On the Dam Problem with Two Fluids Governed by a Nonlinear Darcy's Law

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Abstract

We consider the problem of two fluids flow through a porous medium governed by a nonlinear law. We prove the existence of a weak solution, establish the local Lipschitz continuity of this solution in the zone above the lower fluid, and prove the continuity of the upper free boundary. In the rectangular case, we prove the existence of a monotone solution with respect to the vertical variable, and the continuity of the lower free boundary. Finally, we prove the uniqueness of a monotone solution with respect to x and y, when the dam is rectangular and the flow obeying to the linear Darcy law.

AMS Subject Classifications: 35R35, 76S05, 76Txx.

1 Introduction

The dam problem with one fluid has been studied by many authors (see [9], [10], [11], [12], [13], [1], [2], [3], [16], [7], [17], [27], [19], etc.) However to the best of our knowledge, the two-fluid dam problem has been considered only in [6] in the case where the flow is governed by the well known linear Darcy law. The authors established the existence of a weak solution, which is locally Lipschitz continuous, and that the upper free boundary is an analytic curve $x = k(y)$. In the rectangular case, they proved the existence of a monotone solution with respect to both variables x and y. Finally they proved that the lower free boundary is a continuously differentiable curve $y = \phi(x)$.
In this paper, we would like to reconsider the model studied by Alt-Caffarelli-Friedman, assuming the flow governed by a nonlinear Darcy's law. The dam is represented by the open set Ω (see Figure 1)

$$\Omega = \{(x, y) \in \mathbb{R}^2 \ / \ x \in (0, a), \ s_-(x) < y < s_+(x)\},$$

where s_- and s_+ are C^1 functions defined on $[0, a]$ such that $s_-(x) < s_+(x)$ $\forall x \in (0, a)$, $s_-(0) = s_+(0)$, and $s_-(a) = s_+(a)$. We denote by T the point of $\partial\Omega$ of coordinates $(x_0, s_+(x_0))$

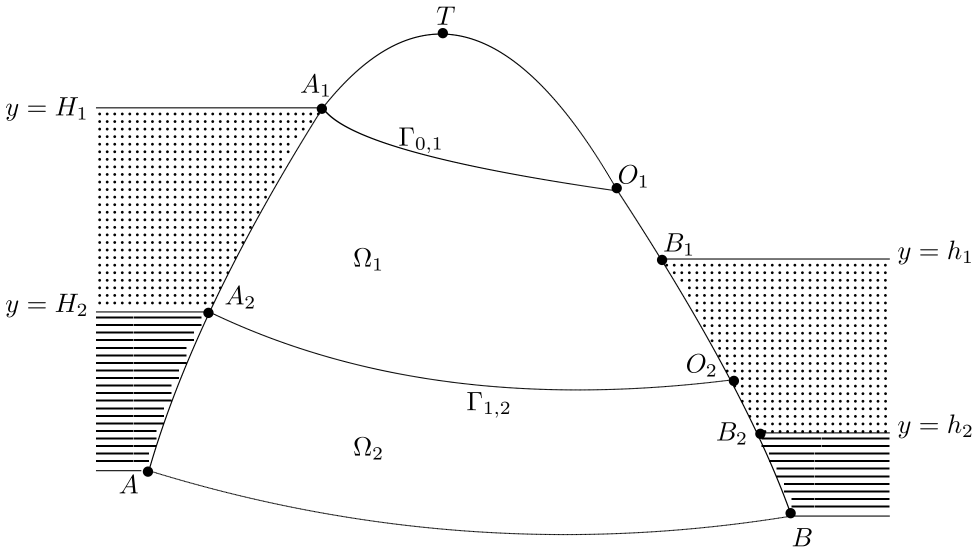

Figure 1

such that $s_+(x_0) = \max_{x\in[0,a]} s_+(x)$, and

$$s'_+(x) \geq 0 \quad \text{for all } x \in (0, x_0) \quad \text{and} \quad s'_+(x) \leq 0 \quad \text{for all } x \in (x_0, a). \tag{1.1}$$

The dam is supplied by two reservoirs. The left (resp. right) one contains two fluids at levels H_1 and H_2 (resp. h_1 and h_2) with $0 < h_2 < H_2 < H_1$ and $h_2 < h_1 < H_1 < s_+(x_0)$.
We assume the flow through the porous medium obeying to the following nonlinear Darcy's law

$$|\mathbf{v}|^{m-1}\mathbf{v} = -k\nabla\phi, \quad m > 0 \tag{1.2}$$

where $\mathbf{v}$ is the fluid velocity, k is the permeability of the medium which we assume constant and equal to 1, $\phi = p + \gamma y$ is the piezometric head, p is the fluids pressure, and γ is given by

$$\gamma = \delta_1\chi(\Omega_1) + \delta_2\chi(\Omega_2), \quad \text{with } \delta_1, \delta_2 > 0$$

where δ_i $(i = 1, 2)$ represents the specific weight of the i^{th} fluid occupying the domain Ω_i of Ω, $\chi(E)$ denotes the characteristic function of the set E. We shall also denote the restriction of a function f to Ω_i by f_i,.

The lower liquid is assumed to be the heavier one, which means that $\delta = \delta_2 - \delta_1 > 0$. Moreover the flow is assumed to be incompressible, that is

$$div(\mathbf{v_i}) = 0 \qquad \text{in} \quad \Omega_i. \tag{1.3}$$

The bottom $\overset{\frown}{AB}$ of the dam is assumed to be impervious. So if $\nu = (\nu_x, \nu_y)$ is the outward unit normal vector to $\partial\Omega$, we have

$$\mathbf{v_2} \cdot \nu = 0 \qquad \text{on} \quad \overset{\frown}{AB}\ . \tag{1.4}$$

Using the continuity of the pressure across $\overset{\frown}{AA_1}$ and $\overset{\frown}{BB_2}$, and the fact that $p + \delta_i y$ is constant in the i^{th} reservoir, we obtain

$$\phi = c^{st} \qquad \text{on} \quad \overset{\frown}{AA_1} \quad \text{and} \quad \text{on} \quad \overset{\frown}{BB_2}\ . \tag{1.5}$$

Assuming there is overflow on $\overset{\frown}{B_1O_1}$, we obtain by taking into account (1.2)

$$\phi = \delta_1 y \qquad \text{and} \qquad -\frac{\partial\phi}{\partial\nu} \geq 0 \qquad \text{on} \quad \overset{\frown}{B_1O_1}\ . \tag{1.6}$$

Since $p + \delta_1 y = \delta_1 h_1$ to the right of $\overset{\frown}{B_2O_2}$, and $\phi = p + \delta_2 y$ to the left of $\overset{\frown}{B_2O_2}$, we obtain

$$\phi = (\delta_2 - \delta_1)y + \delta_1 h_1 \qquad \text{on} \quad \overset{\frown}{B_2O_2}\ . \tag{1.7}$$

We also assume that we have overflow on $\overset{\frown}{B_2O_2}$. Therefore we obtain by taking into account (1.2)

$$-\frac{\partial\phi}{\partial\nu} \geq 0 \qquad \text{on} \quad \overset{\frown}{B_2O_2}\ . \tag{1.8}$$

Since $p + \delta_1 y = \delta_1 h_1$ to the right of $\overset{\frown}{O_2B_1}$, and $\phi = p + \delta_1 y$ to the left of $\overset{\frown}{O_2B_1}$, we obtain

$$\phi = \delta_1 h \qquad \text{on} \quad \overset{\frown}{O_2B_1}\ . \tag{1.9}$$

On the upper free boundary $\Gamma_{0,1}$, separating Ω_1 from the dry region, we have

$$p = 0 \qquad \text{and} \qquad \mathbf{v} \cdot \nu = 0 \qquad \text{on} \quad \Gamma_{0,1} \tag{1.10}$$

that is

$$\phi = \delta_1 y \qquad \text{and} \qquad \frac{\partial\phi}{\partial\nu} = 0 \qquad \text{on} \quad \Gamma_{0,1}. \tag{1.11}$$

The continuity of the pressure at the lower free boundary $\Gamma_{1,2}$ separating Ω_1 from Ω_2, and the immiscibility of the two fluids reads

$$p_1 = p_2 \qquad \text{and} \qquad \mathbf{v_1} \cdot \nu = \mathbf{v_2} \cdot \nu = 0 \qquad \text{on} \quad \Gamma_{1,2} \tag{1.12}$$

which is equivalent to

$$\phi_1 - \phi_2 = (\delta_1 - \delta_2)y \qquad \text{and} \qquad \frac{\partial \phi_1}{\partial \nu} = \frac{\partial \phi_2}{\partial \nu} = 0 \qquad \text{on} \quad \Gamma_{1,2}. \tag{1.13}$$

Assuming that Ω_i $(i = 1, 2)$ is a simply connected domain, we deduce (see [4]) that there exist functions $\psi_i : \Omega_i \to \mathbb{R}$ such that for $q = m + 1$, $r = \dfrac{1}{m} + 1$, $m > 0$

$$\Delta_q \psi_i = 0 \qquad \text{in} \quad \Omega_i. \tag{1.14}$$

$$\mathbf{v_i} = Rot\psi_i = \Big(\frac{\partial \psi_i}{\partial y}, -\frac{\partial \psi_i}{\partial x}\Big) = -|\nabla \phi_i|^{r-2} \nabla \phi_i \qquad \text{in} \quad \Omega_i. \tag{1.15}$$

Taking into account (1.15), we have for $\tau = (-\nu_y, \nu_x)$ the tangent unit vector to $\partial\Omega$

$$\mathbf{v_i}.\tau = -\frac{\partial \psi_i}{\partial \nu} = -|\nabla \phi_i|^{r-2} \frac{\partial \phi_i}{\partial \tau} \qquad \text{and} \qquad \mathbf{v_i}.\nu = \frac{\partial \psi_i}{\partial \tau} = -|\nabla \phi_i|^{r-2} \frac{\partial \phi_i}{\partial \nu}. \tag{1.16}$$

Using (1.16) and the fact that $|\nabla \psi_i| = |\nabla \phi_i|^{r-1}$, we obtain

$$|\nabla \psi_i|^{q-2} \frac{\partial \psi_i}{\partial \nu} = \frac{\partial \phi_i}{\partial \tau}. \tag{1.17}$$

From (1.12) and (1.16), we deduce that ψ_i are constant on $\Gamma_{1,2}$. We normalize ψ_i by choosing the constant to be zero, which determines ψ_i uniquely. Setting $\psi = \chi(\Omega_1)\psi_1 + \chi(\Omega_2)\psi_2$, we get

$$\psi = 0 \qquad \text{on} \quad \Gamma_{1,2}. \tag{1.18}$$

By (1.11), we have

$$\frac{\partial \phi}{\partial \tau} = \delta_1 \frac{\partial y}{\partial \tau} = \delta_1 \nu_x \qquad \text{on} \quad \Gamma_{0,1}.$$

Using (1.17), we obtain

$$|\nabla \psi|^{q-2} \nabla \psi.\nu = \delta_1 \nu_x \qquad \text{on} \quad \Gamma_{0,1}. \tag{1.19}$$

Now from the first formula in (1.13), we obtain

$$\frac{\partial \phi_1}{\partial \tau} - \frac{\partial \phi_2}{\partial \tau} = (\delta_1 - \delta_2) \frac{\partial y}{\partial \tau} = (\delta_1 - \delta_2)\nu_x \qquad \text{on} \quad \Gamma_{1,2}$$

which can be written by (1.17)

$$|\nabla \psi_1|^{q-2} \nabla \psi_1.\nu - |\nabla \psi_2|^{q-2} \nabla \psi_2.\nu = (\delta_1 - \delta_2)\nu_x \qquad \text{on} \quad \Gamma_{1,2}. \tag{1.20}$$

Using (1.4) and (1.16), we deduce that

$$\frac{\partial \psi_2}{\partial \tau} = 0 \qquad \text{on} \quad \widehat{AB}\ .$$

This means that $\psi_2 = c_2$ is constant along $\widehat{AB}$. Similarly we get $\psi_1 = c_1$ is constant along $\Gamma_{0,1}$.

From (1.5) and (1.9), we know that ϕ is constant and therefore $\dfrac{\partial \phi}{\partial \tau} = 0$ along $\widehat{AA_1} \cup \widehat{BB_2} \cup \widehat{O_2B_1}$. It follows from (1.17) that

$$|\nabla\psi|^{q-2}\nabla\psi.\nu = 0 \qquad \text{on} \qquad \widehat{AA_1} \cup \widehat{BB_2} \cup \widehat{O_2B_1}\ . \tag{1.21}$$

Differentiating the first formula in (1.6), we obtain

$$\frac{\partial \phi}{\partial \tau} = \delta_1 \frac{\partial y}{\partial \tau} = \delta_1 \nu_x \qquad \text{on} \qquad \widehat{B_1O_1}\ .$$

Using (1.16)-(1.17) and (1.6), we deduce that

$$\frac{\partial \psi}{\partial \tau} \geq 0 \qquad \text{and} \qquad |\nabla\psi|^{q-2}\nabla\psi.\nu = \delta_1\nu_x \qquad \text{on} \qquad \widehat{B_1O_1}\ . \tag{1.22}$$

Similarly we obtain

$$\frac{\partial \psi}{\partial \tau} \geq 0 \qquad \text{and} \qquad |\nabla\psi|^{q-2}\nabla\psi.\nu = (\delta_2 - \delta_1)\nu_x \qquad \text{on} \qquad \widehat{B_2O_2}\ . \tag{1.23}$$

Finally, we claim that $c_1 > 0$. Indeed assume that $c_1 \leq 0$ and let $m = \min\limits_{\Omega_1} \psi$.
First due to (1.14), (1.22) and the maximum principle(see [30]), we have $\psi > m$ in Ω_1.
Next due to (1.14), (1.21) and the maximum principle(see [30]), ψ cannot achieve its minimum on $\widehat{A_2A_1} \cup \widehat{O_2B_1}$. Moreover since $\psi = 0$ on $\widehat{A_2O_2}$, $\psi \leq 0$ on $\widehat{A_1O_1}$ and $\dfrac{\partial \psi}{\partial \tau} \geq 0$ on $\widehat{B_1O_1}$, ψ achieves necessarily its minimum at B_1. But this leads by the maximum principle to $\dfrac{\partial \psi}{\partial \nu}(B_1) < 0$, which is in contradiction with (1.21)-(1.22).
Arguing as above, one can verify that $c_2 < 0$. Thus there exists $Q_1, Q_2 > 0$ such that

$$\psi = -Q_2 \qquad \text{on} \qquad \widehat{AB} \qquad \text{and} \qquad \psi = Q_1 \qquad \text{on} \qquad \Gamma_{0,1}.$$

Hence we obtain the following strong formulation

$$\begin{cases}
\Delta_q \psi = 0 & \text{in} \quad \Omega_i,\ i = 1,2 \\
\psi = -Q_2 & \text{on} \quad \widehat{AB} \\
|\nabla\psi|^{q-2}\nabla\psi.\nu = 0 & \text{on} \quad \widehat{AA_1} \cup \widehat{BB_2} \cup \widehat{O_2B_1} \\
|\nabla\psi|^{q-2}\nabla\psi.\nu = \delta_1\nu_x \quad \text{and} \quad \dfrac{\partial \psi}{\partial \tau} \geq 0 & \text{on} \quad \widehat{B_1O_1} \\
|\nabla\psi|^{q-2}\nabla\psi.\nu = (\delta_2 - \delta_1)\nu_x \quad \text{and} \quad \dfrac{\partial \psi}{\partial \tau} \geq 0 & \text{on} \quad \widehat{B_2O_2} \\
|\nabla\psi|^{q-2}\nabla\psi.\nu = \delta_1\nu_x \quad \text{and} \quad \psi = Q_1 & \text{on} \quad \Gamma_{0,1} \\
|\nabla\psi_1|^{q-2}\nabla\psi_1.\nu - |\nabla\psi_2|^{q-2}\nabla\psi_2.\nu = (\delta_1 - \delta_2)\nu_x & \\
\text{and} \quad \psi = 0 & \text{on} \quad \Gamma_{1,2}.
\end{cases}$$

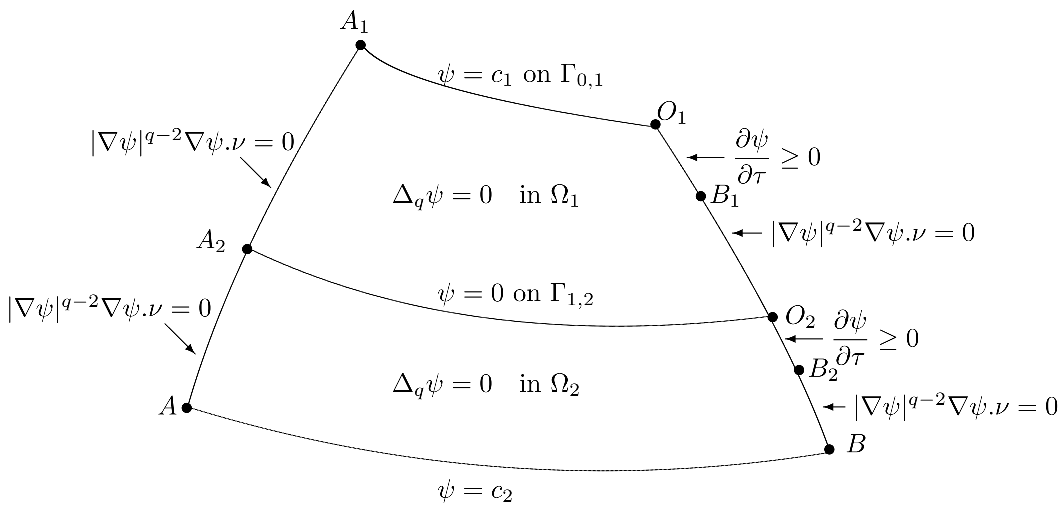

Figure 2

Using the strong formulation and arguing as in [6], we obtain the following weak formulation

$$
(P)\begin{cases}
\text{Find } (\psi,\gamma,\widetilde{\gamma}) \in W^{1,q}(\Omega)\times L^\infty(\Omega)\times L^\infty(\widehat{AT}\cup\widehat{BB_2}) \text{ such that } :\\
(i) \quad \displaystyle\int_\Omega \big(|\nabla\psi|^{q-2}\nabla\psi-\gamma e_x\big).\nabla\zeta + \int_{\widehat{AT}\cup\widehat{BB_2}} \widetilde{\gamma}\zeta\nu_x \;=\; \delta\int_{\widehat{B_2B_1}}\zeta\nu_x + \delta_2\int_{\widehat{B_1T}}\zeta\nu_x\\
\qquad\qquad \forall\zeta\in W^{1,q}(\Omega),\quad \zeta=0 \qquad \text{on} \quad \widehat{AB},\\
(ii) \quad \gamma\in H(\psi) \qquad \text{a.e. in} \quad \Omega, \qquad \widetilde{\gamma}\in H(\psi) \qquad \text{a.e. in} \quad \widehat{AT}\cup\widehat{BB_2},\\
(iii) \quad \psi=-Q_2 \qquad \text{on} \quad \widehat{AB},
\end{cases}
$$

where H is the monotone graph

$$H(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ -\delta & \text{if} \quad 0 < t < Q_1 \\ -\delta_2 & \text{if} \quad t > Q_1 \\ [-\delta, 0] & \text{if} \quad t = 0 \\ [-\delta_2, -\delta] & \text{if} \quad t = Q_1. \end{cases}$$

In the next section, we prove the existence of a solution $(\psi, \gamma, \widetilde{\gamma})$ of the problem (P). In section 3, we prove that ψ is locally Lipschitz continuous above the lower free boundary. In section 4, we prove that the upper free boundary is represented by a curve of a continuous function $\Phi(y)$. In section 5, we give some properties of the set $\psi = Q_1$. In section 6, we specialize in the rectangular case, and show the existence of a monotone solution with respect to y, and then prove that the lower free boundary is represented by a curve of a continuous function $f(x)$. Finally in section 7, we prove the uniqueness of a monotone solution with respect to x and y, when the dam is rectangular and the Darcy law is linear.

2 Existence of a solution

The first step in the existence proof of a solution consists on approximating the problem (P) by a family of problems (P_ϵ). Indeed for $\epsilon > 0$ small enough, we consider the following approximated problem:

$$(P_\epsilon) \begin{cases} \text{Find } \psi_\epsilon \in W^{1,q}(\Omega) \text{ such that :} \\ (i) \quad \displaystyle\int_\Omega \big(|\nabla \psi_\epsilon|^{q-2} \nabla \psi_\epsilon - H_\epsilon(\psi_\epsilon) e_x\big) \nabla \zeta + \int_\Omega \epsilon |\psi_\epsilon|^{q-2} \psi_\epsilon \zeta \\ \qquad + \displaystyle\int_{\widehat{AT} \cup \widehat{BB_2}} H_\epsilon(\psi_\epsilon) \zeta \nu_x = \delta \int_{\widehat{B_2 B_1}} \zeta \nu_x + \delta_2 \int_{\widehat{B_1 T}} \zeta \nu_x \\ \qquad\qquad \forall \zeta \in W^{1,q}(\Omega), \quad \zeta = 0 \quad \text{on} \quad \widehat{AB} \\ (ii) \quad \psi_\epsilon = -Q_2 \quad \text{on} \quad \widehat{AB} \end{cases}$$

where

$$H_\epsilon(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ -\dfrac{\delta}{\epsilon} t & \text{if} \quad 0 \le t \le \epsilon \\ -\delta & \text{if} \quad \epsilon < t < Q_1 \\ -\dfrac{\delta_1}{\epsilon}(t - Q_1) - \delta & \text{if} \quad Q_1 \le t \le Q_1 + \epsilon \\ -\delta_2 & \text{if} \quad t \ge Q_1 + \epsilon. \end{cases} \tag{2.1}$$

Then we have

Theorem 2.1. *There exists a solution* ψ_ϵ *of* (P_ϵ).

Proof. Let $V = \{v \in W^{1,q}(\Omega)/\ v = 0 \ \text{ on } \ \widehat{AB}\}$ and $K = \{v \in W^{1,q}(\Omega)/\ v = -Q_2 \ \text{ on } \ \widehat{AB}\}$. Consider the operator A defined by : $u \in K \longmapsto A(u) \in (W^{1,q}(\Omega))'$ with

$$A(u) : W^{1,q}(\Omega) \longrightarrow \mathbb{R}, \qquad \zeta \longmapsto < A(u), \zeta > = \int_\Omega |\nabla u|^{q-2}\nabla u.\nabla\zeta + \epsilon|u|^{q-2}u\zeta,$$

and the map $f_v : W^{1,q}(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\zeta \longmapsto < f_v, \zeta > = \int_\Omega H_\epsilon(v)\zeta_x - \int_{\widehat{AT}\cup\widehat{BB_2}} H_\epsilon(v)\zeta\nu_x + \delta\int_{\widehat{B_2B_1}} \zeta\nu_x + \delta_2\int_{\widehat{B_1T}} \zeta\nu_x.$$

One can check without difficulty that A is continuous, coercive, monotone and that $f_v \in (W^{1,q}(\Omega))'$.
Then for each $v \in W^{1,q}(\Omega)$, there exists a unique solution ψ_ϵ of the variational problem

$$\psi_\epsilon \in K, \qquad < A(\psi_\epsilon), \zeta > = < f_v, \zeta > \qquad \forall\zeta \in V. \tag{2.2}$$

This defines a map $F_\epsilon : W^{1,q}(\Omega) \longrightarrow K$, $v \longmapsto \psi_\epsilon$. Moreover we have

$F_\epsilon(W^{1,q}(\Omega)) \subset \overline{B}(0, R)$, *where* $\overline{B}(0, R)$ *is the closed ball of* $W^{1,q}(\Omega)$ *of center* 0 *and radius* R *independent of* ϵ.

Indeed $\psi_\epsilon + Q_2 \in V$ is a suitable test function for (2.2). So

$$\begin{aligned}\int_\Omega |\nabla\psi_\epsilon|^q + \epsilon|\psi_\epsilon|^q = -\int_\Omega \epsilon Q_2|\psi_\epsilon|^{q-2}\psi_\epsilon + \int_\Omega H_\epsilon(v)\psi_{\epsilon x}\\ -\int_{\widehat{AT}\cup\widehat{BB_2}} H_\epsilon(v)(\psi_\epsilon + Q_2)\nu_x + \delta\int_{\widehat{B_2B_1}}(\psi_\epsilon + Q_2)\nu_x + \delta_2\int_{\widehat{B_1T}}(\psi_\epsilon + Q_2)\nu_x.\end{aligned} \tag{2.3}$$

Note that for $\lambda = (q')^{1/q'}$, we have by Young's inequality

$$\begin{aligned}\Big|\int_\Omega \epsilon Q_2|\psi_\epsilon|^{q-2}\psi_\epsilon\Big| \le \int_\Omega \lambda\epsilon^{1/q'}|\psi_\epsilon|^{q-1}.\frac{Q_2}{\lambda}\epsilon^{1/q} &\le \frac{1}{q'}\int_\Omega \lambda^{q'}\epsilon|\psi_\epsilon|^q + \frac{1}{q}\int_\Omega\Big(\frac{Q_2}{\lambda}\Big)^q\epsilon\\ &= \int_\Omega \epsilon|\psi_\epsilon|^q + \frac{1}{q}\Big(\frac{Q_2}{\lambda}\Big)^q\epsilon|\Omega|.\end{aligned} \tag{2.4}$$

Using the fact that H_ϵ is uniformly bounded, we obtain by Hölder's inequality

$$\Big|\int_\Omega H_\epsilon(v)\psi_{\epsilon x}\Big| \le C\Big(\int_\Omega |\nabla\psi_\epsilon|^q\Big)^{1/q}. \tag{2.5}$$

Using the continuity of the trace operator, and Poincaré's inequality, we obtain

$$\Big|\int_{\widehat{AT}\cup\widehat{BB_2}} H_\epsilon(v)(\psi_\epsilon+Q_2)\nu_x\Big|,\ \Big|\int_{\widehat{B_2B_1}}(\psi_\epsilon+Q_2)\nu_x\Big|,\ \Big|\int_{\widehat{B_1T}}(\psi_\epsilon+Q_2)\nu_x\Big|\le C\Big(\int_\Omega|\nabla\psi_\epsilon|^q\Big)^{1/q} \tag{2.6}$$

where C is some positive constant independent of ϵ. Now for $\epsilon<1$, we obtain from (2.3)-(2.6), for another positive constant independent of ϵ, still denoted by C

$$\int_\Omega|\nabla\psi_\epsilon|^q\le C\Big(\int_\Omega|\nabla\psi_\epsilon|^q\Big)^{1/q}+\frac{1}{q}\Big(\frac{Q_2}{\lambda}\Big)^q|\Omega|$$

from which we deduce that $\nabla\psi_\epsilon$ is uniformly bounded in $L^q(\Omega)$ and therefore we obtain by Poincaré's inequality applied to $\psi_\epsilon+Q_2$ that $|\psi_\epsilon|_{1,q}\le R$, where R is some positive constant independent of ϵ.

Now we claim that

$F_\epsilon:\overline{B}(0,R)\longrightarrow\overline{B}(0,R)$ *is weakly continuous.*

Indeed, let $(v_i)_{i\in I}$ be a generalized sequence in $\mathcal{C}=\overline{B}(0,R)$ which converges weakly to v in $\mathcal{C}$. Set $\psi_\epsilon^i=F_\epsilon(v_i)$ and $\psi_\epsilon=F_\epsilon(v)$. We would like to prove that $(\psi_\epsilon^i)_{i\in I}$ converges weakly to ψ_ϵ. Since $\mathcal{C}$ is compact with respect to the weak topology, it is enough to show that $(\psi_\epsilon^i)_{i\in I}$ has ψ_ϵ as a unique limit point for the weak topology in $\mathcal{C}$. So let ψ be a weak limit point for $(\psi_\epsilon^i)_{i\in I}$ in $\mathcal{C}$.
Using the compact embedding : $W^{1,q}(\Omega)\subset L^q(\Omega)$, we get a subsequence $(\psi_\epsilon^{i_k})_{k\in\mathbb{N}}$ such that $\psi_\epsilon^{i_k}\rightharpoonup\psi$ in $W^{1,q}(\Omega)$ and $\psi_\epsilon^{i_k}\longrightarrow\psi$ in $L^q(\Omega)$.
Choose $\psi_\epsilon^{i_k}-\psi_\epsilon$ as a test function for (2.2) written for $\psi_\epsilon^{i_k}$ and ψ_ϵ. Subtract the equations, so that

$$\begin{aligned}<A(\psi_\epsilon^{i_k})-A(\psi_\epsilon),\psi_\epsilon^{i_k}-\psi_\epsilon> &=\int_\Omega\Big(H_\epsilon(v_{i_k})-H_\epsilon(v)\Big)(\psi_\epsilon^{i_k}-\psi_\epsilon)_x\\ &-\int_{\widehat{AT}\cup\widehat{BB_2}}\Big(H_\epsilon(v_{i_k})-H_\epsilon(v)\Big)(\psi_\epsilon^{i_k}-\psi_\epsilon)\nu_x.\end{aligned}$$

Note that since H_ϵ is bounded and Lipschitz continuous, and since $\psi_\epsilon^{i_k}$, ψ_ϵ belong to $\overline{B}(0,R)$, we have

$$\begin{aligned}\Big|\int_\Omega\big(H_\epsilon(v_{i_k})-H_\epsilon(v)\big)(\psi_\epsilon^{i_k}-\psi_\epsilon)_x\Big| &\le\Big(\int_\Omega|H_\epsilon(v_{i_k})-H_\epsilon(v)|^{q'}\Big)^{1/q'}\Big(\int_\Omega|\nabla(\psi_\epsilon^{i_k}-\psi_\epsilon)|^q\Big)^{1/q}\\ &\le C(\epsilon)|v_{i_k}-v|_{0,q}^{1/q'},\end{aligned}$$

$$\Big|\int_{\widehat{AT}\cup\widehat{BB_2}}\big(H_\epsilon(v_{i_k})-H_\epsilon(v)\big)(\psi_\epsilon^{i_k}-\psi_\epsilon)\nu_x\Big|\le C(\epsilon)|v_{i_k}-v|_{L^q(\widehat{AT}\cup\widehat{BB_2})}.$$

Then we obtain

$$\lim_{k\to\infty} < A(\psi_\epsilon^{i_k}) - A(\psi_\epsilon), \psi_\epsilon^{i_k} - \psi_\epsilon >= 0.$$

Arguing as in [28], we get $\nabla\psi_\epsilon^{i_k} \to \nabla\psi_\epsilon$ in $L^q(\Omega)$ and $\psi_\epsilon^{i_k} \to \psi_\epsilon$ in $L^q(\Omega)$. It follows that $\psi_\epsilon = \psi$ and therefore ψ_ϵ is the unique weak limit point of (ψ_ϵ^i) in $\mathcal{C}$. Thus $\psi_\epsilon^i = F_\epsilon(v_i) \rightharpoonup \psi_\epsilon = F_\epsilon(v)$ weakly in $\mathcal{C}$. Hence the continuity of F_ϵ holds.
At this step, applying the Tychonoff fixed point theorem (see [29]) in $\mathcal{C}$, we obtain that F_ϵ has a fixed point, which is a solution of (P_ϵ). □

The next step of the existence proof is to pass to the limit in (P_ϵ).

Theorem 2.2. *There exists a solution $(\psi, \gamma, \widetilde{\gamma})$ of the problem (P).*

Proof. From the proof of Theorem 2.1, we know that for some positive constant C independent of ϵ, we have $|\psi_\epsilon|_{1,q} \le C$. Moreover $H_\epsilon(\psi_\epsilon)$ is uniformly bounded. Thus, due to the compact embedding $W^{1,q}(\Omega) \hookrightarrow L^q(\Omega)$ and the complete continuity of the trace operator, there exists a subsequence still denoted by ψ_ϵ and functions $\psi \in W^{1,q}(\Omega)$, $\gamma \in L^\infty(\Omega)$, $\tilde{\gamma} \in L^\infty(\overset{\frown}{AT} \cup \overset{\frown}{BB_2})$ such that

$$\begin{array}{llll} & \psi_\epsilon \rightharpoonup \psi & \text{in } W^{1,q}(\Omega) & \\ \psi_\epsilon \to \psi & \text{in } L^q(\Omega), & \psi_\epsilon \to \psi & \text{in } L^q(\partial\Omega) \\ \psi_\epsilon \to \psi & \text{a.e. in } \Omega, & \psi_\epsilon \to \psi & \text{a.e. in } \partial\Omega \\ H_\epsilon(\psi_\epsilon) \rightharpoonup \gamma & \text{in } L^{q'}(\Omega), & H_\epsilon(\psi_\epsilon) \rightharpoonup \widetilde{\gamma} & \text{in } L^{q'}(\overset{\frown}{AT} \cup \overset{\frown}{BB_2}). \end{array}$$

Note that we can write $H_\epsilon(t) = H_\epsilon^1(t) + H_\epsilon^2(t)$, where H_ϵ^1 and H_ϵ^2 are defined by

$$H_\epsilon^1(t) = \begin{cases} 0 & \text{for } t < 0 \\ -\delta t/\epsilon & \text{for } 0 \le t \le \epsilon \\ -\delta & \text{for } t > \epsilon \end{cases} \quad H_\epsilon^2(t) = \begin{cases} 0 & \text{for } t < Q_1 \\ -\delta_1(t - Q_1)/\epsilon & \text{for } Q_1 \le t \le Q_1 + \epsilon \\ -\delta_1 & \text{for } t > Q_1 + \epsilon. \end{cases}$$

Since $(H_\epsilon^1(\psi_\epsilon))$ and $(H_\epsilon^2(\psi_\epsilon))$ are uniformly bounded, we have up to a subsequence

$$H_\epsilon^1(\psi_\epsilon) \rightharpoonup \gamma^1, \quad H_\epsilon^2(\psi_\epsilon) \rightharpoonup \gamma^2 \quad \text{in } L^{q'}(\Omega)$$

with $\gamma = \gamma^1 + \gamma^2$ a.e. in Ω. Since $H_\epsilon^1(\psi_\epsilon) \in K_1 = \{v \in L^{q'}(\Omega),\ -\delta \le v \le 0 \text{ a.e. in } \Omega\}$ which is weakly closed in $L^{q'}(\Omega)$ (for being closed and convex), we have

$$-\delta \le \gamma^1 \le 0 \qquad \text{a.e. in } \Omega.$$

Moreover, we have

$$H_\epsilon^1(\psi_\epsilon) \to 0 \quad \text{a.e. in} \quad [\psi < 0] \qquad \text{and} \quad H_\epsilon^1(\psi_\epsilon) \to -\delta \qquad \text{a.e. in} \quad [\psi > 0].$$

So by the Lebesgue theorem

$$H^1_\epsilon(\psi_\epsilon) \to 0 \quad \text{in} \quad L^{q'}([\psi<0]) \qquad \text{and} \quad H^1_\epsilon(\psi_\epsilon) \to -\delta \qquad \text{in} \quad L^{q'}([\psi>0]).$$

We deduce that

$$\gamma^1 = 0 \qquad \text{a.e. in} \quad [\psi<0] \qquad \text{and} \qquad \gamma^1 = -\delta \qquad \text{a.e. in} \quad [\psi>0].$$

Now, since $-\delta \le \gamma^1 \le 0$ a.e. in Ω, we obtain

$$\gamma^1 \in H^1(\psi) = \begin{cases} 0 & \text{if } \psi<0 \\ [-\delta,0] & \text{if } \psi=0 \\ -\delta & \text{if } \psi>0. \end{cases}$$

In the same way, we prove that

$$\gamma^2 \in H^2(\psi) = \begin{cases} 0 & \text{if } \psi<Q_1 \\ [-\delta_1,0] & \text{if } \psi=Q_1 \\ -\delta_1 & \text{if } \psi>Q_1. \end{cases}$$

Thus $\gamma = \gamma^1+\gamma^2 \in H^1(\psi)+H^2(\psi) = H(\psi)$ a.e. in Ω.
Similarly, one can prove that $\widetilde{\gamma} \in H(\psi)$ a.e. in $\widehat{AT} \cup \widehat{BB_2}$. Thus $(P)ii)$ holds.
$(P)iii)$ is obtained as a consequence of $(P_\epsilon)ii)$ and the fact that $\psi_\epsilon \to \psi$ in $L^q(\partial\Omega)$.
It remains to prove $(P)i)$. So we take $\psi_\epsilon - \psi$ as a test function for (P_ϵ) and we obtain

$$\begin{aligned}\int_\Omega |\nabla\psi_\epsilon|^q + \epsilon\int_\Omega |\psi_\epsilon|^q &= \int_\Omega |\nabla\psi_\epsilon|^{q-2}\nabla\psi_\epsilon.\nabla\psi + \int_\Omega H_\epsilon(\psi_\epsilon)\partial_x(\psi_\epsilon-\psi) + \epsilon\int_\Omega |\psi_\epsilon|^{q-2}\psi_\epsilon.\psi \\ &- \int_{\widehat{AT}\cup\widehat{BB_2}} H_\epsilon(\psi_\epsilon)(\psi_\epsilon-\psi)\nu_x + \delta\int_{\widehat{B_2B_1}}(\psi_\epsilon-\psi)\nu_x + \delta_2\int_{\widehat{B_1T}}(\psi_\epsilon-\psi)\nu_x. \quad (2.7)\end{aligned}$$

Since $\psi_\epsilon \to \psi$ in $L^q(\partial\Omega)$, the last three integrals in the righthand side of (2.7) converge to 0.
Since ψ_ϵ is bounded in $L^q(\Omega)$, we have

$$\lim_{\epsilon\to 0}\epsilon\int_\Omega |\psi_\epsilon|^{q-2}\psi_\epsilon\psi = 0.$$

To treat the second integral in the righthand side of (2.7), we write

$$\int_\Omega H_\epsilon(\psi_\epsilon)\partial_x(\psi_\epsilon-\psi) = \int_\Omega \partial_x(E^1_\epsilon(\psi_\epsilon)) + \int_\Omega \partial_x(E^2_\epsilon(\psi_\epsilon)) - \int_\Omega H_\epsilon(\psi_\epsilon)\partial_x\psi \qquad (2.8)$$

where

$$E^1_\epsilon(s) = \int_0^s H^1_\epsilon(t)dt \qquad \text{and} \qquad E^2_\epsilon(s) = \int_0^s H^2_\epsilon(t)dt.$$

One can verify that

$$E^1_\epsilon(s) \longrightarrow E^1(s) = -\delta s^+ \qquad \text{and} \qquad E^2_\epsilon(s) \longrightarrow E^2(s) = -\delta_1(s-Q_1)^+, \qquad \text{as} \qquad \epsilon \longrightarrow 0,$$

We claim that

$$\lim_{\epsilon\to 0}\int_\Omega \partial_x(E^1_\epsilon(\psi_\epsilon)) = \int_\Omega \gamma_1\partial_x\psi. \tag{2.9}$$

Indeed, we first have

$$\begin{aligned} |E^1_\epsilon(\psi_\epsilon) - E^1(\psi)| &\le |E^1_\epsilon(\psi_\epsilon) - E^1_\epsilon(\psi)| + |E^1_\epsilon(\psi) - E^1(\psi)| \\ &\le \delta|\psi_\epsilon - \psi| + |E^1_\epsilon(\psi) - E^1(\psi)|. \end{aligned}$$

So $E^1_\epsilon(\psi_\epsilon) \to E^1(\psi)$ a.e. in Ω. By Lebesgue's theorem, we obtain $E^1_\epsilon(\psi_\epsilon) \to E^1(\psi)$ in $L^q(\Omega)$. Then $\partial_x(E^1_\epsilon(\psi_\epsilon)) \to \partial_x(E^1(\psi))$ in $\mathcal{D}'(\Omega)$. But since $|\partial_x(E^1(\psi_\epsilon))|_{L^q(\Omega)}$ is bounded, we get

$$\partial_x(E^1_\epsilon(\psi_\epsilon)) \rightharpoonup \partial_x(E^1(\psi)) \qquad \text{in } L^q(\Omega).$$

Hence

$$\lim_{\epsilon\to 0}\int_\Omega \partial_x(E^1_\epsilon(\psi_\epsilon)) = \int_\Omega \partial_x(E^1(\psi)) = \int_\Omega (E^1)'(\psi)\partial_x\psi = \int_{[\psi>0]} -\delta\partial_x\psi = \int_\Omega \gamma_1\partial_x\psi.$$

In the same way, we establish

$$\lim_{\epsilon\to 0}\int_\Omega \partial_x(E^2_\epsilon(\psi_\epsilon)) = \int_\Omega \gamma_2\partial_x\psi. \tag{2.10}$$

Using (2.8)-(2.10)we get

$$\lim_{\epsilon\to 0}\int_\Omega H_\epsilon(\psi_\epsilon)\partial_x(\psi_\epsilon - \psi) = \int_\Omega \gamma_1\partial_x\psi + \int_\Omega \gamma_2\partial_x\psi - \int_\Omega \gamma\partial_x\psi = 0.$$

Finally, we obtain by letting $\epsilon \to 0$ in (2.7)

$$\limsup_{\epsilon\to 0}\int_\Omega |\nabla\psi_\epsilon|^q \le \limsup_{\epsilon\to 0}\int_\Omega |\nabla\psi_\epsilon|^{q-2}\nabla\psi_\epsilon\nabla\psi$$

from which we deduce that

$$\limsup_{\epsilon\to 0}\Big(\int_\Omega |\nabla\psi_\epsilon|^q\Big)^{1/q} \le \Big(\int_\Omega |\nabla\psi|^q\Big)^{1/q} \qquad \text{and then} \qquad \nabla\psi_\epsilon \to \nabla\psi \quad \text{in } L^q(\Omega).$$

We obtain $(P)i)$ by letting $\epsilon \to 0$ in $(P_\epsilon)i)$ since we have $|\nabla\psi_\epsilon|^{q-2}\nabla\psi_\epsilon \to |\nabla\psi|^{q-2}\nabla\psi$ in $L^{q'}(\Omega)$ and $\epsilon|\psi_\epsilon|^{q-2}\psi_\epsilon \to 0$ in $L^{q'}(\Omega)$. □

3 Regularity of the Solution

The main result of this section is the local Lipschitz continuity of ψ above $\Gamma_{1,2}$. This will be used in the next section to prove the continuity of the function representing $\Gamma_{0,1}$. We first have the following general regularity results.

Proposition 3.1. *Let $(\psi, \gamma, \widetilde{\gamma})$ be a solution of (P). Then we have*

$$div\big(|\nabla\psi|^{q-2}\nabla\psi - \gamma e_x\big) = 0 \qquad in \quad \mathcal{D}'(\Omega). \tag{3.1}$$

$$-Q_2 \le \psi \le Q_1 \qquad in \quad \Omega. \tag{3.2}$$

$$\psi \in C^{0,\alpha}(\Omega \cup \overset{\frown}{AB} \setminus \{A, B\}) \qquad for\ some\ \alpha \in (0,1). \tag{3.3}$$

$$\psi \in C^{1,\beta}([\psi < 0] \cup [0 < \psi < Q_1]) \qquad for\ some\ \beta \in (0,1). \tag{3.4}$$

Proof. $i)$ To prove (3.1), it suffices to take $\zeta \in \mathcal{D}(\Omega)$ as a test function in $(P)i)$.
$ii)$ We first take $(\psi + Q_2)^-$ as a test function in $(P)i)$. We obtain since $\gamma = 0$, $\widetilde{\gamma} = 0$ in $[\psi < -Q_2]$ and $\nu_x \ge 0$ on $\overset{\frown}{BT}$

$$\int_\Omega |\nabla(\psi + Q_2)^-|^q = -\delta \int_{\overset{\frown}{B_2B_1}} (\psi + Q_2)^- \nu_x - \delta_2 \int_{\overset{\frown}{B_1T}} (\psi + Q_2)^- \nu_x \le 0.$$

Since $\psi = -Q_2$ on $\overset{\frown}{AB}$, we get $(\psi + Q_2)^- = 0$ in Ω.
Similarly, we take $(\psi - Q_1)^+$ in $(P)i)$ and obtain since $\gamma = -\delta_2$, $\widetilde{\gamma} = -\delta_2$ in $[\psi > Q_1]$

$$\begin{aligned}\int_\Omega |\nabla\psi|^{q-2}\nabla(\psi - Q_1)^+ + \delta_2(\psi - Q_1)^+_x - \delta_2 \int_{\overset{\frown}{AT} \cup \overset{\frown}{BB_2}} (\psi - Q_1)^+ \nu_x \\ = \delta \int_{\overset{\frown}{B_2B_1}} (\psi - Q_1)^+ \nu_x + \delta_2 \int_{\overset{\frown}{B_1T}} (\psi - Q_1)^+ \nu_x\end{aligned}$$

which can be written after integrating by part

$$\int_\Omega |\nabla(\psi - Q_1)^+|^q + (\delta_2 - \delta) \int_{\overset{\frown}{B_2B_1}} (\psi - Q_1)^+ \nu_x = 0.$$

This gives $(\psi - Q_1)^+ = 0$ in Ω since $(\psi - Q_1)^+ = (-Q_2 - Q_1)^+ = 0$ on $\overset{\frown}{AB}$.

$iii)$ Since $\gamma \in L^\infty(\Omega)$, we deduce (see [21]) that $\psi \in C^{0,\alpha}_{loc}(\Omega \cup Int(\overset{\frown}{AB}))$ for some $\alpha \in (0,1)$.
$iv)$ Since γ is constant in $[\psi < 0]$ and in $[0 < \psi < Q_1]$, we obtain from (3.1) that $div\big(|\nabla\psi|^{q-2}\nabla\psi\big) = 0$ in $\mathcal{D}'([\psi < 0] \cup [0 < \psi < Q_1])$. Therefore $\psi \in C^{1,\beta}_{loc}([\psi < 0] \cup [0 < \psi < Q_1])$ for some $\beta \in (0,1)$ (see [26]) □

Now we state the main result of this section.

Theorem 3.1. *Let $(\psi, \gamma, \widetilde{\gamma})$ be a solution of (P). Then we have*

$$\psi \in C^{0,1}_{loc}([\psi > 0]). \tag{3.5}$$

In order to prove Theorem 3.1, we introduce the following notations

$$\Omega_+ = \Omega \cap [\psi > 0], \qquad u = Q_1 - \psi, \qquad \text{and} \qquad \widehat{\gamma} = \gamma + \delta_2.$$

For $\zeta \in W^{1,q}_0(\Omega_+)$, we remark that $\zeta\chi(\Omega_+)$ is a test function for (P). So we have

$$\begin{aligned} &\int_{\Omega_+} \big(|\nabla\psi|^{q-2}\nabla\psi - \gamma e_x\big).\nabla\zeta \ = \ 0 \\ \Longleftrightarrow &\int_{\Omega_+} \big(|\nabla(Q_1-\psi)|^{q-2}\nabla(Q_1-\psi) + (\gamma+\delta_2)e_x\big).\nabla\zeta \ = \ 0. \end{aligned}$$

Moreover, one has $\widehat{\gamma} = \delta_2 + \gamma \in \delta_2 + H(\psi)$. Therefore $(u, \widehat{\gamma})$ satisfies

$$(P^+)\begin{cases} (i) \quad \displaystyle\int_{\Omega_+} \big(|\nabla u|^{q-2}\nabla u + \widehat{\gamma} e_x\big).\nabla\zeta \ = \ 0 \qquad \forall \zeta \in W^{1,q}_0(\Omega_+) \\ (ii) \quad \widehat{\gamma} \in [0, \delta_1] \qquad \text{if} \quad u = 0 \\ (iii) \quad \widehat{\gamma} = \delta_1 \qquad \text{if} \quad 0 < u < Q_1. \end{cases}$$

To prove Theorem 3.1, we need the following lemma

Lemma 3.1. *Let $B_r(X_0)$ be an open ball of center $X_0 = (x_0, y_0)$ and radius r contained in $\Omega_+ \cap [u > 0]$, satisfying $\overline{B_r(X_0)} \subset \Omega$ and $\partial B_r(X_0) \cap [u = 0] \neq \emptyset$. Then there exists a constant $C > 0$ depending only on q and δ_1 such that*

$$u(X_0) \leq Cr. \tag{3.6}$$

Proof. The proof for $q = 2$ can be adapted from [7]. We shall consider here only the case $q \neq 2$. Let $\epsilon > 0$ such that $B_{r+\epsilon}(X_0) \subset \Omega$. Let $D = B_{r+\epsilon}(X_0) \setminus \overline{B_{r/2}(X_0)}$, $m = \inf\limits_{\partial B_{r/2}(X_0)} u$ and v defined by

$$\begin{cases} v(X) = a\rho^{\frac{q-2}{q-1}} + b \qquad \text{where} \qquad \rho = \sqrt{(x-x_0)^2 + (y-y_0)^2}, \\ a = \dfrac{m}{\left(\frac{r}{2}\right)^{\frac{q-2}{q-1}} - (r+\epsilon)^{\frac{q-2}{q-1}}} \qquad \text{and} \qquad b = \dfrac{-m(r+\epsilon)^{\frac{q-2}{q-1}}}{\left(\frac{r}{2}\right)^{\frac{q-2}{q-1}} - (r+\epsilon)^{\frac{q-2}{q-1}}}. \end{cases}$$

One can verify easily that v satisfies

$$\begin{cases} div(|\nabla v|^{q-2}\nabla v) = 0 & \text{in} \quad D \\ v = m & \text{on} \quad \partial B_{r/2}(X_0) \\ v = 0 & \text{on} \quad \partial B_{r+\epsilon}(X_0). \end{cases}$$

Note that $\zeta = (v-u)^+_{|D} \in W^{1,q}_0(D)$ since $v \leq u$ on ∂D. Then $\zeta\chi(D)$ is a test function for (P^+) and we have

$$\int_D (|\nabla u|^{q-2}\nabla u + \widehat{\gamma} e_x)\nabla(v-u)^+ = 0. \tag{3.7}$$

We have also

$$\int_D |\nabla v|^{q-2}\nabla v \nabla(v-u)^+ = 0. \tag{3.8}$$

We get by subtracting (3.7) from (3.8)

$$\int_D \big(|\nabla v|^{q-2}\nabla v - |\nabla u|^{q-2}\nabla u\big)\nabla(v-u)^+ - \int_D \widehat{\gamma}(v-u)^+_x = 0$$

which can be written

$$\begin{aligned} \int_{D\cap[u>0]} &\big(|\nabla v|^{q-2}\nabla v - |\nabla u|^{q-2}\nabla u\big)\nabla(v-u)^+ \\ &= \int_{D\cap[u=0]} \big((\widehat{\gamma}-\delta_1)v_x - |\nabla v|^q\big) \leq \int_{D\cap[u=0]} |\nabla v|(\delta_1 - |\nabla v|^{q-1}). \end{aligned} \tag{3.9}$$

Assume that

$$\int_{D\cap[u>0]} \big(|\nabla v|^{q-2}\nabla v - |\nabla u|^{q-2}\nabla u\big)\nabla(v-u)^+ = 0.$$

In particular, we obtain

$$\int_{B_r(X_0)} \big(|\nabla v|^{q-2}\nabla v - |\nabla u|^{q-2}\nabla u\big)\nabla(v-u)^+ = 0$$

which leads to $\nabla(v-u)^+ = 0$ in $B_r(X_0)$. Since $v \leq u$ on $\partial B_{r/2}(X_0)$, we get $v \leq u$ in $B_r(X_0)$. This constitutes a contradiction with the fact that $v > 0$ in D and $\partial B_r(X_0) \cap [u=0] \neq \emptyset$. From (3.9), we then get

$$\int_{D\cap[u=0]} |\nabla v|(\delta_1 - |\nabla v|^{q-1}) > 0. \tag{3.10}$$

Now we claim that $|\nabla v| < \delta_1^{\frac{1}{q-1}}$ on $\partial B_{r+\epsilon}(X_0)$. Indeed otherwise, we will have $|\nabla v| \geq \delta_1^{\frac{1}{q-1}}$ in D since $|\nabla v| = |a|\dfrac{|q-2|}{q-1}\dfrac{1}{\rho^{\frac{1}{q-1}}}$ is non-increasing with respect to ρ, and we get a contradiction with (3.10). We deduce that $|\nabla v|_{|\partial B_{r+\epsilon}(X_0)} = |a|\dfrac{|q-2|}{q-1}\dfrac{1}{(r+\epsilon)^{\frac{1}{q-1}}} \leq \delta_1^{\frac{1}{q-1}}$, which can be written

$$\frac{|q-2|}{q-1} \cdot \frac{m}{(r+\epsilon)^{\frac{1}{q-1}} \left| (r+\epsilon)^{\frac{q-2}{q-1}} - \left(\frac{r}{2}\right)^{\frac{q-2}{q-1}} \right|} \leq \delta_1^{\frac{1}{q-1}}.$$

Letting $\epsilon \to 0$, we get

$$\frac{|q-2|}{q-1} \frac{m}{r^{\frac{1}{q-1}} \left| r^{\frac{q-2}{q-1}} - \left(\frac{r}{2}\right)^{\frac{q-2}{q-1}} \right|} \leq \delta_1^{\frac{1}{q-1}} \quad \Longleftrightarrow \quad m \leq \frac{q-1}{|q-2|} \delta_1^{\frac{1}{q-1}} \left| 1 - \left(\frac{1}{2}\right)^{\frac{q-2}{q-1}} \right| r = C_1 r.$$

Since $u > 0$ in $B_{r/2}(X_0)$, we have $\Delta_q u = 0$ in $B_{r/2}(X_0)$. Applying Harnack's inequality (see [22] p. 110), we obtain for a positive constant C_2 depending only on q

$$u(X_0) \leq \max_{B_{r/2}(X_0)} u \leq C_2 \min_{B_{r/2}(X_0)} u = C_2 m \leq C_2 C_1 r = Cr.$$

□

Proof of Theorem 3.1. Let X_1, $X_2 \in \Omega_+$. Without loss of generality, one can choose X_1, X_2 such that

$$|X_1 - X_2| < d/2 \quad \text{and} \quad B_{2d}(X_i) \subset \Omega_+ \quad \text{for some } d > 0.$$

Set $R(X_i) = \min(d, dist(X_i, [u=0]))$. Then clearly we have $B_{R(X_i)}(X_i) \subset [u > 0]$.
If $u(X_1) = 0$ or $u(X_2) = 0$, we argue as in [20].
Assume that $u(X_1) > 0$ and $u(X_2) > 0$. Then if $\frac{1}{2}\max(R(X_1), R(X_2)) < |X_1 - X_2|$, we argue as in [20].

Assume that $\frac{1}{2}\max(R(X_1), R(X_2)) \geq |X_1 - X_2| > 0$, and that for example $R(X_1) \geq R(X_2)$. Then $\frac{1}{2}\max(R(X_1), R(X_2)) = \frac{R(X_1)}{2} \geq |X_1 - X_2|$. We distinguish two cases :

i) $R(X_1) < \frac{d}{2}$.

In this case, we have for $X \in B_1(O)$

$$d(X_1 + R(X_1)X, [u=0]) \leq d(X_1, [u=0]) + R(X_1) = 2R(X_1) < d.$$

So $R(X_1 + R(X_1)X) < d$, and therefore $\partial B_{R(X_1+R(X_1)X)}(X_1 + R(X_1)X) \cap \partial[u > 0] \neq \emptyset$. Applying Lemma 3.1, we get

$$u(X_1 + R(X_1)X) \leq CR(X_1 + R(X_1)X) \leq 2CR(X_1).$$

It follows that the function defined by

$$v(X) = \frac{u(X_1 + R(X_1)X)}{R(X_1)}, \qquad X \in B_1(O)$$

is uniformly bounded in $B_1(O)$, i.e. $v(X) \le 2C \ \forall X \in B_1(O)$. Moreover, it satisfies $\Delta_q v = 0$ in $B_1(O)$. Then by applying for example Theorem 1 of [26], we get for some $\alpha \in (0,1)$ depending only on q

$$|v|_{1,\alpha,\overline{B}_{1/2}(O)} \le C \qquad \text{where } C = C(dist(B_{1/2}(O), \partial B_1(O)).$$

In particular, we deduce that $|\nabla v|$ is uniformly bounded in $\overline{B}_{1/2}(O)$. Now, since $(X_2 - X_1)/R(X_1) \in \overline{B}_{1/2}(O)$, we obtain

$$\Big| v\Big(\frac{X_2 - X_1}{R(X_1)}\Big) - v(0)\Big| \le C\Big|\frac{X_2 - X_1}{R(X_1)}\Big|$$

which leads to

$$|u(X_2) - u(X_1)| \le C|X_2 - X_1|.$$

ii) $R(X_1) \ge \dfrac{d}{2}$.
We consider the same function v defined in the previous case. Here we remark that we have

$$|v|_{0,B_1(O)} \le \frac{|u|_{0,\Omega}}{R(X_1)} \le \frac{2}{d}|u|_{0,\Omega}.$$

Therefore $|\nabla v|_{0,\overline{B}_{1/2}(O)} \le C(d)$, and arguing as before, we get

$$|u(X_2) - u(X_1)| \le C(d)|X_2 - X_1|.$$

□

Remark 3.1. *When $q = 2$, it is showen in [6] that $\psi \in C^{0,1}_{loc}(\Omega)$. The proof of this result near $[\psi = 0]$ relies on the monotonicity formula proved in [5].*

4 Study of the Upper Free Boundary $\Gamma_{0,1}$

The main result of this section is the proof that $\Gamma_{0,1}$ is represented by the graph of a continuous function $\Phi(y)$. First, we prove a monotonicity result for γ in Ω_+ which allows us to define the function $\Phi(y)$. Next, making use of the local Lipschitz continuity of ψ in Ω_+, we prove that Φ is continuous.
We assume that the arc $\widehat{BT}$ (resp. $\widehat{AT}$) can be represented in the form

$$x = \sigma_+(y) \quad (\text{resp. } x = \sigma_-(y)) \qquad \text{for} \quad y_B < y < y_T.$$

Then we have

Theorem 4.1. *Let $(\psi,\gamma,\widehat{\gamma})$ be a solution of (P). The upper free boundary $\Gamma_{0,1} = (\partial[\psi < Q_1]) \cap \Omega_+$ is a y-graph, i.e. there exists a function Φ, $\sigma_-(y) \le \Phi(y) \le \sigma_+(y)$, such that for each $(x,y) \in \Omega_+$*

$$\psi(x,y) = Q_1 \qquad \Longleftrightarrow \qquad \Phi(y) \le x < \sigma_+(y).$$

Moreover Φ is lower semicontinuous on $\pi_y(\Omega_+)$.

The proof of Theorem 4.1 is a consequence of Lemmas 4.1 and 4.2.

Lemma 4.1. *Using the notations of the previous section, we have*

$$\Delta_q u \ge 0, \qquad \widehat{\gamma}_x \le 0 \qquad \text{in} \quad \mathcal{D}'(\Omega_+). \tag{4.1}$$

Proof. Let $\zeta \in \mathcal{D}(\Omega_+)$, $\zeta \ge 0$, and $\epsilon > 0$. Taking $\xi = \min\big(\dfrac{u}{\epsilon}, \zeta\big)$ as a test function in $(P^+)i)$, we obtain since $\widehat{\gamma} = \delta_1$ in $[u > 0]$

$$\int_{\Omega_+\cap[u\ge\epsilon\zeta]} |\nabla u|^{q-2}\nabla u.\nabla\zeta = -\frac{1}{\epsilon}\int_{\Omega_+\cap[u<\epsilon\zeta]} |\nabla u|^q - \int_{\Omega_+} \widehat{\gamma}\xi_x \le -\int_{\Omega_+} \delta_1\xi_x = 0.$$

Letting $\epsilon \to 0$, we get $\displaystyle\int_{\Omega_+} |\nabla u|^{q-2}\nabla u.\nabla\zeta \le 0$ which means that $\Delta_q u \ge 0$. From $(P^+)i)$, we get $\widehat{\gamma}_x = -\Delta_q u \le 0$ in $\mathcal{D}'(\Omega_+)$. □

Lemma 4.2. *Let $(x_0, y_0) \in \Omega_+$. We have*

$$\psi(x_0,y_0) < Q_1 \qquad \Longrightarrow \qquad \psi(x,y_0) < Q_1 \quad \forall x \in (\sigma_-(y_0), x_0). \tag{4.2}$$

Proof. Let $(x_0,y_0) \in \Omega_+$ such that $\psi(x_0,y_0) < Q_1$. By continuity of ψ, there exists $\epsilon > 0$ such that $0 < \psi < Q_1$ in $B_\epsilon(x_0,y_0)$. So $u > 0$ in $B_\epsilon(x_0,y_0)$ and $\widehat{\gamma} = \delta_1$ a.e. in $B_\epsilon(x_0,y_0)$.
Let $x_m = \inf\{\, x \in (\sigma_-(y_0), x_0) \,/\, \psi(x', y_0) < Q_1 \ \forall x' \in (x, x_0] \,\}$.
If $x_m = \sigma_-(y_0)$, then $\psi(x,y_0) < Q_1$ for all $x \in (\sigma_-(y_0), x_0]$.
If $\sigma_-(y_0) < x_m < x_0$, then $\psi(x_m,y_0) = Q_1$. By continuity, there exists $\eta > 0$ such that $B_\eta(x_m,y_0) \subset \Omega_+$. Now let $\rho > 0$ small enough such that $B_\rho(x_m+\eta/2, y_0) \subset B_\eta(x_m,y_0) \cap [\psi < Q_1]$. Then $\widehat{\gamma} = \delta_1$ a.e. in $B_\rho(x_m+\eta/2,y_0)$. But since $\widehat{\gamma}_x \le 0$ in $\mathcal{D}'(B_\eta(x_m,y_0))$ and $0 \le \widehat{\gamma} \le \delta_1$ in $B_\eta(x_m,y_0)$, we deduce that

$$\widehat{\gamma} = \delta_1 \quad \text{a.e. in} \quad C = \big((-\infty, x_m+\eta/2) \times (y_0-\rho, y_0+\rho) \cup B_\rho(x_m+\eta/2,y_0)\big) \cap B_\eta(x_m,y_0).$$

As a consequence, we obtain $\Delta_q u = 0$ in C which leads by the maximum principle, since $u > 0$ in $B_\rho(x_m+\eta/2,y_0)$, to $u > 0$ in C. Therefore $\psi < Q_1$ in C which is in contradiction with $\psi(x_m,y_0) = Q_1$. □

Proof of Theorem 4.1. Taking into account (4.2), we define for each $y \in \pi_y(\Omega_+)$

$$\Phi(y) = \begin{cases} \sigma_-(y) & \text{if} \quad \{x \,/\, \psi(x,y) < Q_1\} = \emptyset \\ \sup\{x/\,(x,y) \in \Omega_+ \text{ and } \psi(x,y) < Q_1\} & \text{elsewhere.} \end{cases} \tag{4.3}$$

Let $(x_0, y_0) \in \Omega_+$ such that $\psi(x_0, y_0) = Q_1$. By Lemma 4.2, we have $\psi(x, y_0) = Q_1$ for all $x \in [x_0, \sigma_+(y_0))$. Consequently we obtain $\Phi(y_0) \le x_0 < \sigma_+(y_0)$.

Conversely, let $(x_0, y_0) \in \Omega_+$ such that $\Phi(y_0) \le x_0 < \sigma_+(y_0)$. Assume that $\psi(x_0, y_0) < Q_1$. By continuity and Lemma 4.2, we have $\psi(x, y_0) < Q_1$ for all $x \in (\sigma_-(y_0), x_1)$ for some $x_1 \in (x_0, \sigma_+(y_0))$. Consequently, we obtain $\Phi(y_0) \ge x_1$, which contradicts the assumption. □

Lemma 4.3. *Let $\Omega_1 = \Omega_+ \cap [y > y_{B_1}]$. We have*

$$\int_{\Omega_1} \big(|\nabla u|^{q-2}\nabla u + \widehat{\gamma} e_x\big).\nabla\zeta \; = \; 0 \qquad \forall \zeta \in W^{1,q}(\Omega_1), \quad \zeta = 0 \text{ on } \partial\Omega_1 \setminus \widehat{B_1T}. \tag{4.4}$$

Proof. Let ζ be as in the lemma. Then $\chi(\Omega_1)\zeta$ is a test function for (P) and we have

$$\int_{\Omega_1} \big(|\nabla\psi|^{q-2}\nabla\psi - \gamma e_x\big).\nabla\zeta = \delta_2 \int_{\widehat{B_1T}} \zeta\nu_x = \delta_2 \int_{\Omega_1} \zeta_x.$$

This leads to

$$\int_{\Omega_1} \big(|\nabla u|^{q-2}\nabla u + \widehat{\gamma} e_x\big).\nabla\zeta = 0.$$

□

The main result of this section is the following theorem.

Theorem 4.2. *Φ is continuous at each $y \in \pi_y(\Omega_1)$ such that $(\Phi(y), y) \in \Omega_1$.*

Remark 4.1. *In section 6.2, we shall prove, for rectangular dams, that we have $\psi < Q_1$ in $\Omega \cap [y < h_1]$. This shows that $\Gamma_{0,1}$ is located in Ω_1.*

To prove Theorem 4.2, we need several lemmas

Lemma 4.4. *Let $(x_0, y_1), (x_0, y_2) \in \Omega_1$ such that $y_1 < y_2$, $u(x_0, y_i) = 0$ for $i = 1, 2$, and let $Z = \big((x_0, +\infty) \times (y_1, y_2)\big) \cap \Omega \subset \Omega_1$ (see Figure 3). Then we have*

$$\int_Z \big(|\nabla u|^{q-2}\nabla u + \widehat{\gamma} e_x\big).\nabla\zeta \; \le \; 0$$
$$\forall \zeta \in W^{1,q}(Z), \quad \zeta \ge 0, \quad \zeta(x_0, y) = 0 \quad a.e. \; y \in (y_1, y_2). \tag{4.5}$$

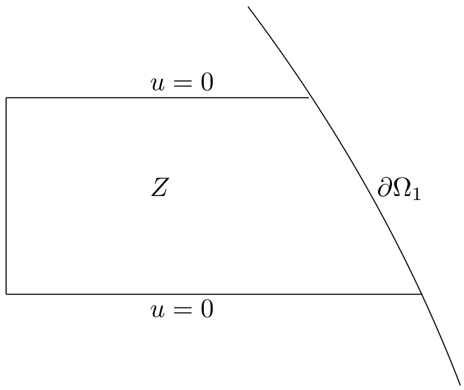

Figure 3

Proof. Let ζ as in Lemma 4.4, $\epsilon > 0$ and $\alpha_\epsilon(y) = \min\big(1, \dfrac{(y-y_1)^+}{\epsilon}\big).\min\big(1, \dfrac{(y_2-y)^+}{\epsilon}\big)$. Since for $\eta > 0$, $\alpha_\epsilon \zeta \chi(Z)$ and $\min\Big(\dfrac{u}{\eta}, (1-\alpha_\epsilon)\zeta\Big)\chi(Z)$ are test functions for (4.4), we have respectively

$$\int_Z \big(|\nabla u|^{q-2}\nabla u + \widehat{\gamma} e_x\big).\nabla(\alpha_\epsilon \zeta) = 0 \tag{4.6}$$

$$\int_Z \big(|\nabla u|^{q-2}\nabla u + \widehat{\gamma} e_x\big).\nabla\Big(\min\Big(\frac{u}{\eta}, (1-\alpha_\epsilon)\zeta\Big)\Big) = 0. \tag{4.7}$$

Taking into account the fact that $\widehat{\gamma} = \delta_1$ in $[u > 0]$, (4.7) becomes

$$\begin{aligned}\int_{Z\cap[\eta(1-\alpha_\epsilon)\zeta \le u]} |\nabla u|^{q-2}\nabla u.\nabla((1-\alpha_\epsilon)\zeta) &\le -\frac{1}{\eta}\int_{Z\cap[\eta(1-\alpha_\epsilon)\zeta > u]} |\nabla u|^q \\ -\delta_1 \int_Z \Big(\min\Big(\frac{u}{\eta}, (1-\alpha_\epsilon)\zeta\Big)\Big)_x &\le -\delta_1 \int_{\partial Z\cap \widehat{B_1 T}} \min\big(\frac{u}{\eta}, (1-\alpha_\epsilon)\zeta\big)\nu_x \le 0.\end{aligned} \tag{4.8}$$

Letting $\eta \to 0$ in (4.8) and adding the result to (4.6), we obtain

$$\int_Z \big(|\nabla u|^{q-2}\nabla u + \alpha_\epsilon \widehat{\gamma} e_x\big).\nabla\zeta \le 0.$$

Finally, we let $\epsilon \to 0$ and obtain (4.5). $\square$

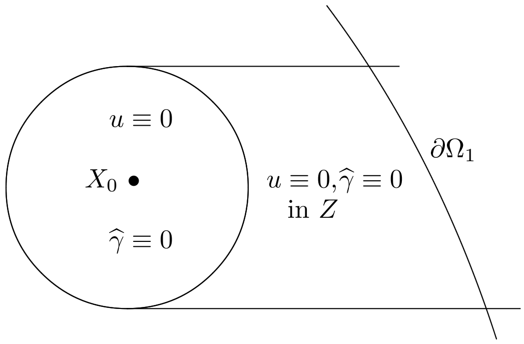

Figure 4

Taking $\zeta = x - x_0$ in (4.5), we obtain

Corollary 4.1. *Under the same assumptions and with the notations of Lemma 4.4, we have*

$$\int_Z |\nabla u|^{q-2} u_x + \widehat{\gamma} \le 0. \tag{4.9}$$

Lemma 4.5. *Assume that $u \equiv 0$ in a ball $B_r(X_0) \subset \Omega_1$, with $r > 0$ and $X_0 = (x_0, y_0)$. Then we have*

$$\widehat{\gamma} \equiv 0 \quad a.e.\ in \quad Z \cup B_r(X_0), \quad where\ (see\ Figure\ 4) \quad Z = \big((x_0, +\infty) \times (y_0 - r, y_0 + r)\big) \cap \Omega \subset \Omega_1.$$

Proof. By Theorem 4.1, we have $u = 0$ in $Z \cup B_r(X_0)$. Applying Corollary 4.1 for each domain $Z' \subset Z \cup B_r(X_0)$ of the form $Z' = \big((x_1, +\infty) \times (y_1, y_2)\big) \cap \Omega$, we obtain $\int_{Z'} \widehat{\gamma} \le 0$. Since $\widehat{\gamma}$ is nonnegative, this leads to $\widehat{\gamma} = 0$ in Z'. Therefore $\widehat{\gamma} \equiv 0$ in $Z \cup B_r(X_0)$. □

Lemma 4.6. *Let $X_0 = (x_0, y_0) \in \Omega_1$ and $r > 0$ such that $B_r(X_0) \subset \Omega_1$. Then we cannot have the following situations in $B_r(X_0)$ (see Figure 5)*

$$\begin{array}{llll} (i) & u = 0 \ \ for\ y = y_0 & and & u > 0 \ \ for\ y \neq y_0 \\ (ii) & u = 0 \ \ for\ y \ge y_0 & and & u > 0 \ \ for\ y < y_0, \\ (iii) & u > 0 \ \ for\ y > y_0 & and & u = 0 \ \ for\ y \le y_0. \end{array}$$

Proof. $i)$ In this case, one has $\widehat{\gamma} = \delta_1$ a.e. in $B_r(X_0)$ and therefore $\Delta_q u = -\widehat{\gamma}_x = 0$ in $\mathcal{D}'(B_r(X_0))$. This is in contradiction with the maximum principle.

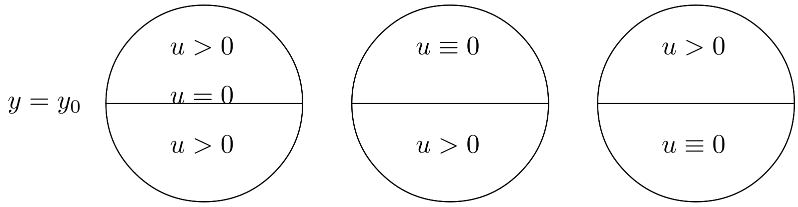

Figure 5

$ii)$ In this case, one has $\widehat{\gamma} = \delta_1$ a.e. in $B_r^-(X_0) = B_r(X_0) \cap [y < y_0]$. Moreover by Lemma 4.5, we have $\widehat{\gamma} = 0$ a.e. in $B_r^+(X_0) = B_r(X_0) \cap [y > y_0]$. It follows that $\widehat{\gamma}_x = 0$ in $\mathcal{D}'(B_r(X_0))$. Hence $\Delta_q u = 0$ in $\mathcal{D}'(B_r(X_0))$, and we obtain again a contradiction.

$iii)$ This case is similar to $ii)$. $\square$

Lemma 4.7. *Let* $(\underline{x}, y_1), (\underline{x}, y_2) \in \Omega_1$ *such that* $y_1 < y_2$ *and* $u(\underline{x}, y_i) = 0$ *for* $i = 1, 2$. *For* $\epsilon > 0$ *small enough, let* $v(x, y) = \delta_1^{\frac{1}{q-1}}(\underline{x} + \epsilon - x)^+$. *Assume that* $[\underline{x} - \epsilon, \underline{x} + \epsilon] \times [y_1, y_2] \subset \Omega_+$, $u(\underline{x}, y) \leq v(\underline{x}, y) \quad \forall y \in (y_1, y_2)$, *and that* $Z = \big((\underline{x}, +\infty) \times (y_1, y_2)\big) \cap \Omega \subset \Omega_1$ *(see Figure 6). Then we have for* $Z_\mu = Z \cap [v > 0] \cap [0 < u - v < \mu]$

$$\lim_{\mu \to 0} \frac{1}{\mu} \int_{Z_\mu} \big(|\nabla u|^{q-2}\nabla u - |\nabla v|^{q-2}\nabla v\big).\nabla(u - v)^+ \ = \ 0. \tag{4.10}$$

Proof. For $\mu, \eta > 0$, we consider $F_\mu(s) = \min\Big(\dfrac{s^+}{\mu}, 1\Big)$, $d_\eta(x) = F_\eta(x - \bar{x})$, $\bar{x} = \underline{x} + \epsilon$. Then $\zeta = F_\mu(u - v) + d_\eta(1 - F_\mu(u))$ is a nonnegative function vanishing on $[x = \underline{x}]$. So by Lemma 4.4, we have

$$\int_Z \big(|\nabla u|^{q-2}\nabla u + \widehat{\gamma} e_x\big).\nabla(F_\mu(u - v)) \leq -\int_Z \big(|\nabla u|^{q-2}\nabla u + \widehat{\gamma} e_x\big).\nabla(d_\eta(1 - F_\mu(u))). \tag{4.11}$$

Moreover, we have

$$\int_Z \big(|\nabla v|^{q-2}\nabla v + \delta_1 \chi([v > 0]) e_x\big).\nabla(F_\mu(u - v)) = 0. \tag{4.12}$$

Subtracting (4.12) from (4.11), we get

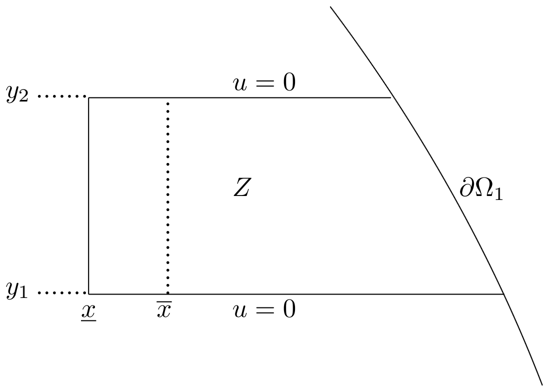

Figure 6

$$\int_Z \big(|\nabla u|^{q-2}\nabla u - |\nabla v|^{q-2}\nabla v\big).\nabla(F_\mu(u-v)) \leq \int_Z (\delta_1\chi([v>0]) - \widehat{\gamma})e_x.\nabla(F_\mu(u-v)) \\ -\int_{Z\cap[v=0]} \big(|\nabla u|^{q-2}\nabla u + \widehat{\gamma}e_x\big).\nabla(d_\eta(1-F_\mu(u)))$$

which can be written

$$\int_{Z\cap[v>0]} F'_\mu(u-v)\big(|\nabla u|^{q-2}\nabla u - |\nabla v|^{q-2}\nabla v\big).\nabla(u-v) \\ \leq \int_{Z\cap[v=0]} \big(|\nabla u|^{q-2}\nabla u + \widehat{\gamma}e_x\big).\nabla((1-d_\eta)(1-F_\mu(u))) = I_1^\mu. \tag{4.13}$$

Note that we can write

$$I_1^\mu = -\int_{Z\cap[v=0]} (1-d_\eta)\big(|\nabla u|^{q-2}\nabla u + \widehat{\gamma}e_x\big).\nabla(F_\mu(u)) \\ -\int_{Z\cap[v=0]} (1-F_\mu(u))\big(|\nabla u|^{q-2}\nabla u + \widehat{\gamma}e_x\big).\nabla d_\eta = I_2^\mu + I_3^\mu. \tag{4.14}$$

Since for $x > \overline{x}$, $d_\eta \to 1$ when $\eta \to 0$, we have

$$\lim_{\eta\to 0} I_2^\mu = 0. \tag{4.15}$$

Moreover

$$I_3^\mu = -\int_{Z\cap[u=v=0]} \widehat{\gamma} e_x.\nabla d_\eta - \int_{Z\cap[u>v=0]} (1-F_\mu(u))\big(|\nabla u|^{q-2}\nabla u + \widehat{\gamma} e_x\big).\nabla d_\eta = I_4^\mu + I_5^\mu. \tag{4.16}$$

$$I_4^\mu = -\int_{Z\cap[u=v=0]} \widehat{\gamma}\partial_x d_\eta = -\frac{1}{\eta}\int_{Z\cap[u=v=0]\cap[\overline{x}<x<\overline{x}+\eta]} \widehat{\gamma} \leq 0. \tag{4.17}$$

Since $u \in C^{0,1}_{loc}(\Omega_+)$, one has for some positive constant C

$$\begin{aligned}
|I_5^\mu| \quad &\leq \frac{1}{\eta}(C^{q-1}+\delta_1)\int_{Z\cap[u>v=0]\cap[\overline{x}<x<\overline{x}+\eta]} (1-F_\mu(u)) \\
&= \frac{C^{q-1}+\delta_1}{\eta}\int_J\int_{\overline{x}}^{\min(\phi(y),\overline{x}+\eta)} (1-F_\mu(u)) \\
&\leq \frac{C^{q-1}+\delta_1}{\eta}\int_J\Big(\int_{\overline{x}}^{\overline{x}+\eta}(1-F_\mu(u))dx\Big)dy,
\end{aligned} \tag{4.18}$$

where $J = \{y \in (y_1,y_2)\ /\ \overline{x} < \phi(y)\}$. Using the continuity of the function $x \mapsto (1-F_\mu(u))(x,y)$, we have

$$\lim_{\eta\to 0}\frac{1}{\eta}\int_{\overline{x}}^{\overline{x}+\eta}(1-F_\mu(u))(x,y)dx = (1-F_\mu(u))(\overline{x},y).$$

Moreover $f_\eta(y) = \dfrac{1}{\eta}\displaystyle\int_{\overline{x}}^{\overline{x}+\eta}(1-F_\mu(u))(x,y)dx$ satisfies $|f_\eta(y)| \leq 1$ for all $y \in (y_1,y_2)$. Then by the Lebesgue theorem, we obtain

$$\lim_{\eta\to 0}\int_J f_\eta(y)dy = \int_J (1-F_\mu(u))(\overline{x},y)dy. \tag{4.19}$$

Combining (4.18) and (4.19), we get

$$\overline{\lim}_{\eta\to 0}|I_5^\mu| \leq (C^{q-1}+\delta_1)\int_J(1-F_\mu(u))(\overline{x},y)dy. \tag{4.20}$$

Taking into account (4.13)-(4.17) and (4.20), we obtain

$$\begin{aligned}
\frac{1}{\mu}\int_{Z\cap[v>0]\cap[0<u-v<\mu]} &\big(|\nabla u|^{q-2}\nabla u - |\nabla v|^{q-2}\nabla v\big).\nabla(u-v)^+ \\
&\leq (C^{q-1}+\delta_1)\int_J(1-F_\mu(u))(\overline{x},y)dy.
\end{aligned} \tag{4.21}$$

But for $y \in J$, we have $u(\overline{x}, y) > 0$. So $\lim\limits_{\mu \to 0}(1 - F_\mu(u))(\overline{x}, y) = 0$. Letting $\mu \to 0$ in (4.21), we get (4.10). $\square$

Lemma 4.8. *Under the assumptions and notations of Lemma 4.7, we have*

$$div(a(x,y)\nabla w) = 0 \qquad in \quad \mathcal{D}'(Z^*) \tag{4.22}$$

where

$$w = \begin{cases} (u-v)^+ & in \quad Z^+ = [v > 0] \\ 0 & in \quad Z^- = (\underline{x} - \epsilon, \underline{x}] \times (y_1, y_2) \end{cases}$$

$Z^* = Z^- \cup Z^+$, *and $a(x,y)$ is a 2-by-2 strictly elliptic and bounded matrix.*

Proof. Let $\zeta \in \mathcal{D}(Z^*)$. Note that

$$\begin{aligned} &\int_{Z^+} \chi([u > v])\big(|\nabla u|^{q-2}\nabla u - |\nabla v|^{q-2}\nabla v\big).\nabla\zeta \\ &= \lim_{\mu \to 0} \int_{Z^+} F_\mu(u-v)\big(|\nabla u|^{q-2}\nabla u - |\nabla v|^{q-2}\nabla v\big).\nabla\zeta \ = \ \lim_{\mu \to 0} I_\mu, \end{aligned} \tag{4.23}$$

where

$$\begin{aligned} I_\mu &= \int_{Z^+} \big(|\nabla u|^{q-2}\nabla u - |\nabla v|^{q-2}\nabla v\big).\nabla(F_\mu(u-v)\zeta) \\ &-\frac{1}{\mu}\int_{Z^+ \cap [0 < u-v < \mu]} \zeta\big(|\nabla u|^{q-2}\nabla u - |\nabla v|^{q-2}\nabla v\big).\nabla(u-v) = I_\mu^1 - I_\mu^2. \end{aligned} \tag{4.24}$$

By Lemma 4.7, we have

$$\lim_{\mu \to 0} I_\mu^2 = 0 \tag{4.25}$$

since

$$|I_\mu^2| \leq \sup_{Z^+}|\zeta|\frac{1}{\mu}\int_{Z^+ \cap [0 < u-v < \mu]} \big(|\nabla u|^{q-2}\nabla u - |\nabla v|^{q-2}\nabla v\big).\nabla(u-v).$$

Moreover, we have

$$\begin{aligned} I_\mu^1 \ &= \int_{Z^+} |\nabla u|^{q-2}\nabla u.\nabla(F_\mu(u-v)\zeta) - \int_{Z^+} |\nabla v|^{q-2}\nabla v.\nabla(F_\mu(u-v)\zeta) \\ &= -\int_{Z^+} \widehat{\gamma}(F_\mu(u-v)\zeta)_x + \int_{Z^+} \delta_1(F_\mu(u-v)\zeta)_x = 0. \end{aligned} \tag{4.26}$$

It follows from (4.23)-(4.26) that

$$\int_{Z^+} \chi([u>v])\big(|\nabla u|^{q-2}\nabla u - |\nabla v|^{q-2}\nabla v\big).\nabla\zeta = 0$$

which can be written (see [17])

$$\int_{Z^+} a(x,y)\nabla(u-v)^+.\nabla\zeta = 0 \tag{4.27}$$

where

$$a(x,y) = (a_{ij}(x,y))_{1\le i,j\le 2}, \qquad a_{ij}(x,y) = \int_0^1 \frac{\partial A^i}{\partial h_j}(\nabla u_t)dt,$$

A^i are the components of the vector function $A(h) = |h|^{q-2}h$, $h \in I\!R^2$ and $u_t = tu + (1-t)v$.

Now arguing as in [17], we can verify that for a.e. $(x,y) \in Z^+$ and for all $\xi \in I\!R^2$

$$\min(1,q-1).\lambda(x,y).|\xi|^2 \le a(x,y).\xi.\xi \le \max(1,q-1).\lambda(x,y).|\xi|^2, \qquad \lambda(x,y) = \int_0^1 |\nabla u_t|^{q-2}(x,y)dt.$$

Since $|\nabla v| = \delta_1^{\frac{1}{q-1}}$ and $|\nabla u| \le C$ in Z^+, we can verify that

$$0 < c_0 \le \lambda(x,y) \le c_1 \qquad \text{in} \quad Z^+, \quad \text{with } c_0, c_1 \text{ constants.}$$

Next, if we extend $a(x,y)$ by $c_0 I_2$ (I_2 being the 2-by-2 identity matrix) into Z^-, we obtain from (4.27)

$$\int_{Z^*} a(x,y)\nabla w.\nabla\zeta = 0 \qquad \forall\zeta \in \mathcal{D}(Z^*)$$

with $a(x,y)$ strictly elliptic and bounded in Z^*. □

Proof of Theorem 4.2. Let $\epsilon > 0$ and $y_0 \in \pi_y(\Omega_1)$ such that $X_0 = (\Phi(y_0), y_0) \in \Omega_1$. For ϵ small enough, we can suppose that $B_{2\epsilon}(X_0) \subset \Omega_1$. Since $u(X_0) = 0$ and u continuous, there exists $\epsilon' \in (0,\epsilon)$ such that

$$u(x,y) \le \epsilon\delta^{\frac{1}{q-1}} \qquad \forall (x,y) \in B_{\epsilon'}(X_0) \subset \Omega_1. \tag{4.28}$$

By Lemma 4.6, one of the following situations is true

$$\begin{array}{llll} i) \ \ \exists X_1 = (x_1,y_1) \in B_{\epsilon'}(X_0) & \text{such that} & y_1 < y_0 & \text{and} \quad u(X_1) = 0 \\ ii) \ \ \exists X_2 = (x_2,y_2) \in B_{\epsilon'}(X_0) & \text{such that} & y_2 > y_0 & \text{and} \quad u(X_2) = 0. \end{array}$$

Let us assume that $i)$ holds. Set $\underline{x} = \max(\Phi(y_0), x_1)$ and $Z = \big((\underline{x}, +\infty) \times (y_1,y_0)\big) \cap \Omega$ (see Figure 7). Since $\{\underline{x}\} \times (y_1,y_0) \subset B_{\epsilon'}(X_0)$, we have by (4.28)

$$u(\underline{x},y) \le \epsilon\delta_1^{\frac{1}{q-1}} \qquad \forall y \in (y_1,y_0). \tag{4.29}$$

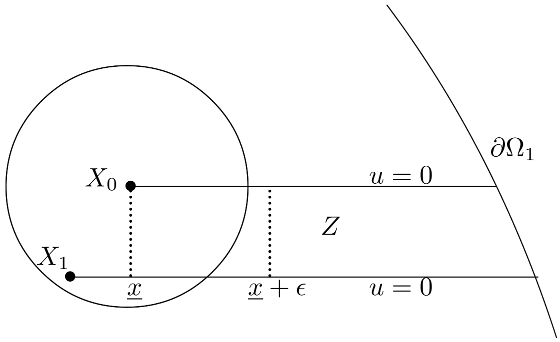

Figure 7

Now it is easy to verify that $u \leq Q_1/2$ in Z (choose $(u - Q_1/2)^+\chi(Z)$ as a test function in (4.4)). This means that $Z \subset [\psi > 0]$. Since we have also $y_1 > y_{B_1}$, we obtain $Z \subset \Omega_1$.
Let $v(x, y) = \delta_1^{\frac{1}{q-1}}(\underline{x} + \epsilon - x)^+$. Then we have from (4.29)

$$u(\underline{x}, y) \ \leq \ v(\underline{x}, y) \qquad \forall y \in (y_1, y_0).$$

Moreover $u(\underline{x}, y_0) = u(\underline{x}, y_1) = 0$. Therefore the assumptions of Lemma 4.7 are fulfilled and we obtain from Lemma 4.8

$$div(a(x, y)\nabla w) = 0 \qquad \text{in} \quad \mathcal{D}'(Z^*), \qquad \text{where} \quad Z^* = (\underline{x} - \epsilon, \underline{x} + \epsilon) \times (y_1, y_0).$$

Since w is nonnegative in Z^*, $w = 0$ in $Z^- = (\underline{x} - \epsilon, \underline{x}] \times (y_1, y_0)$, we obtain from the maximum principle that $w \equiv 0$ in Z^*. This leads to $u \leq v$ in Z^+, and then $u(\underline{x}+\epsilon, y) = 0$ for all $y \in (y_1, y_0)$. By Theorem 4.1, we obtain $u \equiv 0$ in $Z \cap [x \geq \underline{x} + \epsilon]$ and then $u \equiv 0$ in $Z \cap [x \geq x_0 + 2\epsilon]$.

Using Lemma 4.6 again and arguing as before, we deduce that $u \equiv 0$ in $Z' = \big((x_0 + 4\epsilon, +\infty) \times (y_0, y_2)\big) \cap \Omega$. Finally $u(x, y) \equiv 0$ in $\big((x_0 + 4\epsilon, +\infty) \times (y_1, y_2)\big) \cap \Omega$. We deduce that

$$\Phi(y) \leq x_0 + 4\epsilon = \Phi(y_0) + 4\epsilon \qquad \forall y \in (y_1, y_2)$$

Hence Φ is upper semi-continuous at y_0. $\square$

5 Some Properties of the set $[\psi = Q_1]$

In this section, we give some properties of the set $[\psi = Q_1]$. We also show that if the total flux of the two fluids is small enough, then the dam is not entirely wet.

For each point E, we shall denote by (x_E, y_E) the coordinates of E and by l_E the intersection of Ω with the line $[y = y_E]$. E^* will denote the left endpoint of l_E.

Theorem 5.1. *Suppose that $\overset{\frown}{B_1^*T}$ is an $x-$graph and that the interior sphere condition is satisfied at each point of $\overset{\frown}{B_1^*T}$. If $Q_1 + Q_2$ is small enough, then there exists a point $E \in \overset{\frown}{B^*T}$ such that $\psi(E) = Q_1$. We then have $\psi \equiv Q_1$ in $\Omega_E = \Omega \cap [y \geq y_E]$ provided that $l_E \cap \overset{\frown}{AB} = \emptyset$.*

The proof of this Theorem requires three Lemmas.

Lemma 5.1. *Let $E \in \overset{\frown}{AT}$ such that $l_E \cap \overset{\frown}{AB} = \emptyset$ and $y_E \geq y_{B_2}$. If $\psi = Q_1$ on l_E, then $\psi \equiv Q_1$ in Ω_E.*

Proof. The function $\zeta = \min(Q_1 - \psi, Q_1)\chi(\Omega_E)$ is a test function for (P). Given that $\nabla\zeta = 0$ a.e. in $[\zeta = Q_1]$ and $\gamma\zeta_x = -\delta\zeta_x$, we obtain

$$- \int_{\Omega_E \cap [\psi > 0]} |\nabla(Q_1 - \psi)|^q + \int_{\overset{\frown}{AT}} (\delta + \widetilde{\gamma})\zeta\nu_x - \delta_1 \int_{\overset{\frown}{B_1T}} \zeta\nu_x = 0.$$

Since $\delta + \widetilde{\gamma} \geq 0$ a.e. in $\overset{\frown}{AT} \cap [\zeta > 0]$, $\nu_x \leq 0$ on $\overset{\frown}{AT}$ and $\nu_x \geq 0$ on $\overset{\frown}{B_1T}$, we deduce that $Q_1 - \psi$ is constant on each connected component of $\Omega_E \cap [\psi > 0]$. But $\psi = Q_1$ on l_E leads to $\psi = Q_1$ in the connected component of $\Omega_E \cap [\psi > 0]$ that contains l_E on its boundary.
By continuity, there exists $h_0 > y_E$ such that $\Omega_E \cap [y < h_0] \subset \Omega_E \cap [\psi > 0]$. Therefore $\psi = Q_1$ in $\Omega_E \cap [y < h_0]$.
Let now $h_{\max} = \sup\{\, h > y_E \,/\, \psi \equiv Q_1 \text{ in } \Omega_E \cap [y < h]\,\}$ and assume that $h_{\max} < y_T$. Since $\psi(x, h_{\max}) = Q_1$ for all $x \in \pi_x(\Omega_E \cap [y < h_{\max}])$, we obtain as before that $\psi \equiv Q_1$ in $\Omega_E \cap [y < h_1]$ for some $h_1 \in (h_{\max}, y_T)$. But this contradicts the definition of $h_{\max}$. ☐

Lemma 5.2. *Assume that $E \in Int(\overset{\frown}{AT})$ is such that $\psi(E) = Q_1$ and that the interior sphere condition is satisfied at E. Then there exists a sequence of points $E_n \in \Omega$ such that*

$$E_n \longrightarrow E \quad \text{and} \quad \psi(E_n) = Q_1.$$

Proof. If the assertion is not true, then there exists an $\epsilon > 0$ such that $0 \leq \psi < Q_1$ in $B_\epsilon(E) \cap \Omega = V_E$.
Let $\zeta \in W^{1,q}(B_\epsilon(E))$ with $\zeta \geq 0$ and $\zeta = 0$ on $\partial V_E \cap \Omega$. Then for $\eta > 0$, $\xi = \min\Big(\zeta, \dfrac{\psi}{\eta}\Big)\chi(V_E)$ is a test function for (P) and we have

$$\int_{V_E \cap [\psi \geq \eta\zeta]} |\nabla\psi|^{q-2}\nabla\psi.\nabla\zeta \leq - \int_{\overset{\frown}{AT}} (\widetilde{\gamma} + \delta)\xi\nu_x \leq 0$$

since $\nu_x \leq 0$ on $\widehat{AT}$ and $\widetilde{\gamma} \in [-\delta_2, -\delta]$ on $\widehat{AT} \cap \overline{V_E} \cap [\psi > 0]$. Letting $\eta \to 0$, we obtain

$$\int_{V_E} |\nabla\psi|^{q-2}\nabla\psi.\nabla\zeta \leq 0 \quad \forall\zeta \in W^{1,q}(B_\epsilon(E)),\ \zeta \geq 0,\ \zeta = 0 \ \text{ on } \ \partial V_E \cap \Omega. \tag{5.1}$$

Now let v be defined by

$$\begin{cases} \displaystyle\int_{V_E} |\nabla v|^{q-2}\nabla v.\nabla\zeta = 0 \quad \forall\zeta \in W^{1,q}(V_E),\ \zeta = 0 \ \text{ on } \ \partial V_E \cap \Omega \\ v = \psi \ \text{ on } \partial V_E \cap \Omega. \end{cases} \tag{5.2}$$

Taking $\zeta = (\psi - v)^+\chi(V_E)$ in (5.1) and in (5.2), and subtracting (5.2) from (5.1), we get

$$\int_{V_E} |(\nabla\psi|^{q-2}\nabla\psi - |\nabla v|^{q-2}\nabla v).\nabla(\psi - v)^+ \leq 0$$

from which we deduce that $(\psi - v)^+ = 0$ in V_E. In particular, we obtain $v(E) \geq \psi(E) = Q_1$. But by taking $(v - Q_1)^+$ in (5.2), one gets $v \leq Q_1$ in V_E. So $v(E) = Q_1$ and v achieves its maximum at E. By the strong maximum principle we would have $|\nabla v|^{q-2}\nabla v.\nu > 0$ at E which contradicts $|\nabla v|^{q-2}\nabla v.\nu = 0$ on $\partial V_E \cap \widehat{AT}$ by definition of v. □

Lemma 5.3. *Assume that $l_{B_2} \cap \widehat{AB} = \emptyset$. If $E \in \widehat{B_2^*T}$ is such that $\psi(E) = Q_1$ and that the interior sphere condition is satisfied at E, then $\psi \equiv Q_1$ in $\Omega \cap [y > y_E]$.*

Proof. By Lemma 5.2, there exists a sequence of points $E_n \in \Omega$ such that $E_n \to E$ and $\psi(E_n) = Q_1$ for all $n \geq 1$. It follows by Theorem 4.1 that $\psi(x, y_{E_n}) = Q_1$ for all $x > x_{E_n}$. Letting $n \to \infty$, we get $\psi(x, y_E) = Q_1$ for all $x \geq x_E$. This means that $\psi = Q_1$ on l_E and the lemma follows as a consequence of Lemma 5.1. □

Proof of Theorem 5.1. First note that $\psi + Q_2$ is a test function for (P) and we have

$$\begin{aligned} \int_\Omega |\nabla\psi|^q &= -\delta\int_{\Omega\cap[0<\psi<Q_1]} \psi_x - \int_{\widehat{AT}\cup \widehat{BB_2}} \widetilde{\gamma}(\psi + Q_2)\nu_x \\ &\quad +\delta\int_{\widehat{B_2B_1}} (\psi + Q_2)\nu_x + \delta_2\int_{\widehat{B_1T}} (\psi + Q_2)\nu_x. \end{aligned} \tag{5.3}$$

It follows that

$$\int_\Omega |\nabla\psi|^q \leq C\bigg(\Big(\int_\Omega |\nabla\psi|^q\Big)^{1/q} + (Q_2 + Q_1)\bigg)$$

which leads to $|\nabla\psi|_{L^q(\Omega)}$ is bounded. Using again (5.3), we get

$$\int_\Omega |\nabla\psi|^q \leq C\bigg(\Big|[0 < \psi < Q_1]\Big|^{1/q'} + (Q_2 + Q_1)\bigg) \longrightarrow 0 \quad \text{as} \quad Q_2 + Q_1 \longrightarrow 0. \tag{5.4}$$

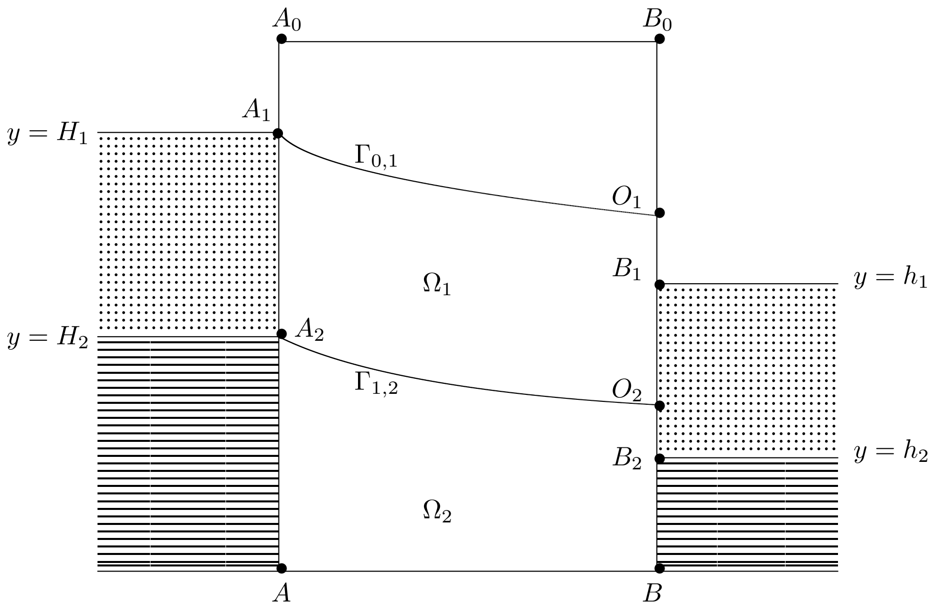

Figure 8

Let now $x = \sigma_-(y)$ be a parameterization of $\widehat{B_1^*T}$ and let $\eta \in C^\infty(\mathbb{R})$ such that $\eta = 0$ for $y < y_{B_1}$. Take $\zeta = \eta(y)$ as a test function for (P), we get by (5.4)

$$\delta_1 \int_{y_{B_1}}^{y_T} \chi([0 < \psi < Q_1])\eta(y)dy \le \int_{y_{B_1}}^{y_T} (\widetilde{\gamma} + \delta_2)\eta(y)\sqrt{1 + \sigma_-'^2(y)}dy = \int_{\Omega_{B_1}} |\nabla\psi|^{q-2}\nabla\psi.\nabla\eta \longrightarrow 0$$

as $Q_2 + Q_1 \to 0$. It follows that the measure of the set $\widehat{B_1^*T} \cap [0 < \psi < Q_1]$ converges to zero when $Q_2 + Q_1 \to 0$. Thus if $Q_2 + Q_1$ is small enough, there exists a point $E \in \widehat{B_1^*T}$ such that $\psi(E) = Q_1$. We conclude by Lemma 5.1. $\square$

6 The Rectangular Case

In this section we assume that $\Omega = (0, a) \times (0, H)$ (see Figure 8) i.e. the dam is rectangular. Then we consider the following version of the problem (P)

$$
(P') \begin{cases} \text{Find } (\psi, \gamma, \widetilde{\gamma}) \in W^{1,q}(\Omega) \times L^\infty(\Omega) \times L^\infty(\widehat{AA_0}) \text{ such that :} \\ (i) \quad \displaystyle\int_\Omega \big(|\nabla\psi|^{q-2}\nabla\psi - \gamma e_x\big).\nabla\zeta - \int_{\widehat{AA_0}} \widetilde{\gamma}\zeta = \delta \int_{\widehat{B_2B_1}} \zeta + \delta_2 \int_{\widehat{B_1B_0}} \zeta \\ \qquad\qquad \forall \zeta \in W^{1,q}(\Omega), \quad \zeta = 0 \quad \text{ on } \quad \widehat{AB} \cup \widehat{A_0B_0}, \\ (ii) \quad \gamma \in H(\psi) \quad \text{ a.e. in } \quad \Omega, \qquad \widetilde{\gamma} \in H(\psi) \quad \text{ a.e. in } \quad \widehat{AA_0}, \\ (iii) \quad \psi = -Q_2 \quad \text{on} \quad \widehat{AB}, \qquad \psi = Q_1 \quad \text{on} \quad \widehat{A_0B_0}, \end{cases}
$$

where $A_0 = (0, H)$ and $B_0 = (a, H)$.

Remark 6.1. *It is not difficult to verify that all properties established in the previous sections for the problem (P), are also true for the problem (P'). Moreover when Q_2+Q_1 is small enough, we know from Theorem 5.1 that $\psi = Q_1$ for $y \geq \overline{H}$, for some $\overline{H} \in (H_1, H)$. In this case, it is easy to show that the two problems are equivalent.*

6.1 Existence of a Monotone Solution

The main result of this section is the existence of a monotone solution with respect to y.

Theorem 6.1. *There exists a solution $(\psi, \gamma, \widetilde{\gamma})$ of (P') such that*

$$
\partial_y \psi \geq 0, \quad \partial_y \gamma \leq 0 \quad \textit{in} \quad \mathcal{D}'(\Omega) \qquad \textit{and} \qquad \partial_y \widetilde{\gamma} \leq 0 \quad \textit{in} \quad \mathcal{D}'(\widehat{AA_0}).
$$

For $\epsilon > 0$, we consider the following approximated problem :

$$
(P'_\epsilon) \begin{cases} \text{Find } \psi_\epsilon \in W^{1,q}(\Omega) \text{ such that :} \\ (i) \quad \displaystyle\int_\Omega \big(|\nabla\psi_\epsilon|^{q-2}\nabla\psi_\epsilon - H_\epsilon(\psi_\epsilon)e_x\big)\nabla\zeta + \int_\Omega \epsilon|\psi_\epsilon|^{q-2}\psi_\epsilon\zeta \\ \qquad\qquad - \displaystyle\int_{\widehat{AA_0}} H_\epsilon(\psi_\epsilon)\zeta = \delta \int_{\widehat{B_2B_1}} \zeta + \delta_2 \int_{\widehat{B_1B_0}} \zeta \\ \qquad\qquad\qquad \forall \zeta \in W^{1,q}(\Omega), \quad \zeta = 0 \text{ on } \widehat{AB} \cup \widehat{A_0B_0}, \\ (ii) \quad \psi_\epsilon = -Q_2 \text{ on } \widehat{AB}, \quad \psi_\epsilon = Q_1 + \epsilon \text{ on } \widehat{A_0B_0} \,. \end{cases}
$$

Then we have

Theorem 6.2. *There exists a solution ψ_ϵ of (P'_ϵ) such that $\partial_y \psi_\epsilon \geq 0$ in $\mathcal{D}'(\Omega)$.*

Proof. For the existence we argue as in section 2. The only difference is the fact that we cannot use $\psi_\epsilon - \psi$ as a test function for (P'_ϵ) since $\psi_\epsilon - \psi \neq 0$ on $\widehat{A_0B_0}$. To overcome this difficulty

one can choose $\rho(y)(\psi_\epsilon - \psi)$ as a test function, where $\rho \in C_0^\infty(0,H)$. Then one can prove with small modifications that $\rho\nabla\psi_\epsilon \to \rho\nabla\psi$ strongly in $L^q(\Omega)$ which leads to $\nabla\psi_\epsilon \to \nabla\psi$ strongly in $L^q_{loc}(\Omega)$.
Moreover one can verify that $-Q_2 \le \psi_\epsilon \le Q_1 + \epsilon$ (choose $(\psi_\epsilon + Q_2)^-$ and $(\psi_\epsilon - Q_1 - \epsilon)^+$ as test functions).

To prove the monotonicity, let $\eta \in (0,\epsilon)$ small enough and set $\psi_\epsilon^\eta(x,y) = \psi_\epsilon(x, y+\eta)$ for $(x,y) \in \Omega_{0\eta} = (0,a)\times(-\eta, H-\eta)$. Then ψ_ϵ^η satisfies

$$(P_\epsilon'^\eta)\begin{cases} \displaystyle\int_{\Omega_{0\eta}} \big(|\nabla\psi_\epsilon^\eta|^{q-2}\nabla\psi_\epsilon^\eta - H_\epsilon(\psi_\epsilon^\eta)e_x\big)\nabla\zeta + \int_{\Omega_{0\eta}} \epsilon|\psi_\epsilon^\eta|^{q-2}\psi_\epsilon^\eta\zeta \\ \displaystyle\qquad - \int_{-\eta}^{H-\eta} H_\epsilon(\psi_\epsilon^\eta(0,y))\zeta(0,y) = \delta\int_{h_2-\eta}^{h_1-\eta}\zeta(a,y) + \delta_2\int_{h_1-\eta}^{H-\eta}\zeta(a,y) \\ \qquad\qquad \forall \zeta \in W^{1,q}(\Omega_{0\eta}), \quad \zeta(x,-\eta) = \zeta(x, H-\eta) = 0 \ \text{ a.e. } x \in (0,a). \end{cases}$$

We are going to show that $\psi_\epsilon \le \psi_\epsilon^\eta$ a.e. in $\Omega_\eta = (0,a)\times(0,H-\eta)$. To do this, we consider for $\mu > 0$, the function $T_\mu : \mathbb{R} \to \mathbb{R}$ defined par

$$T_\mu(s) = \begin{cases} s & \text{if} \quad |s| \le \mu \\ \mu\dfrac{s}{|s|} & \text{if} \quad |s| > \mu. \end{cases}$$

It is clear that $T_\mu \in C^{0,1}(\mathbb{R})$ and therefore

$$\forall u \in W^{1,q}(\Omega), \quad T_\mu(u) \in W^{1,q}(\Omega), \quad \text{with} \quad \nabla(T_\mu o u) = T_\mu'(u)\nabla u = \chi([|u| \le \mu])\nabla u.$$

In particular $\zeta = T_\mu\big((\psi_\epsilon - \psi_\epsilon^\eta)^+\big) \in W^{1,q}(\Omega_\eta)$. Moreover since $-Q_2 \le \psi_\epsilon \le Q_1 + \epsilon$ in Ω, we have

- $\zeta(x,0) = T_\mu\big((-Q_2 - \psi_\epsilon^\eta(x,0))^+\big) = T_\mu(0) = 0 \quad \forall x \in (0,a)$,
- $\zeta(x,H-\eta) = T_\mu\big((\psi_\epsilon(x,H-\eta) - Q_1 - \epsilon)^+\big) = T_\mu(0) = 0 \quad \forall x \in (0,a)$.

It follows that $\zeta = T_\mu\big((\psi_\epsilon - \psi_\epsilon^\eta)^+\big)\chi(\Omega_\eta)$ is a test function for both (P_ϵ') and $(P_\epsilon'^\eta)$. Hence we have

$$\begin{aligned} &\int_{\Omega_\eta} \big(|\nabla\psi_\epsilon|^{q-2}\nabla\psi_\epsilon - H_\epsilon(\psi_\epsilon)e_x\big)\nabla\zeta + \int_{\Omega_\eta} \epsilon|\psi_\epsilon|^{q-2}\psi_\epsilon\zeta \\ &= \int_0^{H-\eta} H_\epsilon(\psi_\epsilon(0,y))\zeta(0,y) + \delta\int_{h_2}^{h_1}\zeta(a,y) + \delta_2\int_{h_1}^{H-\eta}\zeta(a,y) \end{aligned} \tag{6.1}$$

$$\begin{aligned} &\int_{\Omega_\eta} \big(|\nabla\psi_\epsilon^\eta|^{q-2}\nabla\psi_\epsilon^\eta - H_\epsilon(\psi_\epsilon^\eta)e_x\big)\nabla\zeta + \int_{\Omega_\eta} \epsilon|\psi_\epsilon^\eta|^{q-2}\psi_\epsilon^\eta\zeta \\ &= \int_0^{H-\eta} H_\epsilon(\psi_\epsilon^\eta(0,y))\zeta(0,y) + \delta\int_{h_2-\eta}^{h_1-\eta}\zeta(a,y) + \delta_2\int_{h_1-\eta}^{H-\eta}\zeta(a,y). \end{aligned} \tag{6.2}$$

Subtracting (6.2) from (6.1), we get for $\Omega_\eta^\mu = \Omega_\eta \cap [(\psi_\epsilon - \psi_\epsilon^\eta)^+ \le \mu]$

$$\begin{aligned}
\epsilon \int_{\Omega_\eta} & \big(|\psi_\epsilon|^{q-2}\psi_\epsilon - |\psi_\epsilon^\eta|^{q-2}\psi_\epsilon^\eta\big).T_\mu\big((\psi_\epsilon - \psi_\epsilon^\eta)^+\big) \\
&= -\int_{\Omega_\eta^\mu} \big(|\nabla\psi_\epsilon|^{q-2}\nabla\psi_\epsilon - |\nabla\psi_\epsilon^\eta|^{q-2}\nabla\psi_\epsilon^\eta\big).\nabla(\psi_\epsilon - \psi_\epsilon^\eta)^+ \\
&\quad + \int_{\Omega_\eta^\mu} \big(H_\epsilon(\psi_\epsilon) - H_\epsilon(\psi_\epsilon^\eta)\big)e_x.\nabla(\psi_\epsilon - \psi_\epsilon^\eta)^+ \\
&\quad + \int_0^{H-\eta} \big(H_\epsilon(\psi_\epsilon(0,y)) - H_\epsilon(\psi_\epsilon^\eta(0,y))\big).T_\mu\big((\psi_\epsilon - \psi_\epsilon^\eta)^+\big)(0,y) \\
&\quad + \delta \int_{h_1-\eta}^{h_1} T_\mu\big((\psi_\epsilon - \psi_\epsilon^\eta)^+\big)(a,y) - \delta \int_{h_2-\eta}^{h_2} T_\mu\big((\psi_\epsilon - \psi_\epsilon^\eta)^+\big)(a,y) \\
&\quad - \delta_2 \int_{h_1-\eta}^{h_1} T_\mu\big((\psi_\epsilon - \psi_\epsilon^\eta)^+\big)(a,y). \qquad (6.3)
\end{aligned}$$

Using the monotonicity of the functions H_ϵ, T_μ, and $h \to |h|^{q-2}h$, and the fact that $\delta - \delta_2 = -\delta_1$, one derives from (6.3)

$$\int_{\Omega_\eta} \big(|\psi_\epsilon|^{q-2}\psi_\epsilon - |\psi_\epsilon^\eta|^{q-2}\psi_\epsilon^\eta\big).T_\mu\big((\psi_\epsilon - \psi_\epsilon^\eta)^+\big) \le \frac{1}{\epsilon}\int_{\Omega_\eta^\mu} \big(H_\epsilon(\psi_\epsilon) - H_\epsilon(\psi_\epsilon^\eta)\big)e_x.\nabla(\psi_\epsilon - \psi_\epsilon^\eta)^+. \qquad (6.4)$$

The Lipschitz continuity of H_ϵ, leads to

$$\int_{\Omega_\eta^\mu} \big(H_\epsilon(\psi_\epsilon) - H_\epsilon(\psi_\epsilon^\eta)\big)e_x.\nabla(\psi_\epsilon - \psi_\epsilon^\eta)^+ \le \frac{\delta_2\mu}{\epsilon}\int_{\Omega_\eta^\mu} |\nabla(\psi_\epsilon - \psi_\epsilon^\eta)^+|. \qquad (6.5)$$

Using (6.4)-(6.5), and the monotonicity of the function $|u|^{q-2}u$, we obtain

$$\int_{\Omega_\eta \setminus \Omega_\eta^\mu} \big(|\psi_\epsilon|^{q-2}\psi_\epsilon - |\psi_\epsilon^\eta|^{q-2}\psi_\epsilon^\eta\big) \le \frac{\delta_2}{\epsilon^2}\int_{\Omega_\eta^\mu} |\nabla(\psi_\epsilon - \psi_\epsilon^\eta)^+|. \qquad (6.6)$$

Letting $\mu \to 0$ in (6.6), we obtain

$$\int_{\Omega_\eta \cap [\psi_\epsilon > \psi_\epsilon^\eta]} \big(|\psi_\epsilon|^{q-2}\psi_\epsilon - |\psi_\epsilon^\eta|^{q-2}\psi_\epsilon^\eta\big) \le 0,$$

which leads to $\psi_\epsilon \le \psi_\epsilon^\eta$ in Ω_η and the proof is complete. □

Proof of Theorem 6.1. Arguing as in section 2, one can prove that $(\psi_\epsilon, H_\epsilon(\psi_\epsilon), H_\epsilon(\psi_\epsilon)_{|\widehat{AA_0}})$ converges in an appropriate way to a solution $(\psi, \gamma, \widetilde{\gamma})$ of (P'). Moreover one has the following convergences

$$\psi_\epsilon \to \psi \qquad \text{in } W^{1,q}(\Omega) \tag{6.7}$$
$$H_\epsilon(\psi_\epsilon) \rightharpoonup \gamma \qquad \text{in } L^{q'}(\Omega) \tag{6.8}$$
$$H_\epsilon(\psi_\epsilon) \rightharpoonup \widetilde{\gamma} \qquad \text{in } L^{q'}(\widehat{AA_0}). \tag{6.9}$$

We deduce immediately from (6.7) and the monotonicity of ψ_ϵ that $\partial_y\psi \geq 0$ in $\mathcal{D}'(\Omega)$.
Since $\partial_y(H_\epsilon(\psi_\epsilon)) = H'_\epsilon(\psi_\epsilon)\partial_y\psi_\epsilon$, we obtain from (6.8)-(6.9) that $\partial_y\gamma \leq 0$ in $\mathcal{D}'(\Omega)$ and $\partial_y\widetilde{\gamma} \leq 0$ in $\mathcal{D}'(\widehat{AA_0})$. □

6.2 Study of the Free Boundary $\Gamma_{0,1}$

In this section, we propose another proof of the continuity of the upper free boundary. We also prove that Φ is a decreasing function. First we prove that $\Gamma_{0,1}$ is located above the line $[y = h_1]$.

Proposition 6.1. *We have*

$$\psi(x,y) < Q_1 \qquad in \quad \Omega \cap [y < h_1].$$

Proof. Assume that there exists $(x_0,y_0) \in \Omega \cap [y < h_1]$ such that $\psi(x_0,y_0) = Q_1$. By Theorem 3.1 and the monotonicity of ψ, we deduce that $\psi(x,y) = Q_1$ in $(x_0,a) \times (y_0,H)$. In particular we have $\psi(x,y) = Q_1$ in $Z = (x_0,a) \times (y_0,h_1)$.
It follows that $\gamma_x = -\Delta_q\psi = 0$ in $\mathcal{D}'(Z)$, and consequently $\gamma = \gamma(y)$ in Z. Now let $\zeta \in \mathcal{D}(Z \cup ([x = a] \cap \partial Z))$ and take $\chi(Z)\zeta$ as a test function for (P'), we obtain

$$\int_Z -\gamma(y)\zeta_x = \int_{\max(y_0,h_2)}^{h_1} \delta\zeta.$$

Without loss of generality, we can assume that $y_0 > h_2$, so that we can assume that $Z \subset (x_0,a) \times (h_2,h_1)$.
Integrating by part the left integral of the previous identity, we obtain $\gamma = \gamma(y) = -\delta$ in Z.
By continuity there exists $(x_1,y_1) \in (x_0,a) \times (0,y_0)$ such that $0 < \psi(x_1,y_1) < Q_1$. Also by continuity there exists $\epsilon > 0$ small enough such that $0 < \psi < Q_1$ in $(x_1-\epsilon, x_1+\epsilon)\times(y_1-\epsilon, y_1+\epsilon)$.
By monotonicity of ψ, we obtain

$$0 < \psi \leq Q_1 \qquad \text{in} \quad Z' = Z \cup (x_1-\epsilon, x_1+\epsilon) \times (y_1-\epsilon, h_1).$$

We deduce that $\gamma \in [-\delta_2, -\delta]$ in Z'. But since $\gamma = -\delta$ in Z and $\gamma_y \leq 0$, we get

$$\gamma = -\delta \qquad \text{in} \quad Z'.$$

Hence $\Delta_q \psi = 0$ in Z'. But since $\psi \le Q_1$ in Z' and $\psi = Q_1$ in Z, we get by the maximum principle that $\psi = Q_1$ in Z' which is in contradiction with the fact that $\psi < Q_1$ in $(x_1 - \epsilon, x_1 + \epsilon) \times (y_1 - \epsilon, y_1 + \epsilon)$. $\square$

The main result of this section is the following theorem.

Theorem 6.3. *If $[\psi = Q_1] \cap \Omega \neq \emptyset$, then there exists α, β with $h_1 \le \alpha < \beta \le H$, and $a_* \in [0, a)$ such that $\Phi : (\alpha, \beta) \to (a_*, a)$ is continuous and decreasing. Moreover $g = \Phi^{-1} : (a_*, a) \to (\alpha, \beta)$ is also continuous and decreasing, $\lim_{x \to a^-} g(x) = \alpha$, and if we set $g(x) = \beta$ for $x \in (0, a_*]$, we have*

$$[\psi < Q_1] = [y < g(x)].$$

We need the following lemma similar to Lemma 4.6.

Lemma 6.1. *Let $X_0 = (x_0, y_0) \in \Omega_1$ and $r > 0$ such that the ball $B_r(X_0)$ is contained in $\Omega_+ \cap [y > h_1]$. We cannot have the following situations (see Figure 8)*

$$(i) \quad \begin{cases} \psi = Q_1 & \textit{in} \quad S_v = B_r(X_0) \cap [x = x_0] \\ \psi < Q_1 & \textit{in} \quad B_r(X_0) \setminus S_v, \end{cases}$$

$$(ii) \quad \begin{cases} \psi < Q_1 & \textit{in} \quad B_r(X_0) \cap [x < x_0] \\ \psi = Q_1 & \textit{in} \quad B_r(X_0) \cap [x \ge x_0], \end{cases}$$

$$(iii) \quad \begin{cases} \psi = Q_1 & \textit{in} \quad B_r(X_0) \cap [x \le x_0] \\ \psi < Q_1 & \textit{in} \quad B_r(X_0) \cap [x > x_0]. \end{cases}$$

Proof. *i*) Since $0 < \psi < Q_1$ in $B_r(X_0) \setminus S_v$, one has $\gamma = -\delta$ a.e. in $B_r(X_0)$. This leads to $\Delta_q \psi = -\gamma_x = 0$ in $\mathcal{D}'\big(B_r(X_0)\big)$. By the maximum principle, we have either $\psi \equiv Q_1$ or $\psi < Q_1$ in $B_r(X_0)$ which both are in contradiction with the assumption *i*).

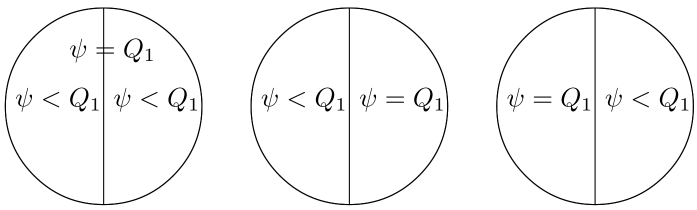

Figure 8

ii) Let $\eta \in (0, r/2)$ and consider $\psi_\eta(x, y) = \psi(x, y - \eta)$ defined in $B_{r/2}(X_0)$.

From the assumptions, we have $\gamma = \gamma(y)$ in $B_r(X_0) \cap [x > x_0]$ and $\gamma = -\delta$ in $B_r(X_0) \cap [x < x_0]$. Moreover by Theorem 4.1 and since $\psi = Q_1$ in $B_r(X_0) \cap [x > x_0]$, we know that $\psi = Q_1$ in $Z = (x_0, a) \times (y_0 - r, y_0 + r)$ and therefore $\gamma = \gamma(y)$ in Z.
Let $\zeta \in \mathcal{D}(Z \cup (\partial Z \cap [x = a]))$ and take $\chi(Z)\zeta$ as a test function for (P'). We obtain

$$-\int_Z \gamma(y)\zeta_x = \int_{y_0-r}^{y_0+r} \delta_2 \zeta(a, y).$$

A simple integration by part shows that $\gamma = -\delta_2$ in Z.
Let now $\zeta \in \mathcal{D}\big(B_{r/2}(X_0)\big)$. We have

$$\begin{aligned}
\int_{B_{r/2}(X_0)} |\nabla\psi_\eta|^{q-2}\nabla\psi_\eta\nabla\zeta &= \int_{B_{r/2}(x_0,y_0-\eta)} (|\nabla\psi|^{q-2}\nabla\psi)(x,y).\nabla\zeta(x, y+\eta) \\
&= \int_{B_{r/2}(x_0,y_0-\eta)} \gamma(x,y)\zeta_x(x,y+\eta) = \int_{B_{r/2}(X_0)} \gamma(x,y-\eta)\zeta_x(x,y) \\
&= \int_{B_{r/2}(X_0)} \gamma(x,y)\zeta_x \qquad \text{since } \gamma \text{ depends only on } x \\
&= \int_{B_{r/2}(X_0)} |\nabla\psi|^{q-2}\nabla\psi\nabla\zeta.
\end{aligned}$$

It follows that

$$\Delta_q\psi_\eta = \Delta_q\psi \quad \text{in} \quad \mathcal{D}'\big(B_{r/2}(X_0)\big).$$

Since $\psi_\eta \leq \psi$ in $B_{r/2}(X_0)$ and $\psi_\eta = \psi = Q_1$ in $B_{r/2}(X_0) \cap [x = x_0]$, we deduce by the maximum principle (see [18] Lemma 2.4) that $\psi_\eta = \psi$ in $B_{r/2}(X_0)$. This holds for all $\eta \in (0, r/2)$. Consequently $\psi(x, y) = \psi(x)$ in $B_{r/2}(X_0)$.
Since $\Delta_q\psi = 0$ in $B_{r/2}(X_0) \cap [x < x_0]$, we deduce that $\psi = mx + n$ in $B_{r/2}(X_0) \cap [x < x_0]$, where m and n are two constants.
Now let C_0 be the connected component of $[0 < \psi < Q_1]$ that contains $B_{r/2}(X_0) \cap [x < x_0]$. We know that ψ is analytic in $C_0 \setminus [\nabla\psi = 0]$ and that because we are in dimension 2, $[\nabla\psi = 0]$ is a discrete set (see [4]). Therefore we obtain by analytic continuation that $\psi = mx + n$ in C_0. In particular we have $\psi(x_0, y_0) = mx_0 + n = Q_1$.
Finally let $y_1 = \inf\{y \in (0, H) \ / \ \psi(x_0, y) > 0\}$. Then it is clear that $(x_0, y_1) \in \partial C_0 \cap \Omega$. This leads to $\psi(x_0, y_1) = mx_0 + n = 0$ which contradicts $mx_0 + n = Q_1$.

iii) This is impossible by Theorem 4.1. $\square$

Proof of Theorem 6.3. Set

$$\begin{aligned}
\alpha &= \sup\{y \ / \ \forall x \in (0, a) \quad \psi(x, y) < Q_1 \ \} \geq h_1 \\
\beta &= \inf\{y \ / \ \forall x \in (0, a) \quad \psi(x, y) = Q_1 \ \} \leq H.
\end{aligned}$$

α is well defined and we have $\alpha \geq h_1$ by Proposition 6.1.

β is well defined since $\psi(x, H) = Q_1$ for all $x \in (0, a)$.

• We claim that $\alpha < \beta$:

Indeed assume first that $\alpha > \beta$. Then

$$\begin{array}{llll} i) & \exists y_0 \in (\beta, \alpha) & \text{such that} & \forall x \in (0,a) \quad \psi(x, y_0) < Q_1 \\ ii) & \exists y_0' \in (\beta, y_0) & \text{such that} & \forall x \in (0,a) \quad \psi(x, y_0') = Q_1. \end{array}$$

But then $ii)$ leads by monotonicity of ψ to $\psi(x, y_0) = Q_1$ for all $x \in (0, a)$, which contradicts $i)$. So $\alpha \leq \beta$.

Now assume that $\alpha = \beta$. We distinguish two cases:

$* \ \alpha = \beta = H$

In this case, we have $\psi < Q_1$ in Ω which contradicts the assumption $[\psi = Q_1] \cap \Omega \neq \emptyset$.

$* \ \alpha = \beta < H$

From the definition of β, one has necessarily $\psi(x, \beta) = Q_1$ for all $x \in (0, a)$ and by monotonicity of ψ, we get

$$\psi(x, y) = Q_1 \quad \text{for all} \quad (x, y) \in (0, a) \times [\beta, H).$$

One can also verify that

$$\psi(x, y) < Q_1 \quad \text{for all} \quad (x, y) \in (0, a) \times (0, \alpha).$$

If we choose a small ball B centered at a point (x_0, α), we will have $\psi = Q_1$ in $B^+ = B \cap [y > \alpha]$ and $\psi < Q_1$ in $B^- = B \cap [y < \alpha]$. But this is in contradiction with Lemma 4.6.
Hence we have $\alpha < \beta$.

• Φ is non-increasing in (α, β):

Let $y_1, y_2 \in (\alpha, \beta)$ such that $y_1 < y_2$. By definition, we have $\psi(x, y_1) = Q_1 \quad \forall x \geq \Phi(y_1)$. By monotonicity of ψ, we deduce that $\psi(x, y) = Q_1$ in $[\Phi(y_1), a) \times [y_1, H]$.
In particular, we obtain $\psi(\Phi(y_1), y_2) = Q_1$ which leads to $\Phi(y_2) \leq \Phi(y_1)$.

• Φ is decreasing in (α, β):

Let $y_1, y_2 \in (\alpha, \beta)$ such that $y_1 < y_2$. Assume that $\Phi(y_2) = \Phi(y_1) = x_0$, with $0 < x_0 < a$. Since Φ is non-increasing, we obtain $\Phi(y) = x_0$ for all $y \in (y_1, y_2)$ which leads to $\psi < Q_1$ in $(0, x_0) \times (y_1, y_2)$ and $\psi = Q_1$ in $(x_0, a) \times (y_1, y_2)$. This is impossible by Lemma 6.1.

• Φ is continuous at each point $y \in (\alpha, \beta)$:

Let $y_0 \in (\alpha, \beta)$. From the definition of α and β, we have necessarily $\Phi(y_0) \in (0, a)$. Since Φ is decreasing, there exist $l^- = \lim\limits_{y \to y_0^-} \Phi(y)$ and $l^+ = \lim\limits_{y \to y_0^+} \Phi(y)$.

By the monotonicity of ϕ, we have $l^- \geq l^+$. Assume that $l^- > l^+$. Again by the monotonicity of Φ, we have

$$0 < \Phi(y) \leq l^+ \quad \forall y > y_0 \qquad \text{and} \qquad a > \Phi(y) \geq l^- \quad \forall y < y_0.$$

We deduce that

$$\psi = Q_1 \quad \text{in} \quad [l^+, a) \times (y_0, H) \qquad \text{and} \qquad \psi < Q_1 \quad \text{in} \quad (0, l^-) \times (0, y_0).$$

Let $\epsilon \in (0, \min(y_0, H - y_0))$. Then we have

$$\begin{aligned} \psi = Q_1 \quad &\text{in} \quad (l^+, l^-) \times (y_0, y_0 + \epsilon) \\ \psi < Q_1 \quad &\text{in} \quad (l^+, l^-) \times (y_0 - \epsilon, y_0). \end{aligned}$$

This is in contradiction with Lemma 4.6.

• Existence of Φ^{-1}

Since Φ is a decreasing function, there exist $b^* = \lim_{y \to \alpha^+} \Phi(y)$ and $a^* = \lim_{y \to \beta^-} \Phi(y)$. Now $\Phi : (\alpha, \beta) \to (a^*, b^*)$ is continuous and decreasing. Therefore Φ is bijective from (α, β) into (a^*, b^*). Consequently $g = \Phi^{-1} : (a^*, b^*) \to (\alpha, \beta)$ is also continuous and decreasing.

• $\lim_{x \to a^-} g(x) = \alpha$:

It is enough to prove that $b^* = a$.
Assume that $b^* < a$. Then from the definition and monotonicity of Φ, we have $\psi = Q_1$ in $(b^*, a) \times (\alpha, H)$ and $\psi < Q_1$ in $(b^*, a) \times (0, \alpha)$. This is in contradiction with Lemma 4.6. □

Remark 6.2. *i) If $[\psi = Q_1] \cap \Omega = \emptyset$, then the dam is entirely wet, and we have $g(x) = H$ for all $x \in (0, a)$.*

ii) If $Q_1 + Q_2$ is small enough, we know from Theorem 5.1, that $\psi = Q_1$ in Ω_E for some point E between B_1^* and A_0. In this case, we have $\beta < H$ and then by Lemma 4.6 we have necessarily $a_* = 0$.

6.3 Study of the Lower Free Boundary $\Gamma_{1,2}$

The main result of this section is that $\Gamma_{1,2}$ is represented by the graph of a continuous function. First by using the continuity of ψ, we define two functions $f_1, f_2 : (0, a) \to \mathbb{R}$ by

$$f_1(x) = \sup\{\, y \;/ \psi(x, y) < 0 \,\} \qquad \text{and} \qquad f_2(x) = \inf\{\, y \;/ \psi(x, y) > 0 \,\}.$$

Then we have

Proposition 6.2. *i) f_1 (resp. f_2) is lower (resp. upper) semicontinuous in $(0, a)$.*
ii) $[\psi(x, y) < 0] = [y < f_1(x)]$ and $[\psi(x, y) > 0] = [y > f_2(x)]$.
iii) $\Gamma_{1,2} = [\psi = 0] = [f_1(x) \le y \le f_2(x)]$.
iv) $[0 < \psi(x, y) < Q_1] = [f_2(x) < y < g(x)]$ is connected.

Proof. The proof of $i)$, $ii)$ and $iii)$ can be obtained by using the continuity and the monotonicity of ψ. To prove $iv)$, let $(x_1, y_1), (x_2, y_2) \in [0 < \psi(x, y) < Q_1]$. By the monotonicity of ψ, we have

$$0 < \psi(x_i, y) < Q_1 \quad \text{for} \quad y_i \le y < g(x_i),\ i = 1, 2.$$

By Theorem 6.3, we deduce that for $\epsilon > 0$ small enough, we have

$$0 < \psi(x, g(x) - \epsilon) < Q_1 \quad \forall x \in [x_1, x_2].$$

Hence the arc $\{x_1\} \times [y_1, g(x_1) - \epsilon] \cup \{\, (x, g(x) - \epsilon) \,/\, x \in [x_1, x_2] \,\} \cup \{x_2\} \times [y_2, g(x_2) - \epsilon]$ connects the points (x_1, y_1), (x_2, y_2) in $[0 < \psi(x, y) < Q_1]$. $\square$

The main result of this subsection is the following theorem.

Theorem 6.4. *There exists a continuous function $f : (0, a) \to \mathbb{R}$ such that $\Gamma_{1,2} = [y = f(x)]$.*

The proof of this theorem is based on several lemmas. The first one (Lemma 6.2) is a non-oscillation lemma. Lemma 6.3 and Lemma 6.4 are used to eliminate possible vertical segments of the free boundary. Note that similar lemmas are proved in [18] (see also [8]).

Lemma 6.2. *Let $0 < y_1 < y_2 < H$, $0 < x_1 < x_2 < x_3 < x_4 < a$. Set*

$$l_i = \{x_i\} \times [y_1, y_2] \qquad i = 1, 2, 3, 4.$$

Suppose that $[x_1, x_4] \times [y_1, y_2] \subset [\psi < Q_1]$ and that $(-1)^i \psi > 0$ on l_i, $i = 1, 2, 3, 4$ (see Figure 10). Then

$$y_2 - y_1 = O(x_4 - x_1) \qquad \textit{i.e.} \quad y_2 - y_1 \to 0 \quad \textit{when} \quad x_4 - x_1 \to 0.$$

Proof. When $q < 2$, we have $\nabla\psi \in C^{0,1}_{loc}([\psi < 0] \cup [0 < \psi < Q_1])$, and one can use the proof of Lemma 2.3 of [18]. When $q > 2$, one can work in $[\psi < 0] \setminus S$ and in $[0 < \psi < Q_1] \setminus S$, where $S = \{(x, y) \in [\psi < 0] \cup [0 < \psi < Q_1] \ / \ \nabla\psi(x, y) = 0\}$. Then one can adapt the proof in [18]. $\square$

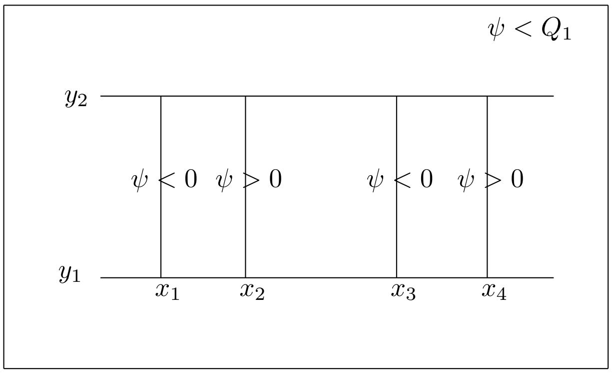

Figure 10

Lemma 6.3. *Let* $m_0 = (x_0, y_0) \in \Omega$, $\rho > 0$ *such that* $B_\rho(m_0) \subset [\psi < Q_1]$. *Set* $S_v = \{x_0\} \times (y_0 - \rho, y_0 + \rho)$. *Then we cannot have the following situation (see Figure 11)*

$$\psi = 0 \quad \text{on } S_v \quad \text{and} \quad \psi \neq 0 \quad \text{in } B_\rho(m_0) \setminus S_v.$$

Proof. *i*) Assume that we have $\psi > 0$ in $B_\rho(m_0) \setminus S_v$. Then $\gamma = -\delta$ a.e. in $B_\rho(m_0)$ and $\Delta_q \psi = 0$ in $B_\rho(m_0)$. But since we have $\psi > 0$ in $B_\rho(m_0) \setminus S_v$ and $\psi = 0$ on S_v, we get a contradiction with the strong maximum principle.

ii) Assume that we have $\psi < 0$ in $B_\rho(m_0) \setminus S_v$. Then we have $\gamma = 0$ a.e. in $B_\rho(m_0)$ and $\Delta_q \psi = 0$ in $B_\rho(m_0)$ which leads to a contradiction as in *i*).

iii) Assume that we have $\psi < 0$ in $B_\rho^-(m_0) = B_\rho(m_0) \cap [x < x_0]$ and $\psi > 0$ in $B_\rho^+(m_0) = B_\rho(m_0) \cap [x > x_0]$. Let $0 < \eta < \rho/2$ and set $\psi_\eta(x, y) = \psi(x, y - \eta)$. Since $\gamma = 0$ a.e. in $B_\rho^-(m_0)$ and $\gamma = -\delta$ a.e. in $B_\rho^+(m_0)$, we obtain by arguing as in the proof of Lemma 6.1, that $\Delta_q \psi_\eta = \Delta_q \psi$ in $B_{\rho/2}(m_0)$. Since moreover $\psi_\eta \leq \psi$ in $B_{\rho/2}(m_0)$ and $\psi_\eta = \psi = 0$ in $B_{\rho/2}(m_0) \cap [x = x_0]$, we obtain by the strong maximum principle (see [18]) that $\psi_\eta = \psi$ in $B_{\rho/2}(m_0)$. Letting η go to 0, we get $\partial_y \psi = 0$ and $\psi(x, y) = \theta(x)$ in $B_{\rho/2}(m_0)$. Since $\Delta_q \psi = 0$ in $B_{\rho/2}^-(m_0)$, we get $\theta(x) = \alpha_0 x + \beta_0$ in $B_{\rho/2}^-(m_0)$. Moreover by the monotonicity of ψ, we have $\psi < 0$ in $D = B_{\rho/2}^-(m_0) \cup (x_0 - \rho/2, x_0) \times (0, y_0)$. Since ψ is analytic in $D \setminus S$, we obtain by unique continuation that $\psi(x, y) = \alpha_0 x + \beta_0$ in D. Using the boundary data of ψ at the bottom of the dam, we get $\alpha_0 = 0$, $\beta_0 = -Q_2$ and $\psi(x, y) = -Q_2$ in D. This contradicts the fact that $\psi(m_0) = 0$.

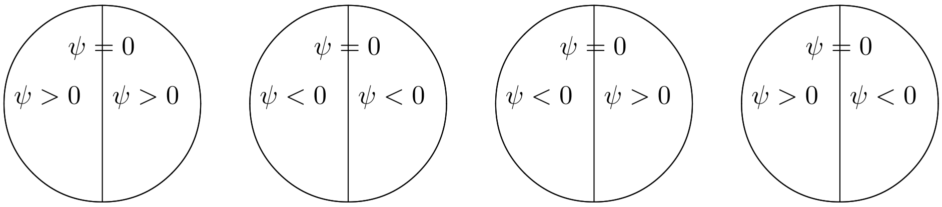

Figure 11

iv) If $\psi > 0$ in $B^-_\rho(m_0)$ and $\psi < 0$ in $B^+_\rho(m_0)$, we get a contradiction as in *iii*). $\square$

Lemma 6.4. *Assume that there exists $x_0 \in (0, a)$, $y_1, y_2 \in (0, H)$ with $y_1 < y_2$ such that*

$$\psi(x_0, y) = 0 \qquad \forall y \in [y_1, y_2].$$

Then $\psi \equiv 0$ in $[x_0, a] \times [y_1, y_2]$.

To prove Lemma 6.4, we need four lemmas

Lemma 6.5. *Let V be a domain in Ω.*

i) If $0 \le \psi < Q_1$ in V, then $\gamma_x \ge 0$ in $\mathcal{D}'(V)$.

ii) If $\psi \le 0$ in V, then $\gamma_x \le 0$ in $\mathcal{D}'(V)$.

Proof. *i*) Let $\zeta \in \mathcal{D}(V)$, $\zeta \ge 0$ and $\eta > 0$. Taking $\min\Big(\zeta, \dfrac{\psi}{\eta}\Big)$ as a test function for (P'), we obtain, since $\gamma = -\delta$ a.e. in $[0 < \psi < Q_1]$

$$\int_{V \cap [\eta\zeta \le \psi]} |\nabla\psi|^{q-2}\nabla\psi.\nabla\zeta \le \int_V \gamma\partial_x\Big(\min\Big(\zeta, \frac{\psi}{\eta}\Big)\Big) = 0.$$

Letting $\eta \to 0$ and using (3.1), we get $\partial_x\gamma \ge 0$ in $\mathcal{D}'(V)$.

ii) Here it is enough to take $\min(\zeta, \dfrac{-\psi}{\eta})$ as a test function for (P), where $\zeta \in \mathcal{D}(V)$, $\zeta \ge 0$ and $\eta > 0$. Then we argue as in *i*). $\square$

Lemma 6.6. *Let $X_0 = (x_0, y_0) \in \Omega$, $\epsilon \in (0, \min(x_0, y_0)$, $x_1 \in (0, x_0 - \epsilon)$, and $C = B_\epsilon(X_0) \cup \big((x_1, x_0) \times (y_0 - \epsilon, y_0 + \epsilon)\big) \subset \Omega$ (see Figure 12).*

i) If $0 \le \psi < Q_1$ in C and $\psi > 0$ in $B^+_\epsilon(X_0) = B_\epsilon(X_0) \cap [x > x_0]$, then $\psi > 0$ in C.

ii) If $\psi \le 0$ in C and $\psi < 0$ in $B^+_\epsilon(X_0)$, then $\psi < 0$ in C.

Proof. *i*) We deduce from Lemma 6.5 *i*) that $\gamma_x \ge 0$ in C. Moreover since $C \subset [0 \le \psi < Q_1]$, we have $-\delta \le \gamma \le 0$ in C. But $\psi > 0$ in $B^+_\epsilon(X_0)$ leads to $\gamma = -\delta$ in $B^+_\epsilon(X_0)$. Therefore we have

$\gamma = -\delta$ in C and then $\Delta_q \psi = \gamma_x = 0$. Since $\psi \geq 0$ in C and $\psi > 0$ in $B_\epsilon^+(X_0)$, we conclude by the maximum principle that $\psi > 0$ in C.

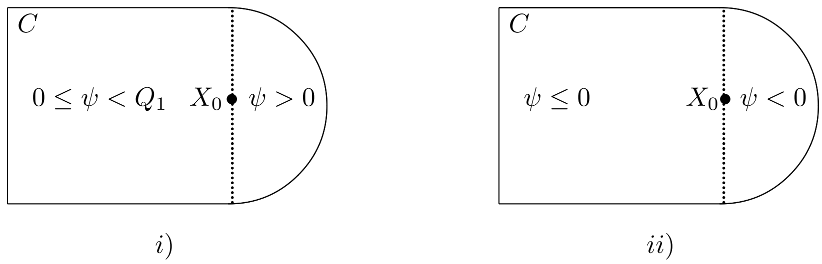

Figure 12

ii) We deduce from Lemma 6.5 *ii)* that $\gamma_x \leq 0$ in C. Moreover since $C \subset [\psi \leq 0]$, we have $-\delta \leq \gamma \leq 0$ in C. But $\psi < 0$ in $B_\epsilon^-(X_0)$ leads to $\gamma = 0$ in $B_\epsilon^+(X_0)$. Therefore we have $\gamma = 0$ in C and then $\Delta_q \psi = \gamma_x = 0$. Since $\psi \leq 0$ in C and $\psi < 0$ in $B_\epsilon^+(X_0)$, we conclude by the maximum principle that $\psi < 0$ in C. ☐

Lemma 6.7. *Let $X_0 = (x_0, y_0) \in \Omega$, and $B_{\epsilon_0}(X_0) \subset [\psi < Q_1]$ ($\epsilon_0 > 0$).*

If $\psi \geq 0$ in $B_{\epsilon_0}^+(X_0) \cap [y \leq y_0]$ and $\psi = 0$ in $B_{\epsilon_0}(X_0) \cap [y \leq y_0] \cap [x = x_0]$, then $\psi \equiv 0$ in $B_{\epsilon_0}^+(X_0) \cap [y \leq y_0]$.

Proof. We argue by contradiction and assume that there exists $X_1 = (x_1, y_1) \in V = B_{\epsilon_0}^+(X_0) \cap [y < y_0]$ such that $\psi(X_1) > 0$. By continuity of ψ, there exists $\epsilon_1 > 0$ small enough such that $\psi > 0$ in $B_{\epsilon_1}(X_1) \subset V$.
Let $C = B_{\epsilon_1}(X_1) \cup \big((x_0, x_1) \times (y_1 - \epsilon_1, y_1 + \epsilon_1)\big)$. Since $C \subset [0 \leq \psi < Q_1]$ and $\psi > 0$ in $B_{\epsilon_1}(X_1)$, we deduce from Lemma 6.6 *i)* that $\psi > 0$ in C.

Consider the points $P = (x_0, y_1 + \epsilon_1/2)$ and $P' = (x_0, y_1 - \epsilon_1/2)$ (see Figure 13). Then one of the following situations holds :

$$\begin{array}{lllll} a) & \exists \epsilon_2 \in (0, \epsilon_1/2) & \text{such that} & \psi \geq 0 & \text{in} \quad B_{\epsilon_2}^-(P), \\ b) & \exists \epsilon_2' \in (0, \epsilon_1/2) & \text{such that} & \psi \leq 0 & \text{in} \quad B_{\epsilon_2'}^-(P'). \end{array}$$

Indeed, otherwise we will have a sequence $X_n = (x_n, y_n) \to P$ such that $x_n < x_0$ and $\psi(X_n) < 0$ and a sequence $X_n' = (x_n', y_n') \to P'$ such that $x_n' < x_0$ and $\psi(X_n') > 0$. But by Lemma 6.2, this is impossible.

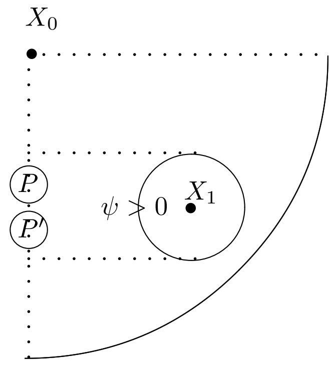

Figure 13

Assume that a) holds. Since we have $\psi \geq 0$ in $B_{\epsilon_2}(P)$, we obtain by applying Lemma 6.6 *i)* to each small ball centered at a point of $\{x_1\} \times (y_P - \epsilon_2, y_P + \epsilon_2)$, that $\psi > 0$ in $B_{\epsilon_2}(P)$ which contradicts $\psi = 0$ on $B_{\epsilon_2}(P) \cap [x = x_0]$.

- Suppose that b) holds. We distinguish two cases

1^{st} Case: There exists $\epsilon'_3 \in (0, \epsilon'_2)$ such that $\psi \equiv 0$ in $B^-_{\epsilon'_3}(P')$
In this case, we have $\psi \geq 0$ in $B_{\epsilon'_3}(P')$ and we get a contradiction as before.

2^{nd} Case: $\forall \eta \in (0, \epsilon'_2)$, $\exists X_\eta \in B^-_\eta(P')$ such that $\psi(X_\eta) < 0$.

In this case, there exists a sequence $X_n = (x_n, y_n) \in B^-_{\epsilon'_2}(P')$ such that $X_n \to P'$ and $\psi(X_n) < 0$. By continuity there exists $\epsilon_n > 0$ such that $\psi < 0$ in $B_{\epsilon_n}(X_n) \subset B^-_{\epsilon'_2}(P')$. Since $\psi \leq 0$ in $B^-_{\epsilon'_2}(P')$, we deduce from Lemma 6.6 *ii)* that $\psi < 0$ in $C_n = \big(B_{\epsilon_n}(X_n) \cup (x_0 - \epsilon'_2, x_n) \times (y_n - \epsilon_n, y_n + \epsilon_n)\big) \cap B^-_{\epsilon'_2}(P')$. By monotonicity we deduce that $\psi < 0$ in $D_n = \big((x_0 - \epsilon'_2, x_n) \times (y_1 - \epsilon_1, y_n + \epsilon_n)\big) \cap B^-_{\epsilon'_2}(P')$. Since for each $X \in B^-_{\epsilon'_2}(P') \cap [y < y_{P'}]$, there exists $n \geq 1$ such that $X \in D_n$, we obtain $\psi < 0$ in $B^-_{\epsilon'_2}(P') \cap [y < y_{P'}]$. We have reached a contradiction with Lemma 6.3, since $\psi > 0$ in $B^+(P', \epsilon'_2)$. $\square$

Lemma 6.8. *Let $X_0 = (x_0, y_0) \in \Omega$, and $B_\epsilon(X_0) \subset [\psi < Q_1]$ $(\epsilon_0 > 0)$.*

If $\psi \leq 0$ in $B^+_{\epsilon_0}(X_0) \cap [y \geq y_0]$ and $\psi = 0$ in $B_{\epsilon_0}(X_0) \cap [y \geq y_0] \cap [x = x_0]$, then $\psi \equiv 0$ in $B^+_{\epsilon_0}(X_0) \cap [y \geq y_0]$.

Proof. We argue by contradiction and assume that there exists $X_1 = (x_1, y_1) \in V = B^+_\epsilon(X_0) \cap [y > y_0]$ such that $\psi(X_1) < 0$. By continuity of ψ, there exists $\epsilon_1 > 0$ small enough such that

$\psi < 0$ in $B_{\epsilon_1}(X_1) \subset V$.
Let $C = B_{\epsilon_1}(X_1) \cup (x_0, x_1) \times (y_1 - \epsilon_1, y_1 + \epsilon_1)$. Since $C \subset [\psi \le 0]$ and $\psi < 0$ in $B_{\epsilon_1}(X_1)$, we deduce from Lemma 6.6 *ii)* that $\psi < 0$ in C.

Consider the points $P = (x_0, y_1 + \epsilon_1/2)$ and $P' = (x_0, y_1 - \epsilon_1/2)$ (see Figure 14). Then as in the proof of Lemma 6.7, one of the following situations holds

$$\begin{array}{llllll} a) & \exists \epsilon_2 \in (0, \epsilon_1/2) & \text{such that} & \psi \ge 0 & \text{in} & B^-_{\epsilon_2}(P), \\ b) & \exists \epsilon'_2 \in (0, \epsilon_1/2) & \text{such that} & \psi \le 0 & \text{in} & B^-_{\epsilon'_2}(P'). \end{array}$$

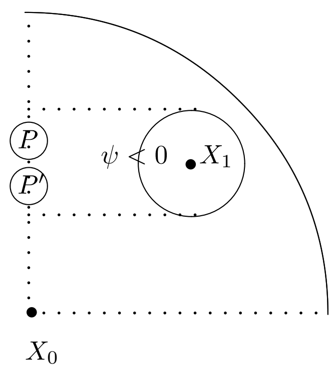

Figure 14

Assume that b) holds. Since $\psi \le 0$ in $B_{\epsilon'_2}(P')$, we obtain by applying Lemma 6.6 *ii)* to each small ball centered at a point of $\{x_1\} \times (y_{P'} - \epsilon'_2, y_{P'} + \epsilon'_2)$, that $\psi < 0$ in $B_{\epsilon'_2}(P')$ which contradicts $\psi = 0$ on $B_{\epsilon'_2}(P') \cap [x = x_0]$.

- Suppose that a) holds. We distinguish two cases

1^{st} Case: There exists $\epsilon_3 \in (0, \epsilon_2)$, such that $\psi \equiv 0$ in $B^-_{\epsilon_3}(P)$.
In this case, we have $\psi \le 0$ in $B_{\epsilon_3}(P)$ and we get a contradiction as above.

2^{nd} Case: $\forall \eta \in (0, \epsilon_2)$, $\exists X_\eta \in B^-_\eta(P)$ such that $\psi(X_\eta) > 0$.

So there exists a sequence $(X_n)_n$ in $B^-_{\epsilon_2}(P)$ such that $\psi(X_n) > 0$ for all $n \ge 1$ and $X_n \to P$. Using the monotonicity and the continuity of ψ, one can prove as in the proof of Lemma 6.7 that $\psi > 0$ in $B^-_{\epsilon_2}(P) \cap [y > y_P]$ which is in contradiction with Lemma 6.3 since $\psi < 0$ in $B^+_{\epsilon_2}(P)$ and $\psi = 0$ on $B^+_{\epsilon_2}(P) \cap [x = x_0]$. $\square$

Proof of Lemma 6.4. First of all, we deduce from the continuity and the monotonicity of ψ that there exists $\epsilon_0 \in (0, y_2 - y_1)$ such that $(x_0 - \epsilon_0, x_0 + \epsilon_0) \times (y_1 - \epsilon_0, y_2 + \epsilon_0) \subset [\psi < Q_1]$.
Let $T_1 = (x_0, y_1)$ and $T_2 = (x_0, y_2)$. Arguing as in the proof of Lemma 6.7, one can prove that one of the following situations holds

$$\begin{array}{llllll} i) & \exists \epsilon \in (0, \epsilon_0) & \text{such that} & \psi \geq 0 & \text{in} & B_\epsilon^+(T_2), \\ ii) & \exists \epsilon \in (0, \epsilon_0) & \text{such that} & \psi \leq 0 & \text{in} & B_\epsilon^+(T_1). \end{array}$$

We claim that

$$\psi \equiv 0 \quad \text{in} \quad (x_0, x_0 + \epsilon) \times (y_1, y_2). \tag{6.10}$$

Indeed let us assume that i) holds. Then by Lemma 6.7, we have $\psi \equiv 0$ in $B_\epsilon^+(T_2) \cap [y \leq y_2]$. By the monotonicity of ψ, we obtain $\psi \leq 0$ in $(x_0, x_0 + \epsilon) \times (0, y_2)$. Applying Lemma 6.8, we obtain $\psi \equiv 0$ in $B_\epsilon^+(T_1) \cap [y \geq y_1]$. We conclude by the monotonicity of ψ that (6.10) holds.

Now assume that *ii*) holds. Then by Lemma 6.8, we have $\psi \equiv 0$ in $B_\epsilon^+(T_1) \cap [y \geq y_1]$. By the monotonicity of ψ, we obtain $\psi \geq 0$ in $(x_0, x_0 + \epsilon) \times (y_1, H)$. Applying Lemma 6.7, we obtain $\psi \equiv 0$ in $B_\epsilon^+(T_2) \cap [y \leq y_2]$.
We conclude by the monotonicity of ψ that (6.10) holds.

It remains to prove that $\psi \equiv 0$ in $(x_0, a) \times [y_1, y_2]$.
Set $I = \{\epsilon \in (0, a - x_0)$ such that $\psi \equiv 0$ in $(x_0, x_0 + \epsilon) \times [y_1, y_2]\ \}$. I is a bounded nonempty set because of (6.10). Let $\rho = \sup I$. We have $0 < \rho \leq a - x_0$ and it is not difficult to verify that $\psi \equiv 0$ in $(x_0, x_0 + \rho) \times [y_1, y_2]$.

Now assume that $\rho < a - x_0$. So $x_0 + \rho < a$. Arguing as above, there exists $\eta_1 > 0$ such that we have $\psi \equiv 0$ in $(x_0 + \rho, x_0 + \rho + \eta_1) \times [y_1, y_2]$. Then $\psi \equiv 0$ in $(x_0, x_0 + \rho + \eta_1) \times [y_1, y_2]$ which contradicts $\rho = \sup I$. Thus $\rho = a - x_0$ and $\psi \equiv 0$ in $(x_0, a) \times [y_1, y_2]$. □

Lemma 6.9. *Assume that ψ is constant in $Z = (x_0, a) \times (y_1, y_2) \subset \Omega$. Then we have :*

i) If $y_2 \leq h_2$, then $\gamma \equiv 0$ in Z.

ii) If $h_2 < y_1 < y_2 \leq h_1$, then $\gamma \equiv -\delta$ in Z.

iii) If $y_1 > h_1$, then $\gamma \equiv -\delta_2$ in Z.

Proof. Since ψ is constant in Z, we deduce from (3.1) that $\gamma = \gamma(y)$ in Z.

i) Assume that $y_2 \leq h_2$ and let $\zeta \in C^\infty(Z)$ such that $\zeta = 0$ on $\partial Z \cap \Omega$. Using $\chi(Z)\zeta$ as a test function for (P'), we obtain $\displaystyle\int_Z -\gamma(y)\zeta_x = 0$ which leads to $\displaystyle\int_{y_1}^{y_2} \gamma(y)\zeta(a, y)dy = 0$. Therefore $\gamma = \gamma(y) \equiv 0$ in Z.

In the same way we prove *ii*) and *iii*). □

Proof of Theorem 6.4. Assume that there exists $x_0 \in (0, a)$ such that $f_1(x_0) < f_2(x_0)$. We claim that

$$
\begin{aligned}
&i) \quad \forall x \in [x_0, a) \quad f_1(x) < f_2(x)\\
&ii) \quad f_1 \text{ is non-increasing in } [x_0, a)\\
&iii) \quad f_2 \text{ is non-decreasing in } [x_0, a).
\end{aligned}
$$

Indeed, we have
i)

$$\psi(x_0, y) = 0 \quad \forall y \in [f_1(x_0), f_2(x_0)].$$

From Lemma 6.4, we deduce that $\psi \equiv 0$ in $(x_0, a) \times [f_1(x_0), f_2(x_0)]$. In particular, we have for $x \geq x_0$, $f_1(x) \leq f_1(x_0)$ and $f_2(x) \geq f_2(x_0)$. Hence $f_1(x) < f_2(x)$.

ii), iii) Let $x_1, x_2 \in [x_0, a)$ with $x_1 < x_2$. By *i)* and Lemma 6.4, we have $\psi \equiv 0$ in $[x_1, a) \times [f_1(x_1), f_2(x_1)]$. In particular we have $\psi(x_2, f_1(x_1)) = 0$ which leads to $f_1(x_2) \leq f_1(x_1)$.

We also have $\psi(x_2, f_2(x_1)) = 0$ which leads to $f_2(x_1) \leq f_2(x_2)$.

So $l_1 = \lim_{x \to a^-} f_1(x)$ and $l_2 = \lim_{x \to a^-} f_2(x)$ exist with $l_1 \leq l_2$. Moreover since $l_1 \leq f_1(x) < f_2(x) \leq l_2$ for all $x \in [x_0, a)$, we deduce that $l_1 < l_2$. We will distinguish three cases:

a) $\underline{h_2 < l_2 \leq h_1}$:

Let $\epsilon > 0$ small enough such that $l_2 - \epsilon > \max(h_2, l_1 + \epsilon)$. There exists $\eta > 0$ such that

$$\begin{cases} l_1 \leq f_1(x) \leq l_1 + \epsilon < l_2 - \epsilon \leq f_2(x) \leq l_2 \quad \forall x \in (a - \eta, a) \\ \psi < Q_1 \quad \text{in } Z_\eta = (a - \eta, a) \times (l_2 - \epsilon, l_2). \end{cases}$$

Since $\psi \equiv 0$ in $Z_\eta \cap [y < f_2(x)]$ and $h_2 < l_2 - \epsilon < l_2 \leq h_1$, we have by Lemma 6.9 $\gamma = -\delta$ in $Z_\eta \cap [y < f_2(x)]$.
Now since $0 \leq \psi < Q_1$ in Z_η, we deduce from Lemma 6.5 *i)* that $\gamma_x \geq 0$ in Z_η. Moreover $-\delta \leq \gamma \leq 0$ in Z_η. Hence $\gamma = -\delta$ in Z_η. But this leads to $\Delta_q \psi = 0$ in Z_η. By the maximum principle, we obtain $\psi \equiv 0$ in Z_η or $\psi > 0$ in Z_η which is impossible.

b) $\underline{l_2 \leq h_2}$:
Let $\epsilon > 0$ small enough such that $l_1 + \epsilon < l_2 - \epsilon$. There exists $\eta > 0$ such that

$$l_1 \leq f_1(x) \leq l_1 + \epsilon < l_2 - \epsilon \leq f_2(x) \leq l_2 \quad \forall x \in (a - \eta, a).$$

Let $Z_\eta = (a - \eta, a) \times (l_1, l_1 + \epsilon)$. Since $\psi \equiv 0$ in $Z_\eta \cap [y > f_1(x)]$ and $l_1 + \epsilon < l_2 \leq h_2$, we have by Lemma 6.9, $\gamma = 0$ in $Z_\eta \cap [y > f_1(x)]$.
Now since $\psi \leq 0$ in Z_η, we deduce from Lemma 6.5 *ii)* that $\gamma_x \leq 0$ in Z_η. Moreover $-\delta \leq \gamma \leq 0$ in Z_η. Hence $\gamma = 0$ in Z_η. But this leads to $\Delta_q \psi = 0$ in Z_η and we get a contradiction with the maximum principle.

c) $\underline{l_2 > h_1}$:
Let $\epsilon > 0$ small enough such that $l_2 - \epsilon > h_1$ and $l_1 + \epsilon < l_2 - \epsilon$. There exists $\eta > 0$ such that

$$\begin{cases} l_1 \le f_1(x) \le l_1 + \epsilon < l_2 - \epsilon \le f_2(x) \le l_2 \quad \forall x \in (a-\eta, a) \\ \psi < Q_1 \quad \text{in } Z_\eta = (a-\eta, a) \times (l_2 - \epsilon, l_2). \end{cases}$$

Arguing as in $i)$, we obtain $\gamma = -\delta_2$ in $Z_\eta \cap [y < f_2(x)]$ which leads to a contradiction with $\gamma_y \le 0$ and $\gamma = -\delta$ in $Z_\eta \cap [y > f_2(x)]$.

Hence $f_1(x_0) = f_2(x_0)$ for all $x_0 \in (0, a)$. □

Proposition 6.3. $\lim_{x \to 0^+} f(x) = f(0_+)$ *and* $\lim_{x \to a^-} f(x) = f(a_-)$ *exist and belong to* $(0, H)$.

Proof. $i)$ Let $l_1 = \liminf_{x \to 0^+} f(x)$ and $l_2 = \limsup_{x \to 0^+} f(x)$. We have $l_1 \le l_2$ and there exists in $(0, a)$ two sequences (x_n^1) and (x_n^2) such that

$$\lim_{n \to +\infty} x_n^i = 0 \quad \text{and} \quad \lim_{n \to +\infty} f(x_n^i) = l_i, \quad i = 1, 2.$$

Since $\psi \in C^{0,\alpha}(\bar{\Omega})$, we have $\psi(0, l_i) = \lim_{n \to +\infty} \psi(x_n^i, f(x_n^i)) = 0$. So $l_i \in (0, H)$.
Assume that $l_1 < l_2$ and let $\epsilon \in (0, (l_2 - l_1)/2)$. There exists $n_0 \ge 1$ such that $\forall n \ge n_0$ $f(x_n^1) < l_1 + \epsilon$ and $f(x_n^2) > l_2 - \epsilon$. We get $\psi(x_{n_0}^1, y) > 0$ for $y > l_1 + \epsilon > f(x_{n_0}^1)$ and $\psi(x_{n_1}^2, y) < 0$ for $y < l_2 - \epsilon < f(x_{n_1}^2)$, where $n_1 > n_0$ and $x_{n_1}^2 < x_{n_0}^1$.
Let $n_3 > n_2 > n_1$ such that $x_{n_3}^2 < x_{n_2}^1 < x_{n_1}^2 < x_{n_0}^1$. Then we have $\psi(x_{n_2}^1, y) > 0$ for $y > l_1 + \epsilon > f(x_{n_2}^1)$ and $\psi(x_{n_3}^2, y) < 0$ for $y < l_2 - \epsilon < f(x_{n_3}^2)$.
This is a contradiction with Lemma 6.2. Hence $l_1 = l_2$.

$ii)$ In the same way we prove that $\lim_{x \to a^-} f(x) = f(a_-)$ exists and belongs to $(0, H)$. □

Corollary 6.1. *Let* $(\psi, \gamma, \widetilde{\gamma})$ *be a solution of the Problem* (P'). *Then we have*

$i)$ $\gamma = -\delta\chi([0 < \psi < Q_1]) - \delta_2\chi([\psi = Q_1])$ *a.e. in* Ω.

$ii)$ $\widetilde{\gamma} = -\delta\chi([f(0_-) < y < g(0_-)]) - \delta_2\chi([y > g(0_-)])$ *a.e. in* $(0, H)$.

6.4 Comparison and Uniqueness of the solution

In this last section, we assume that $q = 2$. We know from [6] that there exists a monotone solution $(\psi, \gamma, \widetilde{\gamma})$ in the sense that $\psi_x \ge 0$, $\psi_y \ge 0$, $\gamma_x \le 0$, $\gamma_y \le 0$ in $\mathcal{D}'(\Omega)$. It is then not difficult to establish that the function $f(x)$ describing the lower free boundary is a decreasing function and therefore a one-to-one function from $(0, a)$ to $(f(0_+), f(a_-))$. We shall prove here that such monotone solutions of the problem (P') decrease with respect to Q_2, and as a consequence, we obtain the uniqueness of this type of solution. Let us denote the problem corresponding to Q_2, by $(P'(Q_2))$. Then we have the following comparison result

Theorem 6.5. *Assume that $q = 2$ and let $(\psi_1, \gamma_1, \widetilde{\gamma}_1)$ and $(\psi_2, \gamma_2, \widetilde{\gamma}_2)$ be two monotone solutions of the problems $(P'(Q_2^1))$ and $(P'(Q_2^2))$ respectively. If $Q_2^1 \geq Q_2^2$, then we have $\psi_1 \leq \psi_2$, $\gamma_1 \geq \gamma_2$ a.e. in Ω, and $\widetilde{\gamma}_1 \geq \widetilde{\gamma}_2$ a.e. in $\widehat{AA_0}$.*

To prove Theorem 6.5, we need three Lemmas.

Lemma 6.10. *Let $(\psi, \gamma, \widetilde{\gamma})$ be a monotone solution of (P'). Then we have*

$$\int_\Omega \big(|\nabla\psi|^{q-2}\nabla\psi - \gamma e_x\big).\nabla\zeta - \int_{\widehat{AA_0}} \widetilde{\gamma}\zeta \;\geq\; \delta\int_{\widehat{B_2B_1}} \zeta + \delta_2\int_{\widehat{B_1B_0}} \zeta$$
$$\forall\zeta \in W^{1,q}(\Omega), \quad \zeta = 0 \quad on \quad \widehat{AB}, \qquad \zeta \geq 0 \quad on \quad \widehat{A_0B_0}\ .$$

Proof. Let $\zeta \in W^{1,q}(\Omega)$, $\zeta = 0$ on $\widehat{AB}$, $\zeta \geq 0$ on $\widehat{A_0B_0}$. Then for $\epsilon > 0$, $\min\Big(\zeta, \dfrac{H-y}{\epsilon}\Big)$ is a test function for (P') and we have

$$\int_\Omega \big(|\nabla\psi|^{q-2}\nabla\psi - \gamma e_x\big).\nabla\Big(\min\Big(\zeta, \frac{H-y}{\epsilon}\Big)\Big) - \int_{\widehat{AA_0}} \widetilde{\gamma}\min\Big(\zeta, \frac{H-y}{\epsilon}\Big) =$$
$$= \delta\int_{\widehat{B_2B_1}} \min\Big(\zeta, \frac{H-y}{\epsilon}\Big) + \delta_2\int_{\widehat{B_1B_0}} \min\Big(\zeta, \frac{H-y}{\epsilon}\Big)$$

which can be written since $\psi_y \geq 0$ a.e. in Ω

$$\int_{[H-y\geq\epsilon\zeta]} \big(|\nabla\psi|^{q-2}\nabla\psi - \gamma e_x\big).\nabla\zeta - \int_{\widehat{AA_0}} \widetilde{\gamma}\min\Big(\zeta, \frac{H-y}{\epsilon}\Big) =$$
$$= \frac{1}{\epsilon}\int_{[H-y<\epsilon\zeta]} |\nabla\psi|^{q-2}\psi_y + \delta\int_{\widehat{B_2B_1}} \min\Big(\zeta, \frac{H-y}{\epsilon}\Big) + \delta_2\int_{\widehat{B_1B_0}} \min\Big(\zeta, \frac{H-y}{\epsilon}\Big)$$
$$\geq \delta\int_{\widehat{B_2B_1}} \min\Big(\zeta, \frac{H-y}{\epsilon}\Big) + \delta_2\int_{\widehat{B_1B_0}} \min\Big(\zeta, \frac{H-y}{\epsilon}\Big)$$

Letting $\epsilon \to 0$, we obtain

$$\int_\Omega \big(|\nabla\psi|^{q-2}\nabla\psi - \gamma e_x\big).\nabla\zeta - \int_{\widehat{AA_0}} \widetilde{\gamma}\zeta \;\geq\; \delta\int_{\widehat{B_2B_1}} \zeta + \delta_2\int_{\widehat{B_1B_0}} \zeta.$$

□

Lemma 6.11. *Under the assumptions of Theorem 6.5, we have*

$$\mathcal{T}(\zeta) = \int_\Omega \big(\nabla(\psi_1 - \psi_m) - (\gamma_1 - \gamma_M)e_x\big).\nabla\zeta \leq \delta_1\int_I \zeta(f_2(y), y)dy \quad \forall\zeta \in \mathcal{D}(\mathbb{R}^2), \quad \zeta \geq 0,$$

where $\psi_m = \min(\psi_1, \psi_2)$, $\gamma_M = \max(\gamma_1, \gamma_2)$ *and* $I = \{y \in (0,H) \,/\, \max(f_2^{-1}(y), \phi_1(y)) < \phi_2(y)\,\}$.

Proof. Let $\zeta \in \mathcal{D}(\mathbb{R}^2)$, $\zeta \geq 0$ and $\epsilon > 0$. Then if we take $\xi = \min\Big(\zeta, \dfrac{\psi_1 - \psi_m}{\epsilon}\Big)$ as a test function in $(P'(Q_2^i))$, $i = 1, 2$ and subtract the two equations from one another, we obtain

$$\int_\Omega \big(\nabla(\psi_1 - \psi_2) - (\gamma_1 - \gamma_2)e_x\big).\nabla\xi = \int_{\widehat{AA_0}} (\widetilde{\gamma_1} - \widetilde{\gamma_2})\xi.$$

Since $\widetilde{\gamma_i} \in H(\psi_i)$ and $-H$ is a maximal monotone graph, we have $(\widetilde{\gamma_1} - \widetilde{\gamma_2}).(\psi_1 - \psi_2) \leq 0$ a.e. in $\widehat{AA_0}$. Then

$$\int_\Omega \big(\nabla(\psi_1 - \psi_m) - (\gamma_1 - \gamma_M)e_x\big).\nabla\xi \leq 0$$

which we can write as

$$\begin{aligned}
&\int_{\Omega\cap[\psi_1 - \psi_m \geq \epsilon\zeta]} \nabla(\psi_1 - \psi_m).\nabla\zeta - \int_\Omega (\gamma_1 - \gamma_M)e_x.\nabla\zeta \\
&\leq -\int_\Omega (\gamma_1 - \gamma_M)\Big(\zeta - \frac{\psi_1 - \psi_m}{\epsilon}\Big)^+_x = I_{1\epsilon} + I_{2\epsilon} + I_{3\epsilon} + I_{4\epsilon} + I_{5\epsilon} + I_{6\epsilon}.
\end{aligned} \tag{6.11}$$

In $[\psi_m = Q_1]$, we have $\psi_1 = \psi_2 = Q_1$ and then $\gamma_1 = \gamma_2 = \gamma_M = -\delta_2$ in $[\psi_m = Q_1]$. So

$$I_{1\epsilon} = -\int_{[\psi_m = Q_1]} (\gamma_1 - \gamma_M)\Big(\zeta - \frac{\psi_1 - \psi_m}{\epsilon}\Big)^+_x = 0. \tag{6.12}$$

In $[\psi_1 < 0]$, one has $\gamma_1 = 0$ and then $\gamma_M = \max(\gamma_1, \gamma_2) = 0$ in $[\psi_1 < 0]$. So

$$I_{2\epsilon} = -\int_{[\psi_1 < 0]} (\gamma_1 - \gamma_M)\Big(\zeta - \frac{\psi_1 - \psi_m}{\epsilon}\Big)^+_x = 0. \tag{6.13}$$

In $[\psi_m < 0] \cap [0 < \psi_1 < Q_1]$, one has $\gamma_1 = -\delta$, $\psi_m = \min(\psi_1, \psi_2) = \psi_2 < 0$, and then $\gamma_M = \max(\gamma_1, \gamma_2) = 0$ in $[\psi_m < 0] \cap [0 < \psi_1 < Q_1]$. So

$$\begin{aligned}
I_{3\epsilon} = -\int_{[\psi_m < 0]\cap[0<\psi_1<Q_1]} (\gamma_1 - \gamma_M)\Big(\zeta - \frac{\psi_1 - \psi_m}{\epsilon}\Big)^+_x &= \delta \int_{[\psi_2 < 0]\cap[0<\psi_1<Q_1]} \Big(\zeta - \frac{\psi_1 - \psi_2}{\epsilon}\Big)^+_x \\
&= \delta \int_{I_1} \Big(\int_{f_1^{-1}(y)}^{\min(\phi_1(y), f_2^{-1}(y))} \Big(\zeta - \frac{\psi_1 - \psi_2}{\epsilon}\Big)^+_x dx\Big)dy \\
&\leq \delta \int_{I_1} \Big(\zeta - \frac{\psi_1 - \psi_2}{\epsilon}\Big)^+ (\min(\phi_1(y), f_2^{-1}(y)), y)dy
\end{aligned}$$

where $I_1 = \{y \in (0,H) \,/\, f_1^{-1}(y) < \min(\phi_1(y), f_2^{-1}(y))\,\}$.

Note that for $y \in I_1$, we have $\psi_1(\min(\phi_1(y), f_2^{-1}(y)), y) > 0$ and $\psi_2(\min(\phi_1(y), f_2^{-1}(y)), y) \leq 0$, and then $(\psi_1 - \psi_2)(\min(\phi_1(y), f_2^{-1}(y)), y) > 0$. This leads to

$$\limsup_{\epsilon \to 0} I_{3\epsilon} \leq 0. \tag{6.14}$$

In $[\psi_m < 0] \cap [\psi_1 = Q_1]$, one has $\gamma_1 = -\delta_2$, $\psi_m = \psi_2 < 0$, $\gamma_2 = 0$, and then $\gamma_M = 0$ in $[\psi_m < 0] \cap [\psi_1 = Q_1]$. So

$$\begin{aligned} I_{4\epsilon} = -\int_{[\psi_m<0]\cap[\psi_1=Q_1]} (\gamma_1 - \gamma_M)\Big(\zeta - \frac{\psi_1 - \psi_m}{\epsilon}\Big)_x^+ = \delta_2 \int_{[\psi_m<0]\cap[\psi_1=Q_1]} \Big(\zeta - \frac{\psi_1 - \psi_m}{\epsilon}\Big)_x^+ \\ = \delta_2 \int_{I_2} \Big(\int_{\phi_1(y)}^{f_2^{-1}(y)} \Big(\zeta - \frac{Q_1 - \psi_2}{\epsilon}\Big)_x^+ dx\Big) dy \leq \delta_2 \int_{I_2} \Big(\zeta - \frac{Q_1}{\epsilon}\Big)^+ (f_2^{-1}(y), y) dy \end{aligned}$$

where $I_2 = \{y \in (0, H) \,/\, \phi_1(y) < f_2^{-1}(y)\,\}$.
It is then clear that

$$\limsup_{\epsilon \to 0} I_{4\epsilon} \leq 0. \tag{6.15}$$

In $[0 < \psi_m < Q_1] \cap [0 < \psi_1 < Q_1]$, one has $\gamma_1 = -\delta$. Since $\psi_2 \geq \psi_m > 0$, $\gamma_2 \in [-\delta_2, -\delta]$, and so $\gamma_M = \max(\gamma_1, \gamma_2) = -\delta$ in $[0 < \psi_m < Q_1] \cap [0 < \psi_1 < Q_1]$. Then

$$I_{5\epsilon} = -\int_{[0<\psi_m<Q_1]\cap[0<\psi_1<Q_1]} (\gamma_1 - \gamma_M)\Big(\zeta - \frac{\psi_1 - \psi_m}{\epsilon}\Big)_x^+ = 0. \tag{6.16}$$

In $[0 < \psi_m < Q_1] \cap [\psi_1 = Q_1]$, one has $\gamma_1 = -\delta_2$, $\psi_m = \psi_2 \in (0, Q_1)$, $\gamma_2 = -\delta$, and then $\gamma_M = -\delta$ in $[0 < \psi_m < Q_1] \cap [\psi_1 = Q_1]$. So

$$\begin{aligned} I_{6\epsilon} &= -\int_{[0<\psi_m<Q_1]\cap[\psi_1=Q_1]} (\gamma_1 - \gamma_M)\Big(\zeta - \frac{\psi_1 - \psi_m}{\epsilon}\Big)_x^+ \\ &= \delta_1 \int_{[0<\psi_m<Q_1]\cap[\psi_1=Q_1]} \Big(\zeta - \frac{\psi_1 - \psi_m}{\epsilon}\Big)_x^+ \\ &= \delta_1 \int_{I_3} \Big(\int_{\max(f_2^{-1}(y), \phi_1(y))}^{\phi_2(y)} \Big(\zeta - \frac{Q_1 - \psi_2}{\epsilon}\Big)_x^+ dx\Big) dy \\ &\leq \delta_1 \int_{I_3} \zeta(\phi_2(y), y) dy \end{aligned}$$

where $I_3 = \{y \in (0, H) \,/\, \max(f_2^{-1}(y), \phi_1(y)) < \phi_2(y)\,\}$.
Then we have

$$\limsup_{\epsilon \to 0} I_{6\epsilon} \leq \delta_1 \int_{I_3} \zeta(\phi_2(y), y) dy. \tag{6.17}$$

Finally, we get from (6.11)-(6.17) that

$$\mathcal{T}(\zeta) \leq \delta_1 \int_{I_3} \zeta(\phi_2(y), y)dy.$$

$\square$

Lemma 6.12. *Under the assumptions of Theorem 6.5, we have*

$$\mathcal{T}(\zeta) = \int_{\Omega} \big(\nabla(\psi_1 - \psi_m) - (\gamma_1 - \gamma_M)e_x\big).\nabla\zeta = 0 \qquad \forall \zeta \in H^1(\Omega). \tag{6.18}$$

Proof. Let $\zeta \in \mathcal{D}(\mathbb{R}^2)$, $\zeta \geq 0$ and $\epsilon > 0$. Let $\alpha_\epsilon(x,y) = \Big(1 - \dfrac{d((x,y),\Lambda)}{\epsilon}\Big)^+$, with $\Lambda = [\psi_1 < Q_1]$. We have $1 - \alpha_\epsilon = 0$ in $\bar{\Lambda}$ and $\mathcal{T}(\zeta) = \mathcal{T}(\alpha_\epsilon\zeta) + \mathcal{T}((1-\alpha_\epsilon)\zeta)$. By the previous lemma which is also true for $\zeta \in H^1(\Omega) \cap C^0(\bar{\Omega})$, $\zeta \geq 0$, we have

$$\mathcal{T}(\alpha_\epsilon\zeta) \leq \delta_1 \int_{I_3} (\alpha_\epsilon\zeta)(\phi_2(y), y)dy.$$

Note that $\Lambda = [x < \phi_1(y)]$, and for $y \in I_3$, we have $\phi_2(y) > \max(f_2^{-1}(y), \phi_1(y)) \geq \phi_1(y)$. It follows that for $y \in I_3$, we have $(\phi_2(y), y) \notin \bar{\Lambda}$ and then

$$\limsup_{\epsilon \to 0} \mathcal{T}(\alpha_\epsilon\zeta) \leq 0. \tag{6.19}$$

Moreover, we have by Lemma 6.10 applied to $(\psi_2, \gamma, \widetilde{\gamma}_2)$ and $(1-\alpha_\epsilon)\zeta$

$$\begin{aligned}
\mathcal{T}((1-\alpha_\epsilon)\zeta) &= \int_{\Omega} \big(\nabla\psi_1 - \gamma_1 e_x\big).\nabla((1-\alpha_\epsilon)\zeta) - \int_{\Omega} \big(\nabla\psi_m - \gamma_M e_x\big).\nabla((1-\alpha_\epsilon)\zeta) \\
&= \delta_2 \int_{\Omega} ((1-\alpha_\epsilon)\zeta)_x - \int_{\Omega} \big(\nabla\psi_2 - \gamma_2 e_x\big).\nabla((1-\alpha_\epsilon)\zeta) \\
&\leq \delta_2 \int_{\partial\Omega} (1-\alpha_\epsilon)\zeta\nu_x - \int_{\widehat{AA_0}} \widetilde{\gamma}_2(1-\alpha_\epsilon)\zeta \\
&\quad -\delta \int_{\widehat{B_2B_1}} (1-\alpha_\epsilon)\zeta - \delta_2 \int_{\widehat{B_1B_0}} (1-\alpha_\epsilon)\zeta \\
&= -\int_{\widehat{AA_0}} (\widetilde{\gamma}_2 + \delta_2)(1-\alpha_\epsilon)\zeta \leq 0
\end{aligned} \tag{6.20}$$

since $\widetilde{\gamma}_2 \geq -\delta_2$ a.e. in $\widehat{AA_0}$ and $\psi_1 < Q_1$ on $\widehat{BB_1}$.
It follows from (6.19)-(6.20) that $\mathcal{T}(\zeta) \leq 0$.
Now let $\zeta \in \mathcal{D}(\mathbb{R}^2)$, $\zeta \geq 0$. Let $M = \sup_{\mathbb{R}^2}\zeta$ and $\xi \in \mathcal{D}(\mathbb{R}^2)$ such that $\xi \geq 0$ and $\xi = 1$ in $\bar{\Omega}$. Since $\xi.(M-\zeta) \in \mathcal{D}(\mathbb{R}^2)$ and $\xi.(M-\zeta) \geq 0$, we obtain $\mathcal{T}(\xi.(M-\zeta)) \leq 0$ which can be written $\mathcal{T}(\zeta) \geq 0$. So we obtain $\mathcal{T}(\zeta) = 0$ $\forall\zeta \in \mathcal{D}(\mathbb{R}^2)$, $\zeta \geq 0$ and by density for $\zeta \geq 0$ in $H^1(\Omega)$.

If $\zeta \in H^1(\Omega)$, we write $\zeta = \zeta^+ - \zeta^-$. Then $\mathcal{T}(\zeta) = \mathcal{T}(\zeta^+) - \mathcal{T}(\zeta^-) = 0$. $\square$

Proof of Theorem 6.5. Let $(\psi_1, \gamma_1, \widetilde{\gamma_1})$ and $(\psi_2, \gamma, \widetilde{\gamma_2})$ be two solutions of the Problems $(P'(Q_{2,1}))$ and $(P'(Q_{2,2}))$ with $Q_{2,1} \geq Q_{2,2}$. Writing (6.18) for $\zeta = \frac{y^2}{2}$, we get

$$\int_\Omega y(\psi_1 - \psi_m)_y = 0.$$

Integrating by part and using the fact that $\psi_1 - \psi_m = 0$ for $y = 0$ and for $y = H$, we obtain

$$\int_\Omega (\psi_1 - \psi_m) = 0.$$

Since $\psi_1 - \psi_m \geq 0$ in Ω, we obtain $\psi_1 = \psi_m$ in Ω. This means that $\psi_1 \leq \psi_2$ in Ω. As a consequence, we obtain by Corollary 6.1 that $\gamma_1 \geq \gamma_2$ in Ω and $\widetilde{\gamma}_1 \geq \widetilde{\gamma}_2$ in $\frown{AA_0}$. $\square$

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