi_{j,A,k}$, j = 0,1,2,….. , be the system of orthogonal 0-interpolants at A with respect to ω on the interval [*c,d*]. *If* $f \in K_A[c,d]$ (cf ii, Section 5.2), *then the polynomial* $\mathbb{L}_{n,k+1}(.,A,f)$ *given by (5.5) interpolates f at the zeros of* $\phi_{j,A,k}$. *In addition, we have*

$$\lim_{n\to\infty} \|\mathbb{L}_{n,k+1}(.,A,f) - f\|_{[c,d],\omega} = 0. \qquad (5.17)$$

Remark 5.1. Theorem 5.2 can be proved exactly by following the steps given in the proof of Theorem 5.1 with minor adjustments, and therefore, omitted. In this process, we have to replace the linear Lagrange interpolant $L_1(x,A,f)$ by the k^{th} degree interpolant $L_k(x,A,f)$ in the proof of Theorem 5.1.

6. Extension of Erdös-Turan Theorem (Multiple Nodes)

This chapter deals with construction of interpolants that include a finite number of simple and/or multiple nodes $x_1, x_2, \ldots x_k \in \mathrm{Dom(f)} \setminus (c,d)$. Here, we discuss an extended version of Problems I and II (cf Chapter 5):

Problem III: Let $f:[a,b] \to \Re$. Consider an interval $[c,d] \subset [a,b]$ with A a finite subset of $[a,b] \setminus (c,d)$ where each point is considered as a simple or multiple node. Then construct a sequence of polynomials h_n such that

1. h_n interpolates f at the zeros of orthogonal polynomials with respect to appropriate weight function on $[c,d]$.
2. $\{h_n\}$ is L_2-convergent to f over $[c,d]$.
3. Each h_n interpolates f in the sense of Hermite at the points of A.

To determine the solution of this problem, we need some additional notations. For a given set of nodes $A := \{x_1, x_2, \ldots, x_{k+1}\}$ lying outside the interval (c,d) where each x_i has multiplicity m_i, we set (cf (4.10)-(4.11))

$$s(k) := \{m_i\}_{i=1}^{k+1}, \tag{6.1}$$

$$W_{A,s(k)}(x) := \prod_{j=1}^{k+1} (x - x_j)^{m_j}. \tag{6.2}$$

$$S(k) := \sum_{i=1}^{k+1} m_i \tag{6.3}$$

6.1. System of orthogonal Hermite 0-interpolants at A

As before, we consider the $(n+2)$-dimensional subspace $\pi_{n+1}(W_{A,s(k)})$ generated by $x^i W_{A,s(k)}(x)$, $i = 0,1,\ldots,n+1$, and its orthogonal basis, say $\{\phi_{j,s(k)}\}_{j=0}^{n+1}$ with respect to a weight function ω over the interval $[c,d]$. The orthogonal polynomials thus constructed will be referred to as the system of orthogonal Hermite 0-interpolants at A. To avoid replication (cf Chapter 5), we briefly state some properties of $\phi_{j,s(k)}, j = 0,1,2,\ldots$:

1. $\phi_{n+1,s(k)}(x) = p_{n+1,s(k)}(x)W_{A,s(k)}(x)$ for some $p_{n+1,s(k)}(x) \in \pi_{n+1}$.

2. $Z_{n+1,s(k)} := \{z_1, z_2, \ldots, z_{n+1}\}$ will denote the set of $n+1$ distinct zeros of the factor polynomial $p_{n+1,s(k)}$. These zeros lie in $[c,d]$.

3. Each orthogonal polynomial $\phi_{j,s(k)}$ has additional $k+1$ fixed zeros x_i each with multiplicity $m_i, i = 1,2\ldots,k+1$.

4. If $l_{i,s(k)}, i = 1,2,\ldots,n+1,$ denote the fundamental polynomials of the Lagrange interpolant based on the points of $Z_{n+1,s(k)}$, then an analogue of Lemma 5.1 holds for $\{W_{A,s(k)} l_{i,s(k)}\}_{i=1}^{n+1}$, i.e.,

$$\left\langle W_{A,s(k)} l_{i,s(k)}, W_{A,s(k)} l_{j,s(k)} \right\rangle_{[c,d],\omega} = 0,\ i \neq j; i, j = 1,2,\ldots,n+1. \tag{6.4}$$

6.2. Modification of function and interpolants

Define a class of real valued functions $K_{A,s(k)}[c,d]$ in which any function f has the following attributes:

a. $f \in C[c,d]$,

b. Dom (f) = $[a,b]$,

c. f is $(m_i - 1)$-times differentiable at x_i $i = 1,2,\ldots,k+1$.

Subject to admissible differentiability conditions on f conditions, the notation $H_{S(k)-1}(.,A,f)$ will denote the polynomial of degree $\leq S(k)-1$ satisfying the conditions:

$$H_{S(k)-1}^{(j)}(x_i, A, f) = f^{(j)}(x_i),\ i = 1,2,\ldots k+1;\ j = 0,1,\ldots m_i. \tag{6.5}$$

For an $f \in K_{A,s(k)}[c,d]$, we define a function on the lines of the one given in (5.4):

$$f_{W_{A,s(k)}}(x) := \begin{cases} \dfrac{f_{A,s(k)}(x)}{W_{A,s(k)}(x)}, & x \notin A \\ \lim\limits_{t \to x} \dfrac{f_{A,s(k)}(t)}{W_{A,s(k)}(t)}, & x \in A \end{cases} \tag{6.6}$$

where

$$f_{A,s(k)}(x) := f(x) - H_{S(k)-1}(x, A, f). \tag{6.7}$$

Remark 6.1. *The differentiability conditions at the multiple nodes x_i's) as prescribed in the definition of $K_{A,s(k)}[c,d]$ assures the existence of the limit considered in (6.6).*

Recall that $W_{A,s(k)}$ is a polynomial of degree $S(k)$ (cf (i), Section 6.1) having k distinct zeros at $A := \{x_1, x_2, \ldots, x_{k+1}\}$ each with multiplicity m_i. Counting all the zeros up to their multiplicity, we define an interpolant $\mathbb{H}_{n,s(k)}(x, A, f)$ to f at $(S(k) + n + 1)$ zeros of the orthogonal polynomial $\phi_{n+1,s(k)}(x)$ as follows:

$$\mathbb{H}_{n,s(k)}(x, A, f) := H_{S(k)-1}(x, A, f) + W_{A,s(k)}(x) L_n(x, Z_{n+1,s(k)}, f_{W_{A,s(k)}}), \tag{6.8}$$

where $n = 0,1,2,\ldots$

Remark 6.2. *If* $s(k) = \{m_i\}_{i=1}^{k}$ with each m_i = 1, then $H_{S(k)-1}(x, A, f)$ reduces to $L_k(x, A, f)$ (see Theorem 5.2). In addition if k =2 with $A = \{c, d\}$, it further reduces to $L_1(x, A, f)$ (see Theorem 5.1). Therefore, the interpolant $\mathbb{H}_{n,s(k)}(x, A, f)$, in these cases, will take the form $\mathbb{L}_{n,k+1}(x, A, f)$ and $\mathbb{L}_{n,2}(x, A, f)$ respectively.

6.3. Main result

With the notations and terminology given in the preceding two sections, we state our main result that provides a solution of Problem III:

Theorem 6.1. *Let $A := \{x_1, x_2, \ldots, x_{k+1}\} \subseteq (-\infty, \infty) \setminus (c, d)$ and let $s(k) := \{m_i\}_{i=1}^{k+1}$ be a finite sequence of positive integers. Let $\phi_{j,s(k)}$, j =0,12, ,..., be the system of orthogonal Hermite 0-interpolants at A with respect to ω on the interval [c,d] as defined in section 6.1. If $f \in K_{A,s(k)}[c,d]$ (cf Section 6.2), then the polynomial $\mathbb{H}_{n,s(k)}(., A, f)$ given by*

(6.8) interpolates f at the $(S(k)+n+1)$ *zeros of* $\phi_{n+1,s(k)}(x)$ *in the sense of Hermite. In addition,*

$$\lim_{n\to\infty}\left\|\mathbb{H}_{n,s(k)}(.,A,f)-f\right\|_{[c,d],\omega}=0, \tag{6.9}$$

Proof. We shall divide the proof into two parts.

Part I. Interpolation characteristic of $\mathbb{H}_{n,s(k)}(x,A,f)$ at the zeros of $\phi_{n+1,s(k)}$:

To justify this, note that $A\cup Z_{n+1,s(k)}$ is the set of all *zeros of* $\phi_{n+1,s(k)}$ whereas A and $Z_{n+1,s(k)}$ are disjoint sets. If $u\in A$ then $u=x_i$ for some i leads to $W_{A,s(k)}^{(j)}(u)=W_{A,s(k)}^{(j)}(x_i)=0, j=0,1,\ldots,m_i-1$. Thus,

$$\mathbb{H}_{n,s(k)}(x,A,f)\Big]_{x=u}^{(j)}=H_{S(k)-1}(x,A,f)\Big]_{x=u}^{(j)}+W_{A,s(k)}(x)L_n(x,Z_{n+1,s(k)},f_{W_{A,s(k)}})\Big]_{x=u}^{(j)}$$

implies that $\mathbb{H}_{n,s(k)}^{(j)}(u,A,f)=H_{S(k)-1}^{(j)}(u,A,f)$. On the other hand, if $u\in Z_{n+1,s(k)}$, then

$$L_n(u,Z_{n+1,s(k)},f_{W_{A,s(k)}})=f_{W_{A,s(k)}}(u)=\frac{f(a)-H_{S(k)-1}(u,A,f)}{W_{A,s(k)}(u)}, \text{ i.e,}$$

$$\mathbb{H}_{n,s(k)}(u,A,f)=H_{S(k)-1}(u,A,f)+W_{A,s(k)}(u)L_n(u,Z_{n+1,s(k)},f_{W_{A,s(k)}})=f(u).$$

Thus, the polynomial $\mathbb{H}_{n,s(k)}(.,A,f)$ interpolates *f* at the simple as well as multiple zeros of $\phi_{n+1,s(k)}$

Part II. Convergence of $\mathbb{H}_{n,s(k)}(.,A,f)$ to *f*

The convergence proof here is similar to that given for Theorem 5.1. For the sake of completeness, we shall provide all relevant explanations.

Step 1. By a slight rearrangement of terms in (6.8) and using 6.6)-(6.7), we have

$$\begin{aligned} f-\mathbb{H}_{n,s(k)}(.,A,f)&=W_{A,s(k)}\left[\left\{\frac{\left(f-H_{S(k)-1}(.,A,f)\right)}{W_{A,s(k)}}\right\}-L_n(.,Z_{n+1,s(k)},f_{W_{A,s(k)}})\right] \\ &=W_{A,s(k)}\left[f_{W_{A,s(k)}}-L_n(.,Z_{n+1,s(k)},f_{W_{A,s(k)}})\right]\end{aligned} \tag{6.10}$$

Step 2. Since $f_{W_{A,s(k)}}\in C[c,d]$, there exists $Q_{n,s(k)}\in\pi_n$ which best approximates $f_{W_{A,s(k)}}$ in the uniform norm [21]. In addition [21],

$$\lim_{n\to\infty}\left\|f_{W_{A,s(k)}}-Q_{n,s(k)}\right\|_{[c,d],\infty}=0 \tag{6.11}$$

which implies that

$$\lim_{n\to\infty}\left\|f_{W_{A,s(k)}}-Q_{n,s(k)}\right\|_{[c,d],\omega}=0. \tag{6.12}$$

By triangle inequality, we have

$$\begin{aligned}&\left\|W_{A,s(k)}\left[f_{W_{A,s(k)}}-L_n(.,Z_{n+1,s(k)},f_{W_{A,s(k)}})\right]\right\|_{[c,d],\omega}\\&\le\left\|W_{A,s(k)}\left[f_{W_{A,s(k)}}-Q_{n,s(k)}\right]\right\|_{[c,d],\omega}+\left\|W_{A,s(k)}\left[Q_{n,s(k)}-L_n(.,Z_{n+1,s(k)},f_{W_{A,s(k)}})\right]\right\|_{[c,d],\omega}.\end{aligned} \tag{6.13}$$

Recall that $W_{A,s(k)}(x)$ is a polynomial independent of *n*. Thus, by (6.11)

$$\left\|W_A\left[f_{W_A}-Q_n\right]\right\|_{[c,d],\omega}\to 0 \text{ as } n\to\infty. \tag{6.14}$$

Step 3. $Q_{n,s(k)}$ being a polynomial of degree n implies that $Q_{n,s(k)}=L_n(.,Z_{n+1,s(k)},Q_{n,s(k)})$. Moreover, L_n is a linear operator. Therefore, the second term on the right side of (6.13) can be written as

$$\left\|W_{A,s(k)}\left[Q_{n,s(k)}-L_n(.,Z_{n+1,s(k)},f_{W_{A,s(k)}})\right]\right\|_{[c,d],\omega}=\left\|W_{A,s(k)}L_n(.,Z_{n+1,s(k)},Q_{n,s(k)}-f_{W_{A,s(k)}})\right\|_{[c,d],\omega}. \tag{6.15}$$

Using the representation of L_n in terms of its fundamental polynomials based on $Z_{n+1,s(k)}$, we have

$$W_{A,s(k)}(x)L_n(x,Z_{n+1,s(k)},Q_{n,s(k)}-f_{W_{A,s(k)}})=\sum_{i=1}^{n+1}(Q_{n,s(k)}-f_{W_{A,s(k)}})(z_i)W_{A,s(k)}(x)l_i(x)$$

Hence,

$$\left\|W_{A,s(k)}L_n(.,Z_{n+1,s(k)},Q_{n,s(k)}-f_{W_{A,s(k)}})\right\|^2_{[c,d],\omega}=\left\|\sum_{i=1}^{n+1}(Q_{n,s(k)}-f_{W_{A,s(k)}})(z_i)W_{A,s(k)}l_i\right\|^2_{[c,d],\omega}. \tag{6.16}$$

Step 4. Considering the orthogonality of $W_{A,s(k)}\,l_i, i=1,2,\ldots,n+1$, we have

$$\begin{aligned}&\left\|\sum_{i=1}^{n+1}(Q_{n,s(k)}-f_{W_{A,s(k)}})(z_i)W_{A,s(k)}l_i\right\|^2_{[c,d],\omega}\\&=\sum_{i=1}^{n+1}\left(Q_{n,s(k)}(z_i)-f_{W_{A,s(k)}}(z_i)\right)^2\int_c^d\left[W_{A,s(k)}(x)l_i(x)\right]^2\omega(x)dx.\end{aligned} \tag{6.17}$$

By using the continuity argument as given in case of (5.16), we get to the following relation:

$$\sum_{i=1}^{n+1}\left(Q_{n,s(k)}(z_i)-f_{W_{A,s(k)}}(z_i)\right)^2\int_c^d\left[W_{A,s(k)}(x)l_i(x)\right]^2\omega(x)dx$$
$$\leq\left\|Q_{n,s(k)}-f_{W_{A,s(k)}}\right\|_{[c,d],\infty}^2\sum_{i=1}^{n+1}\int_c^d\left[W_{A,s(k)}(x)l_i(x)\right]^2\omega(x)dx\,. \tag{6.18}$$

Step 5. Since $\sum_{i=1}^{n+1}l_i(x)=1$, on the lines of (5.17) we have

$$\int_c^d\left[W_{A,s(k)}(x)\right]^2\omega(x)dx=\int_c^d W_{A,s(k)}^2(x)\sum_{i=1}^{n+1}[l_i(x)]^2\omega(x)dx \tag{6.19}$$

Step 6. Combining (6.14)-(6.18), we arrive at

$$\left\|W_{A,s(k)}\left[L_n(.,Z_{n+1,s(k)},Q_{n,s(k)}-f_{W_{A,s(k)}})\right]\right\|_{[c,d],\omega}$$
$$\leq\left\|Q_{n,s(k)}-f_{W_{A,s(k)}}\right\|_{[c,d],\infty}\sqrt{\int_c^d\left[W_{A,s(k)}(x)\right]^2\omega(x)dx}\rightarrow 0 \text{ as } n\rightarrow\infty$$

This along with (6.10), (6.12) and (6.13) shows that $\left\|f-\mathbb{H}_{n,s(k)}(.,A,f)\right\|_{[c,d],\omega}\rightarrow 0$ as as $n\rightarrow\infty$.

6.4. On the choice of additional nodes

In Problems II (cf Chapter 5) and Problem III, any choice of additional nodes in the intervals (*a,c*) and (*d,b*) does not affect the convergence results (5.7) and (6.9) . However, this choice does affect the amount of deviation of the approximating polynomial from the function *f*(*x*) outside the interval of convergence. On the other hand the choice is made out of the shifted Chebyshev zeros, the behavior of approximating polynomial improves outside the interval of convergence. This phenomenon will be observed in the simulation results given in Chapter 7.

7. Computational Aspects

The solutions of Problems I-III (cf Chapters 5 & 6) mainly depend on the computation of orthogonal 0-interpolants (*OZI*) (cf Section 4.2) and their respective zeros. As observed in the 3-term recurrence relation, the computation of *OZI* are entirely based on integrals of the form $\langle g,h\rangle_{\tilde{w}} = \int_a^b g(x)h(x)\tilde{w}(x)dx$. It may be observed that the polynomials g and h grow to higher degree with the successive application of the recurrence relation. Thus, the propagation of round-off error in the computation of these integrals causes a severe ill-conditioning effect on the 3-term recurrence relation. This is a similar situation which we encounter in computing the classical orthogonal polynomials [8], [21]. To overcome this problem, approximation of inner products with a suitable quadrature rule is highly recommended [17]. Here, we adopt discretized Stieltjes procedure based on n-point Fejer Quadrature rule.

7.1. Steiltjes procedure

The orthogonal exponential 0-interpolant computed at each stage by 3-term recurrence relation helps us to compute the recursion coefficients. These coefficients are utilized in the computation of next stage *OEZI*. The process of computing recursion coefficients with this strategy is due to Steiltjes [17].

7.2. Transformation of Chebyshev points

The process of discretiztion usually involves Chebyshev zeros, say t_i, which lie in the interval $(-1,1)$. In case, the underlying interval (c,d), we shift these zeros by the transformation:

$$t_i \to x_i := \frac{d-c}{2} t_i + \frac{d+c}{2}. \tag{7.1}$$

7.3. Discretization by Fejer quadrature rule

The integrals $\int_a^b (.)dt$ involved in the inner products or otherwise are discretized by the following quadrature rule [17]:

$$\int_a^b F(x)\omega(x)dx \approx \sum_{j=1}^{M} w_j^M F(x_j^M)\omega(x_j^M) \tag{7.2}$$

with nodes $x_j^M \in (a,b)$:

$$t_i^M := \cos\theta_i^M \text{ with } \theta_i^M = \frac{2i-1}{2n}; \quad x_i^M = \frac{d-c}{2}t_i^M + \frac{d+c}{2} \tag{7.3}$$

and weights $w_j^M > 0$ given by

$$w_i^M := \frac{2}{M}\left\{1 - 2\sum_{j=1}^{[\![M/2]\!]} \frac{\cos\left(2j\theta_i^M\right)}{4j^2-1}\right\} \tag{7.4}$$

The approximation of integral by (7.2) subject to (7.3) and (7.4) is known as Fejer quadrature rule.

7.4. Computation of simple zeros of orthogonal 0-interpolants

Major part of the required interpolant is based on the simple zeros of the orthogonal 0-interpolants. For this, we

1. select the set of additional nodes lying out side the interval of convergence: $A = \{ x_i : i = 1,2,\ldots,k+1\}$
2. construct the product of linear factors $(x - x_i)^{n_i}$ where n_i is the multiplicity of x_i: $\boldsymbol{W}(x)$.
3. compute the recurrence coefficients in 3-term recurrence relation (3.12) for "orthogonal polynomials based on weight function $\tilde{\omega}(x) = \omega(x)\,\boldsymbol{W}^2(x)$ and $\tilde{\omega}(x) = \omega(x)\,\boldsymbol{W}^4(x)$" separately.
4. compute the set of zeros of orthogonal polynomials, say $\boldsymbol{Z}$, which emerge as the eigenvalues of the tridiagonal matrix based on α 's and β 's: .

7.5. Computation of required interpolants

The interpolant which appear as the solution of Problems I-III (cf Chapters 5-6) is comprised of 3 components:

1. Lagrange or Hermite interpolant $LH(x,A,f)$ to a given function f at the set of additional nodes $A = \{ x_i : i = 1,2,\ldots,k + 1\}$ lying out side the interval of convergence.
2. Modified function: $f_{\mathrm{WA}}(x) =[f(x) - LH(x,A,f)]/\boldsymbol{W_{LH}}(x)$.
3. Construction of Lagrange interpolant to $f_{\mathrm{WA}}(x)$ at $\boldsymbol{Z}$: $L(x,Z, f_{\mathrm{WA}})$.
4. The required interpolant: $\boldsymbol{IntP_n}(x,f) := LH(x,A,f) + \boldsymbol{W_{LH}}(x)\, L(x,Z, f_{\mathrm{WA}})$.

Note that $\boldsymbol{W_{LH}}(x)$ in the presence of additional nodes is either $\prod_{w \in A}(x - w)$ which is the case of simple additional nodes or $\prod_{w \in A}(x - w)^2$ in case of double nodes. Also, in the absence of additional nodes, we shall have $\boldsymbol{IntP_n}(x,f) := L(x,Z, f)$

7.6. Measurement of Error of approximation and Graphs

In order to determine the level of accuracy of approximating polynomial "$\boldsymbol{IntP_n}(x,f)$", we consider standard types of errors:

Definition 3.1. Pointwise error "Err" with respect to mesh points $x = u_i$, $1 \le i \le N$, is defined as

$$\mathrm{Err}(x) = f(x) - \boldsymbol{IntP_n}(x,f). \tag{3.5}$$

Definition 3.2. Maximum error "M-Err" with respect to mesh points t_i , $1 \le i \le N$, is given by

$$\text{M-Err} = \max_{1 \le i \le N} \left| f(u_i) - \boldsymbol{IntP_n}(u_i, f) \right|. \tag{3.6}$$

Definition 3.3. Root Mean squared error "RMS-Err" with respect to mesh points t_i , $1 \le i \le N$, is defined as

$$\text{RMS} = \sqrt{\frac{\sum_{i=1}^{N} \left| f(u_i) - \boldsymbol{IntP_n}(u_i, f) \right|^2}{N}}. \tag{3.7}$$

The graphs of test functions f and their approximants $\boldsymbol{IntP_n}(x,f)$ on the interval of convergence $[c,d]$ as well on the extended interval $[a,b]$ are drawn in the MATLAB environment.

7.7. Interpolating curve out side the interval of convergence

In order to see the behavior of interpolating curve outside the interval of convergence $[c,d]$, we compute f and $\boldsymbol{IntP_n}(x,f)$ on the interval $[a,b]$ that contains both $[c,d]$ and the additional nodes. The graphs of f and $\boldsymbol{IntP_n}(x,f)$ are drawn over the interval $[a,b]$.

8. Simulation Results

We have tested the proposed methods for Runge function $f(x) = 1/(x^2+1)$ on $[c,d] = [-2.5,2.5]$, the interval of convergence and on the extended interval $[a,b] = [-5, 5]$ with the set of preassigned $A = \{-5, -4, -3, 3, 4, 5\}$. Our objective is to compare the two types of errors namely, point-wise error and root mean squared error, that arise from the error function '$f(.) - IntP_n(.)$' for n = 5, 10, 20. Here, n represents the number of orthogonal polynomials involved in the interpolation process. More precisely, we shall consider $\boldsymbol{IntP_n}(x,f)$ relevant to n zeros of orthogonal polynomials over $[c,d] = [-2.5,2.5]$ when

i. there is no preassigned node involved in the interpolation process, i.e.,

   $\boldsymbol{IntP_n}(x,f) = L_n(x,Z,f)$

   as considered in Erdos-Turan Theorem (cf Theorem 3.1).

ii. a finite set A of preassigned simple or double nodes lying outside (c,d) is involved in the interpolation process, i.e.,

   (a) $\boldsymbol{IntP_n}(x,f) = \mathbb{L}_{n,6}(z,f)$

   as considered in our main result related to simple nodes (cf Theorem 5.2),

   (b) $\boldsymbol{IntP_n}(x,f) = \mathbb{H}_{n,s(6)}(z,f)$

   as considered in the last main result related to double nodes (cf Theorem 6.1).

8.1. Explanation of simulation results

For the explanation of simulation results, we recall that the set of all zeros of an orthogonal 0-interpolant ϕ_n depends on the nature of the preassigned zeros which may be simple or double. In case of simple zeros, $\phi_n = Wp_n$ where $W(x) = \prod_{w \in A}(x-w)$ and for the second case $\phi_n = W^2\hat{p}_n$. In both cases, p_n and $\hat{p}_n$ have n simple zeros lying within (c,d) $W^2(x) = (-2.5,2.5)$. However, p_n and $\hat{p}_n$ are orthogonal polynomials over $[c,d]$ with respect to weight functions $W^2(x)$ and $W^4(x)$ respectively. On the other hand,

$\phi_n = p_n = \hat{p}_n$ in the absence of preassinged nodes and therefore ϕ_n, p_n and $\hat{p}_n$ are orthogonal over $[c,d]$ with respect to weight function 1. Thus, it is natural to discuss the errors due to

i. $\boldsymbol{IntP_n}$ = $L_n(., Z, f)$ in Erdos-Turan Theorem based on the zeros of orthogonal polynomial over $[c,d]$ with respect to

   a. weight function $w(x) = 1$,

   b. weight function $w(x) = W^2(x)$,

   c. weight function $w(x) = W^4(x)$,

ii. $\boldsymbol{IntP_n} = \mathbb{L}_{n,6}(z, f)$ in Theorem 5.2,

iii. $\boldsymbol{IntP_n}(x,f) = \mathbb{H}_{n,s(6)}(z, f)$ in Theorem 6.1.

It may be noted that in all the three cases a-c of (i) we do not consider interpolation at any preassigned zero, whereas in case of (ii) and (iii), the interpolant $\boldsymbol{IntP_n}$ respectively interpolates f at 6 simple zeros and 6 double zeros.

8.2. Some abbreviations

We have divided our computational work into two parts:

Part I: Error '$f - \boldsymbol{IntP_n}$' on the interval of convergence, i.e., $[c,d]$

Part II: Error '$f - \boldsymbol{IntP_n}$' on the extended interval, i.e., $[a,b]$.

Two types of errors are computed over a set of mesh points in both intervals:

i. Maximum of absolute value of pointwise error,

ii. Root mean squared error.

These errors '$f - \boldsymbol{IntP_n}$' are separately tabulated. The graphs of Runge function and the respective interpolant are provided below each table. For the sake of simplicity we have used uniform abbreviations in the tables. They are as follows:

i. Er[E-T(1)]: Error based on Erdos-Turan Theorem when $w(x) = 1$,

ii. Er[E-T(W^2)]: Error based on Erdos-Turan Theorem when $w(x) = W^2(x)$,

iii. Er[E-T(W^4)]: Error based on Erdos-Turan Theorem when $w(x) = W^4(x)$,

iv. Main Er[$\mathbb{L}_n$]: Error based on one of our main Theorems (cf Theorem 5.2)

v. Main Er[$\mathbb{H}_n$]: Error based on another main result (cf Theorem 6.1).

In addition, we used the following abbreviation for the two errors:

i. Max(Abs Er) : Maximum of the absolute values of the pointwise error based on mesh points of the underlying interval,

ii. RMS: Root mean squared error based on the mesh points of the underlying interval.

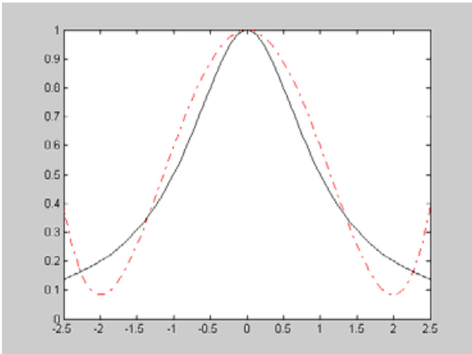
8.3. Simulation results and graphs

Based on the above explanation, the outcome of our computational work is provided in the tables and graphs in the next pages. The computational work is subdivided into two parts. Part I deals with the convergence of interpolating processes over the interval $[c,d] = [-2.5,2.5]$ and Part II takes care of other aspects like interpolation at preassidgned nodes and behavior of error outside $[c,d]$.

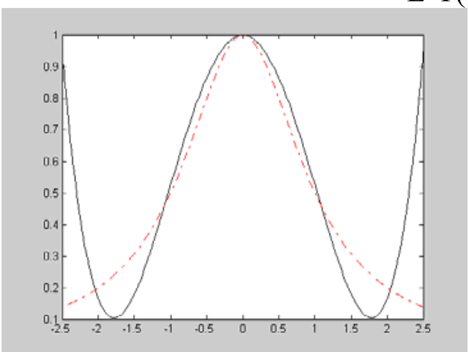
Part I. Interval [-2.5, 2.5]
Table I-(i) ***n*****=5**

	Max(Abs Er)	RMS
Er[E-T(1)]	0.2478	0.1155
Er[E-T(W^2)]	0.8078	0.3451
Er[E-T(W^4)]	1.9709	0.8981
Main Er[$\mathbb{L}_n$]	0.0919	0.0548
Main Er[$\mathbb{H}_n$]	0.0349	0.0247

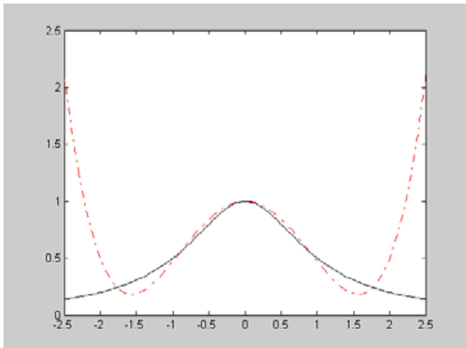
Graphs of Function and Interpolating Polynomial



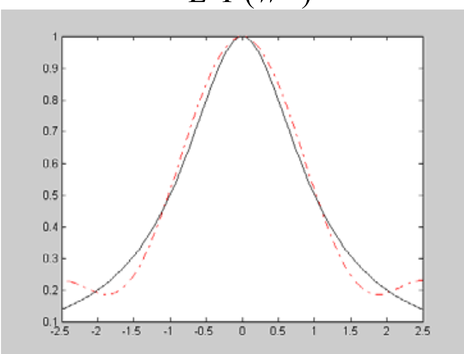
E-T(1)



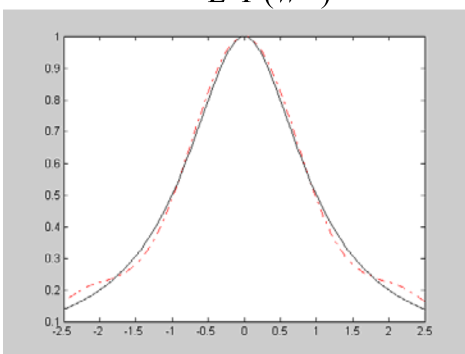
E-T (W^2)



E-T (W^4)



Main [$\mathbb{L}_n$]

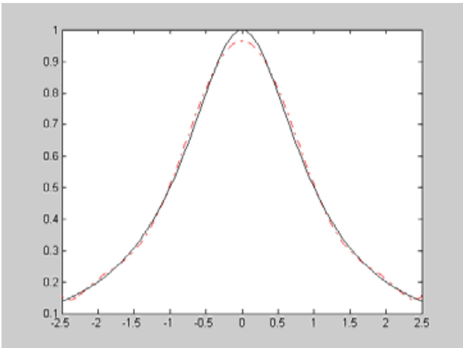


Main [$\mathbb{H}_n$]

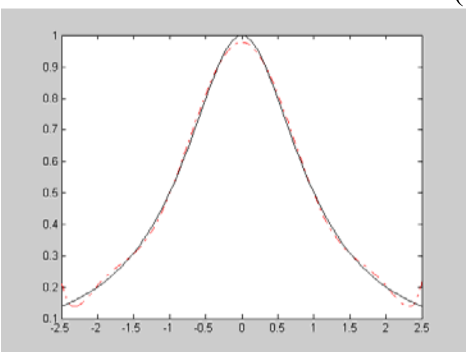
Table I-(ii) $\quad n = 10$

	Max(Abs Er)	RMS
Er[E-T(1)]	0.0342	0.0127
Er[E-T(W^2)]	0.0762	0.0251
Er[E-T(W^4)]	0.2849	0.0939
Main Er[$\mathbb{L}_n$]	0.0176	0.0065
Main Er[$\mathbb{H}_n$]	0.0091	0.0033

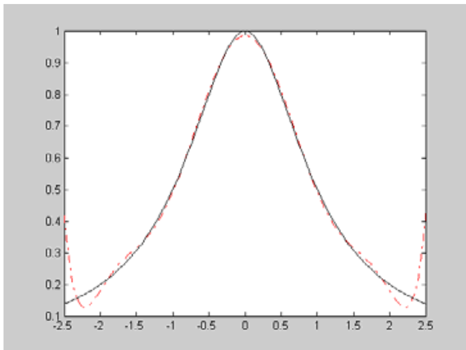
Graphs of Function and Interpolating Polynomial



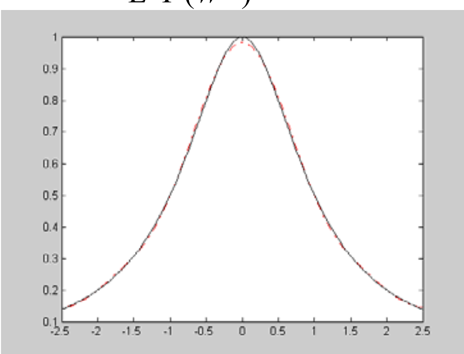
E-T(1)



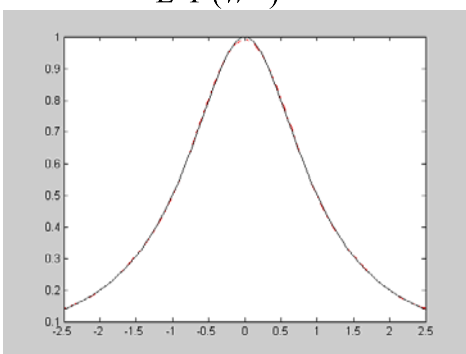
E-T (W^2)



E-T (W^4)



Main [$\mathbb{L}_n$]

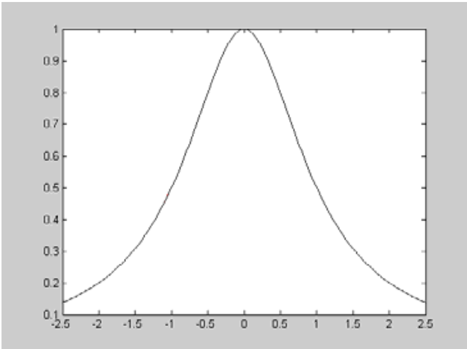


Main [$\mathbb{H}_n$]

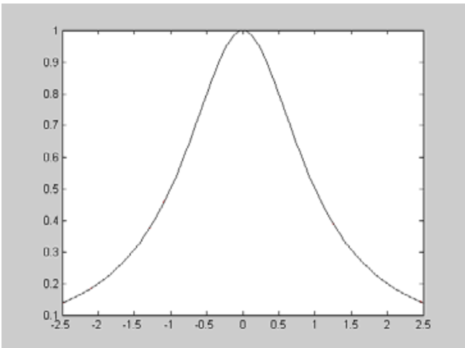
Table I-(iii)   *n*=20

	Max(Abs Er)	RMS
Er[E-T(1)]	6.8120e-004	2.6183e-004
Er[E-T(W^2)]	0.0023	5.5617e-004
Er[E-T(W^4)]	0.0097	0.0022
Main Er[$\mathbb{L}_n$]	3.5070e-004	1.3493e-004
Main Er[$\mathbb{H}_n$]	1.8047e-004	6.9494e-005

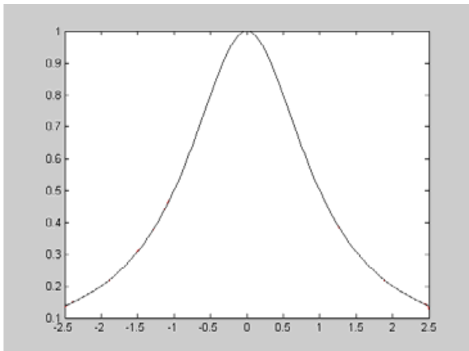
Graphs of Function and Interpolating Polynomial



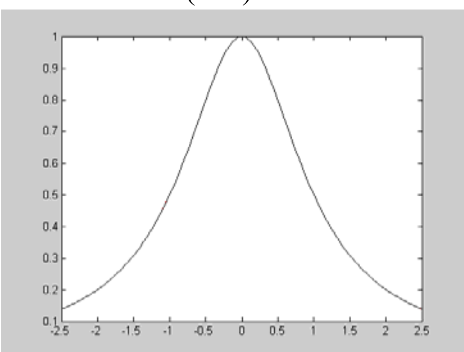
E-T(1)



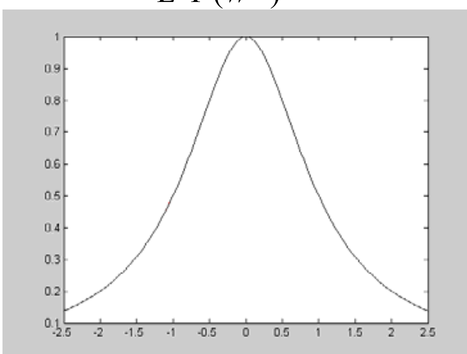
E-T (W^2)



E-T (W^4)



Main [$\mathbb{L}_n$]

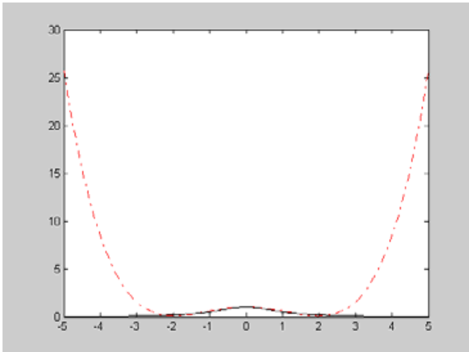


Main [$\mathbb{H}_n$]

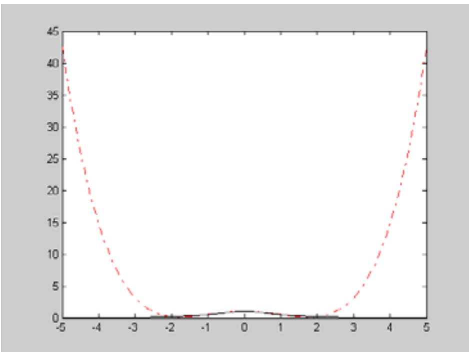
Part II. Interval [-5, 5]
Table II-(i) ***n*=5**

	Max(Abs Er)	RMS
Er[E-T(1)]	25.6868	7.9663
Er[E-T(W^2)]	42.4778	15.5527
Er[E-T(W^4)]	68.4779	22.0419
Main Er[$\mathbb{L}_n$]	1.6752	0.5056
Main Er[$\mathbb{H}_n$]	0.1456	0.0422

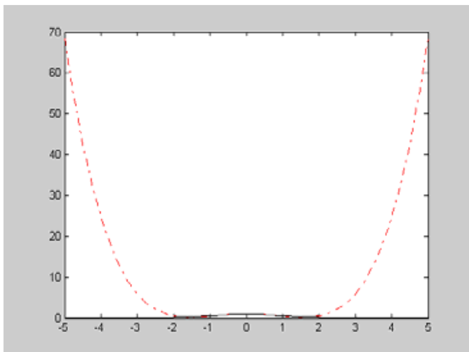
Graphs of Function and Interpolating Polynomial



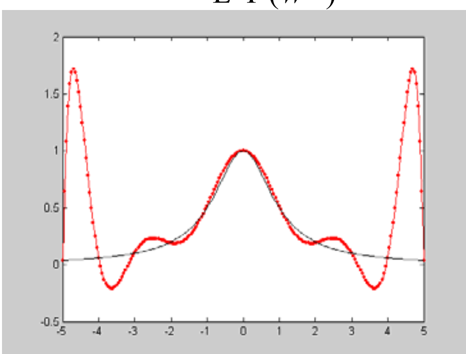
E-T(1)



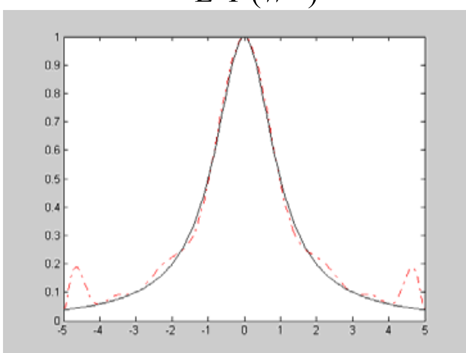
E-T (W^2)



E-T (W^4)



Main [$\mathbb{L}_n$]

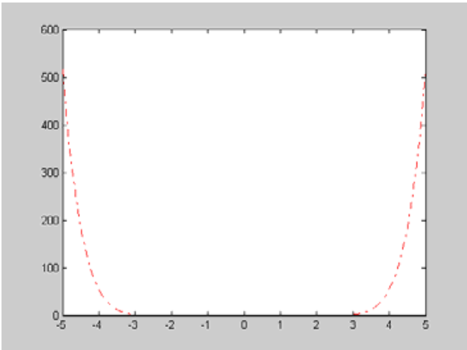


Main [$\mathbb{H}_n$]

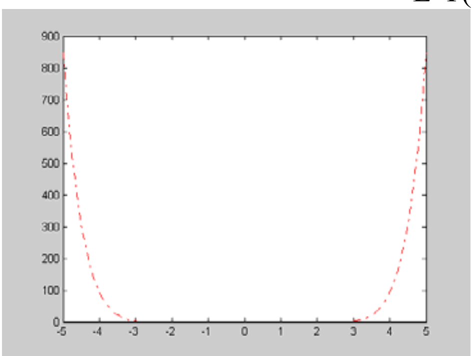
Table II-(ii) n=10

	Max(Abs Er)	RMS
Er[E-T(1)]	516.3618	119.6932
Er[E-T(W^2)]	848.3145	229.5151
Er[E-T(W^4)]	1.3997e+003	329.9662
Main Er[$\mathbb{L}_n$]	25.6391	7.0482
Main Er[$\mathbb{H}_n$]	2.1698	0.5509

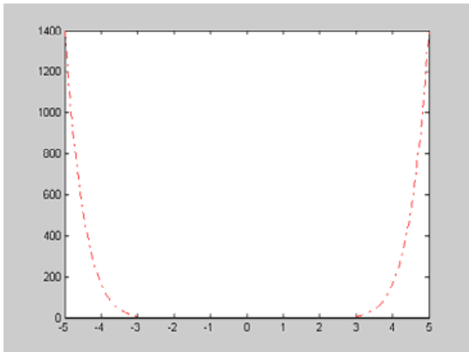
Graphs of Function and Interpolating Polynomial



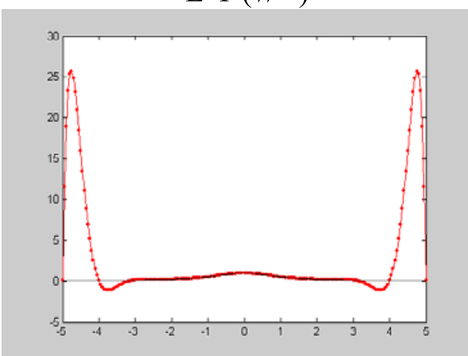
E-T(1)



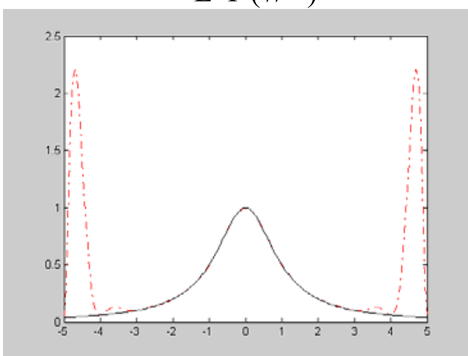
E-T (W^2)



E-T (W^4)



Main [$\mathbb{L}_n$]

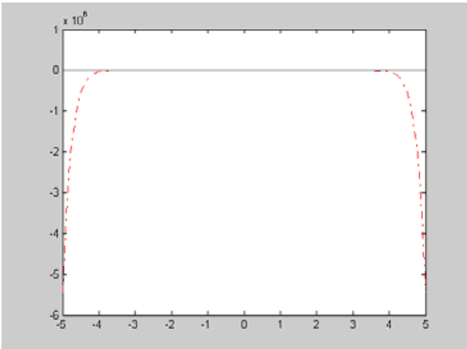


Main [$\mathbb{H}_n$]

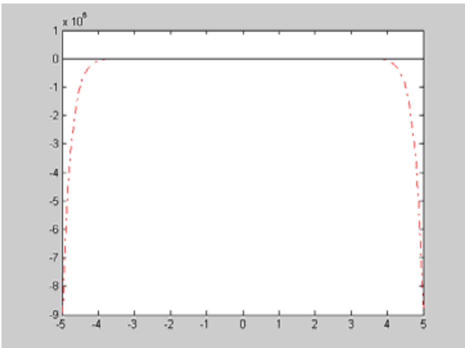
Table II-(iii) n=20

	Max(Abs Er)	RMS
Er[E-T(1)]	5.4506e+006	9.1502e+005
Er[E-T(W^2)]	8.9261e+006	1.7401e+006
Er[E-T(W^4)]	1.4621e+007	2.4703e+006
Main Er[$\mathbb{L}_n$]	1.7179e+005	3.9627e+004
Main Er[$\mathbb{H}_n$]	1.2037e+004	2.7845e+003

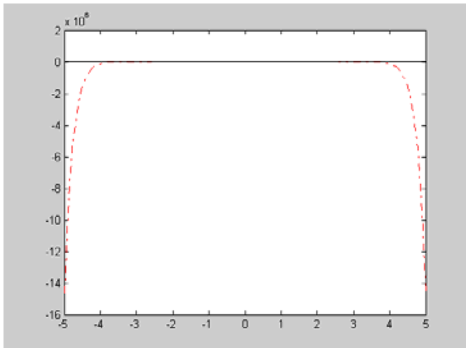
Graphs of Function and Interpolating Polynomial



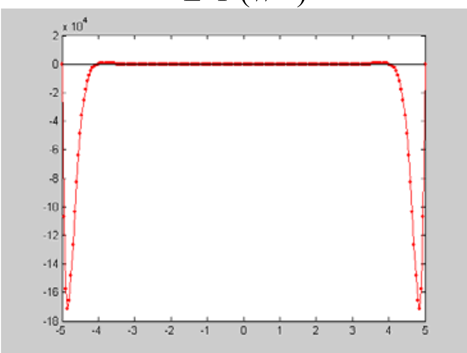
E-T(1)



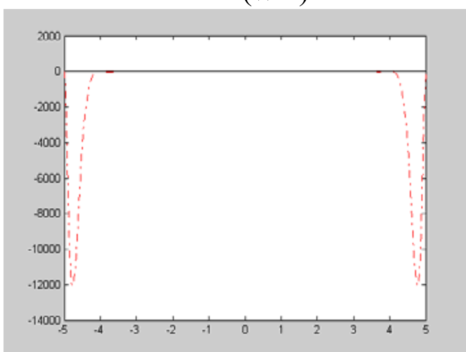
E-T (W^2)



E-T (W^4)



Main [$\mathbb{L}_n$]



Main [$\mathbb{H}_n$]

9. Concluding Remarks

We present our observations related to the simulation results provided in the Tables and graphs (cf Section 8.3). We subdivide our discussion into two parts.

9.1. A note on convergence of various interpolating processes

According to Theorems 3.1, 5.2 and 6.1, the error over the interval [*c,d*], in all the interpolation processes decreases to zero as *n,* the number of zeros of orthogonal polynomial over [*c,d*] increases . This is what we observe in the simulation results given in part I of Section 8.3. However, a comparative study of Er[E-T(1)], Er[E-T(W^2)] and Er[E-T(W^4)] in worth to note. In fact, all the errors in the three cases are related to Erdos-Turan interpolating polynomials. Based on Tables I-(i)-(iii) and the corresponding graphs we note that the Er[E-T(1)] is better to Er[E-T(W^2)] and the latter is better to Er[E-T(W^4)]. On the other side, we note that with the involvement of preassigned simple or double nodes in the Lagrange or Hermite interpolants respectively leads to an improvement in the reduction of error over the earlier three cases, the Hermite interpolant considered in Theorem 6.1 is the best one .

9.2. Behavior of interpolants outside the interval of convergence

It may be noted that convergence outside (*c,d*), the interval of convergence, is not an objective of our work. In case of extended interval [*a,b*], we are merely considerate about interpolation at additional points of the polynomials $\mathbb{L}_n$ and $\mathbb{H}_n$ (cf Theorem 5.1, 5.2 and 6.1). We observe this phenomenon in all the graphs given in Part II of Section 8.3. As far as comparative error is concerned, we note from the Tables II-i-iii that the Main Er[$\mathbb{H}_n$] is better to Main [$\mathbb{L}_n$] and Main [$\mathbb{L}_n$] is better to the errors due to rest of the interpolating processes.

9.3. The case of shifted Chebyshev points as additional nodes

We have noticed the selection of Chebyshev points when considered in the interpolating processes provides an improvement in the reduction of error as compared to any other choice, in particular, the uniformly distributed points. It addition, this choice preserves all the properties discussed in the preceding two sections.

10. References

[1] M. A. Bokhari and M. Iqbal, *L_2-Approximation of real valued functions with interpolatory constraints*, J. Comp. Appl Math, 70(1996), 201-205.

[2] M. A. Bokhari, A variant of Gauss quadrature rule, ICNAAM (Greece)-2006, Wiley- VCH Verlag GmbH & Co. KgaA, Weinheim, 2006

[3] M. A. Bokhari, "On Hermite interpolating L_2-approximants," Dynamical Systems and applications, Vol. 16 (2007), pp. 203-216.

[4] M. A. Bokhari, "Gauss Quadrature Formula: An Extension via Interpolating Orthogonal Polynomials", J. Fanklin Institute, Vol. 344/5 (2007), pp 637-645.

[5] M. A. Bokhari and H. Al-Attas, "Interpolating orthogonal polynomials: *Theoretical & Computational Aspects*", KFUPM FT/2005-08 (2006).

[6] M. A. Bokhari and H. Al-Attas, "Combination of Orthogonal Collocation Points with Pre-assigned Knots: *A Study of Numerical Solutions of Boundary Value Problems*" FT-060002 (2008)

[7] M.R. Capobianco, M.G. Russo, Extended interpolation with additional nodes in some Sobolev-type spaces, *Studia Scientiarum Mathematicarum Hungarica* 35 (1-2), pp. 81-97 (1999)

[8] E. W. Cheney, Introduction to approximation theory, McGraw Hill, (1966).

[9] G. Criscuolo, G. Mastroianni, P. Vértesi, Pointwise simultaneous convergence of extended Lagrange interpolation with additional knots *Math. Comp.*, 59, pp. 515-531 (1992) .

[10] Criscuolo, G., Mastroianni, G., Occorsio, D. Convergence of extended Lagrange interpolation *Math. Comp.*, 55, pp. 197-212. (1990)

[11] S.B. Damelin,, D.S. Lubinsky, Necessary and sufficient conditions for mean convergence of Lagrange interpolation for Erdos weights *Canadian Journal of Mathematics*, 48 (4), pp. 710-736 (1996)

[12] S.B.Damelin, H.S. Jung, K.H. Kwon, Mean convergence of extended Lagrange interpolation for exponential weights, *Acta Applicandae Mathematicae 76 (1), pp. 17-36* (2003)

[13] F. Dautsch, *On uniform approximation with interpolatory constraints*, J. Math. Anal. Appl., 24(1968), 62-79

[14] J.Van Deun, A.Bultheel, An interpolation algorithm for orthogonal rational functions, *Journal of Computational and Applied Mathematics 164 (1), pp. 749-762*(2004)

[15] P. Erdos and P. Turan, *On Interpolation,* AM 38(1937), 142-155.

[16] W.Gander , Change of basis in polynomial interpolation *Numerical Linear Algebra with Applications 12 (8), pp. 769-778*(2005)

[17] W. Gautschi, Orthogonal Polynomials: Computations and Applications, Claredon Press, (2004)

[18] G. H. Golub and J. H. Welsch, *Calculation of Gauss Quadrature Rule*, Math. Comp. 23(1969), 221-230.

[19] R. Lobatto, Lessen over de Differentiaal- en Iintegraal-Rekening. Part II. Iintegraal-Rekening, The Hague: Van Cleef*,* (1852).

[20] E. Meijeing, *A Chronology of Interpolation: From Ancient Astronomy to Modern Signal and Image Processing*, Proceedings of IEEE, v. 90, 3(2002), 319-342.

[21] M.J.D. Powell, Approximation Theory and Methods. Cambridge University Press, Cambridge, (1981).

[22] R Radau, *Etude sur le formules d'approximation qui servent a calculer la valeur numerique d'une integrale definie*, J. Math Pures Appl. *6*, (1880) 283-336,

[23] J. Stoer and R Bulirsch, An Introduction to Numerical Analysis*,* Springer, 2nd Ed., (1993).

[24] I.P. Streltsov, Application of Chebyshev and Legendre polynomials on discrete point set to function interpolation and solving Fredholm integral equations, *Computer Physics Communications* 126 (1), pp. 178-181 (2000)

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11. Presentations & Publications

1. The work carried out in the project has been presented in the following conference:

   7th International ISAAC Congress, Imperial College of London, UK, July 16, 2009.

   Title of Talk: *Interpolation beyond the interval of convergence: AN extension of Erdos-Turan Theorem*

2. M.A. Bokhari and H. Al-Attas, *Interpolation beyond the interval of convergence: AN extension of Erdos-Turan Theorem* for Publication in the refereed Proceedings of 7th ISAAC Congress to be held at Imperial College, London, UK (July 13 – 18, 2009), *World Scientific Publication Co.* (accepted).