The Dam Problem

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Introduction

The steady state dam problem consists of studying the filtration of a fluid (say water) through a porous medium Ω assuming an equilibrium has been reached. Then we look for the saturated region S (see Figure 1) and the fluid pressure p inside Ω. We are also concerned with the regularity of the interface that separates wet and dry regions, called free boundary.

The Variational Inequalities Approach by Baiochi

The study of this problem goes back to the early seventies with the pioneering work of Baiocchi [7], [8], who is probably the first one to solve this problem in the case of a rectangular dam. By introducing the transformation $u(x_1,x_2)=\int_0^H p(x_1,t)dt$, where H is the height of the dam, he showed that u is the unique solution of the variational inequality

$$\begin{cases} \text{Find } u \in I\!\!K_1 \text{ such that:} \\ \int_\Omega \nabla u.\nabla(\zeta-u) \geq -\int_\Omega(\zeta-u) \qquad \forall\zeta\in I\!\!K_1, \end{cases}$$

where $I\!\!K_1=\{\zeta\in H^1(\Omega)\,/\zeta=g \text{ on } \partial\Omega, \zeta\geq 0 \text{ in } \Omega\}$, and g is a given Lipschitz continuous function. He also proved that the free boundary is an analytic curve $x_2=\Phi(x_1)$. In [9] and [10], he generalized these results to dams with horizontal bottoms.

For heterogeneous dams, the first results where also established in the rectangular case via the theory of variational inequalities. Indeed using the following generalized Baiocchi's transformation $u(x_1,x_2)=\int_{x_2}^H k_2(s)p(x_1,s)ds$, allowed authors to handle the case of a matrix permeability of the form $k(x_1,x_2)I_2$, where I_2 is the 2×2 unit matrix and $k(x_1,x_2)=k_1(x_1)k_2(x_2)$. Hence Benci proved in [14] that u is the unique solution of the following variational inequality

$$\begin{cases} \text{Find } u \in I\!\!K_2 \text{ such that:} \\ \int_\Omega \frac{k_1(x_1)}{k_2(x_2)}\nabla u.\nabla(\zeta-u) \geq -\int_\Omega k_1(x_1)(\zeta-u) \qquad \forall\zeta\in I\!\!K_2, \end{cases}$$

where $I\!\!K_2=\{\zeta\in H^1(\Omega)\,/\zeta=h \text{ on } \partial\Omega, \zeta\geq 0 \text{ in } \Omega\}$, and h is a given function. He also proved that the free boundary is a curve $x_1=\Psi(x_2)$. When $k_2'(x_2)\leq 0$, he proved that it is a curve of a continuous decreasing function $\Phi(x_1)$. Baiocchi and Friedman [13] extended these results assuming only that $\ln\left(\displaystyle\int_{x_2}^H k_2(t)dt\right)$ is concave. In [19], Caffarelli and Friedman proved that the free boundary is a curve $x_1=\Psi(x_2)$ provided that $k(x_1,x_2)=k_2(x_2)$ is a non-increasing step function.

Note that Baiocchi's transformation was used locally by Alt in [3] to prove the regularity of the free boundary.

The Weak Formulation of Alt and Brezis-Kinderlehrer-Stampacchia

Given that the variational approach is not possible for dams with general geometry, Brezis, Kinderlehrer, and Stampacchia in [17] and independently Alt in [2] introduced the following formulation:

$$
(P_1)\left\{\begin{array}{l}
\text{Find } (p,\chi)\in H^1(\Omega)\times L^\infty(\Omega) \text{ such that :}\\
(i)\quad p\geq 0,\quad 0\leq\chi\leq 1,\quad p(1-\chi)=0\ \text{ a.e. in } \Omega\\
(ii)\quad p=\varphi\quad \text{ on } S_2\cup S_3\\
(iii)\quad \int_\Omega \big(\nabla p+\chi e\big).\nabla\xi dx\ \leq\ 0,\qquad e=(0,1)\\
\qquad\quad \forall\xi\in H^1(\Omega),\quad \xi=0 \text{ on } S_3,\quad \xi\geq 0 \text{ on } S_2
\end{array}\right.
$$

where χ is a bounded function characterizing the wet part of the dam, S_1 is the impervious part of the dam, S_2 is the part in contact with air and S_3 is the part in contact with the reservoirs, φ represents the exterior pressure.

Existence of a solution was proved by approaching χ with an approximation of the Heaviside graph. Regarding the the free boundary, Alt proved in [3], that it is an analytic surface $x_n=\Phi(x_1,...,x_{n-1})$ when Ω is a Lipschitz domain of $I\!R^n$ with $n\geq 2$. The uniqueness of the so-called S_3-connected or reservoirs-connected solution was proved by Carrillo and Chipot in [21] and also by Alt and Gilardi in [6].

In [23], Carrillo and Lyaghfouri considered this problem, assuming the flow governed by the following nonlinear Darcy law [33] $|v|^{m-1}v=-\nabla(p+x_2)$, $m>0$. The authors formulated the problem in terms of the hydrostatic head $u=p+x_2$ instead of the pressure, which led to the following problem:

$$
(P_2)\left\{\begin{array}{l}
\text{Find } (u,g)\in W^{1,q}(\Omega)\times L^\infty(\Omega) \text{ such that :}\\
(i)\quad u\geq x_2,\quad 0\leq g\leq 1,\quad g(u-x_2)=0\ \text{ a.e. in } \Omega\\
(ii)\quad u=\varphi+y\quad \text{ on } S_2\cup S_3\\
(iii)\quad \int_\Omega \big(|\nabla u|^{q-2}\nabla u-ge\big).\nabla\xi dx\ \leq\ 0\\
\qquad\quad \forall\xi\in W^{1,q}(\Omega),\quad \xi=0 \text{ on } S_3,\quad \xi\geq 0 \text{ on } S_2.
\end{array}\right.
$$

Then they showed the existence of a solution, proved the continuity of the free boundary $x_2=\Phi(x_1)$ and the uniqueness of the reservoirs-connected solution in dimension 2. For dimension $n\geq\max(2,q)$, they proved the existence and uniqueness of a minimal solution.

The general heterogeneous dam with general geometry was formulated first in [2] by Alt. In [30] and [43], the authors showed that for a permeability matrix $a(x)=k(x_1,x_2)I_2$ with $\frac{\partial k}{\partial x_2}\geq 0$ in $\mathcal{D}'(\Omega)$, the free boundary is a continuous curve $x_2=\Phi(x_1)$ and the reservoirs-connected solution is unique. These results were generalized by Lyaghfouri [37] to the case where

$$
a(x)=\begin{pmatrix} a_{11}(x) & 0\\ a_{21}(x) & a_{22}(x)\end{pmatrix}\quad \text{and}\quad \frac{\partial a_{22}}{\partial x_2}\geq 0 \text{ in } \mathcal{D}'(\Omega).
$$

The model with leaky boundary condition i.e. when the flow through the reservoirs' bottoms is equal to a function of the difference between exterior and interior pressures,

$$
v.\nu=-\beta(x,\varphi-p)
$$

was considered first in [10] in the rectangular case. The general situation was considered in [22], [41] and in [26] via the following formulation:

$$(P_3)\begin{cases} \text{Find } (p,\chi)\in H^1(\Omega)\times L^\infty(\Omega) \text{ such that :}\\ (i)\quad p\geq 0,\quad 0\leq\chi\leq 1,\quad p(1-\chi)=0 \ \text{ a.e. in } \Omega\\ (ii)\quad p=\varphi \quad \text{on } S_2\\ (iii)\quad \int_\Omega a(x)\big(\nabla p+\chi e\big).\nabla\xi dx\ \leq\ \int_{S_3}\beta(x,\varphi-p)\xi d\sigma(x)\\ \qquad\qquad \forall\xi\in H^1(\Omega),\quad \xi\geq 0 \text{ on } S_2. \end{cases}$$

The continuity of the free boundary and the uniqueness of the reservoirs-connected solution were established in [26] for the two dimensional and homogenous case. In [40], the continuity of the free boundary was extended to heterogeneous media with linear or nonlinear Darcy's laws together with various uniqueness results including the situation corresponding to the linear Darcy law with a diagonal permeability matrix.

The main difference between the model corresponding to Dirichlet condition and the one with leaky condition is the fact that the region below the reservoirs is always saturated in the first model while it is not necessarily the case for the second one. However we know [26], [40], that if $\beta(x,\varphi)\geq a_{22}\nu_2$ on a connected subset T of S_3, where ν_2 is the second entry of the outward unit normal vector ν to $\partial\Omega$, then the region below T is completely saturated provided its lower boundary is impervious.

Differentiability of the free boundary is still an open problem for the heterogeneous case even in dimension 2 except when the permeability does not depend on the variable x_2 [26]. In dimension $n\geq 3$, the continuity of the free boundary is also an open problem except for the homogeneous case with Dirichlet conditions.

Outline of the Chapter

In this study, we shall be concerned mainly with a two dimensional heterogeneous dam with general geometry, assuming the flow governed by a nonlinear Darcy's law. We refer to [24] or [31] for the homogeneous case with flow obeying to the linear Darcy law. For the variational inequalities approach, we refer to [11]. The paper is organized as follows: in Section 1, we derive the weak formulations of the dam problem with Dirichlet Boundary condition and with leaky boundary condition respectively. Then we prove the existence of a solution (u,g,γ) to a unified formulation of the problem. Moreover we show that u is uniformly bounded and locally Hölder continuous in Ω. Under additional assumptions on the permeability, we also establish a monotonicity result for g, which with the continuity of u allows us to define the free boundary as an x_1x_2-graph. In Section 2, we address the case of Dirichlet condition. We prove the continuity of the free boundary and show that g is the characteristic function of the dry part of the dam. Then we prove the uniqueness of the reservoirs-connected solution. In Section 3, we consider the leaky condition case. We give a sufficient condition for saturation under the reservoirs and prove the continuity of the free boundary. As a consequence, we obtain the expression of g which unlike the previous case is not equal to the characteristic function of the dry part. Finally, the uniqueness of the reservoirs-connected solution is established in two situations.

Notation

A typical point in $\mathbb{R}^2$ is $x = (x_1, x_2)$ and $|x| = \sqrt{x_1^2 + x_2^2}$.

$\overline{S}$, $Int(S)$ and ∂S are respectively the closure, the interior and the boundary of the set S.

$\chi(S)$ is the characteristic function of S.

$\Omega \setminus S = \{x \in \Omega \,/\, x \notin S\,\}$.

$|S|$ is the Lebesgue measure of S in $\mathbb{R}^2$.

$B_r(x)$ is the open ball with center x and radius r.

If u is a real valued function, then $u^+ = \max(u, 0)$, $u^- = (-u)^+$, $u_{x_i} = \frac{\partial u}{\partial x_i}$ and $\nabla u = (u_{x_1}, u_{x_2})$.

$C^{0,\alpha}_{loc}(\Omega)$ is the set of all functions $u : \Omega \to \mathbb{R}$ that are locally Hölder continuous in Ω.

$C^{0,\alpha}_{loc}(\Omega \cup S)$, where $S \subset \partial\Omega$, is the set of all functions $u : \Omega \cup S \to \mathbb{R}$ that possess an extension to $C^{0,\alpha}_{loc}(\Omega')$ for some open set Ω' which contains $\Omega \cup S$.

$C^{1,\alpha}_{loc}(\Omega \cup S)$, where $S \subset \partial\Omega$, is the set of all functions $u : \Omega \cup S \to \mathbb{R}$ whose first partial derivatives are in $C^{0,\alpha}_{loc}(\Omega \cup S)$.

$C^1(\Omega)$ is the set of all continuously differentiable functions $u : \Omega \to \mathbb{R}$.

$C^1(\overline{\Omega})$ is the set of all functions $u : \overline{\Omega} \to \mathbb{R}$ that possess an extension to $C^1(\mathbb{R}^2)$.

$\mathcal{D}(\Omega)$ is the set of all indefinitely continuously differentiable functions $u : \Omega \to \mathbb{R}$ with compact support.

$\mathcal{D}'(\Omega)$ is the set of distributions.

$L^p(\Omega)$ is the set of all Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ such that $|u|_{p,\Omega} < \infty$, where $|u|_{p,\Omega} = \left(\int_\Omega |u(x)|^p dx\right)^{1/p}$ if $1 \le p < \infty$ and $|u|_{\infty,\Omega} = ess\, sup_\Omega |u|$.

$\mathbb{L}^p(\Omega) = (L^p(\Omega))^3$.

$L^{p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, is the dual space of $L^p(\Omega)$.

$W^{1,p}(\Omega) = \{u : \Omega \to \mathbb{R} \,/\, u, u_{x_1}, u_{x_2} \in L^p(\Omega)\,\}$.

$|u|_{1,p} = \left(|u|^p_{p,\Omega} + |u_{x_1}|^p_{p,\Omega} + |u_{x_2}|^p_{p,\Omega}\right)^{1/p}$ is the norm of $W^{1,p}(\Omega)$.

$H^1(\Omega) = W^{1,2}(\Omega)$.

1 A Unified Formulation of the Dam Problem

1.1 Formulation of the Problem

A porous medium that we denote by Ω is supplied by several reservoirs of a fluid which infiltrates through Ω. We assume that Ω is a bounded locally Lipschitz domain of $\mathbb{R}^2$ with boundary $\partial\Omega = S_1 \cup S_2 \cup S_3$, where S_1 is the impervious part, S_2 is the part in contact with air and $S_3 = \bigcup_{i=1}^{i=N} S_{3,i}$ with $S_{3,i}$ the part in contact with the bottom of the i^{th} reservoir. We assume that the flow in Ω has reached a steady state and we look for the fluid pressure p and the saturated region S of the porous medium. The boundary of S is divided into four parts (see Figure 1):

- $\Gamma_1 \subset S_1$: the impervious part,
- $\Gamma_2 \subset \Omega$: the free boundary,
- $\Gamma_3 \subset S_3$: the part covered by fluid,
- $\Gamma_4 \subset S_2$: the part where the fluid flows out of Ω.

In the saturated region, the flow is governed by the following nonlinear Darcy law

$$v = -\mathcal{A}(x, \nabla(p + x_2)) = -\mathcal{A}(x, \nabla u) \tag{1.1.1}$$

where v is the fluid velocity, $u = p + x_2$ is the hydrostatic head and $\mathcal{A} : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a mapping that satisfies the following assumptions for some constants $q > 1$ and $0 < \lambda \leq M < \infty$:

$$\begin{cases} i) & x \longmapsto \mathcal{A}(x,\xi) \text{ is measurable } \forall \xi \in \mathbb{R}^2 \\ ii) & \xi \longmapsto \mathcal{A}(x,\xi) \text{ is continuous for a.e. } x \in \Omega \\ iii) & \text{for all } \xi \in \mathbb{R}^2 \text{ and for a.e. } x \in \Omega \\ & \qquad \mathcal{A}(x,\xi).\xi \geq \lambda|\xi|^q \quad \text{and} \quad |\mathcal{A}(x,\xi)| \leq M|\xi|^{q-1} \\ iv) & \text{for all } \xi, \zeta \in \mathbb{R}^2 \text{ and for a.e. } x \in \Omega \\ & \qquad (\mathcal{A}(x,\xi) - \mathcal{A}(x,\zeta)).(\xi - \zeta) \geq 0. \end{cases} \tag{1.1.2}$$

A first example of such an operator $\mathcal{A}$ corresponds to the classical Darcy law [18]

$$\mathcal{A}(x,\xi) = a(x)\xi,$$

where $a(x)$ is the permeability matrix of the medium.
Another example corresponds to the nonlinear Darcy law [33]

$$\mathcal{A}(x,\xi) = |a(x)\xi.\xi|^{\frac{q-2}{2}} a(x)\xi.$$

In the following, we shall derive weak formulations of the problem with Dirichlet condition on $S_2 \cup S_3$ and leaky condition on S_3 respectively. Then we give a unified formulation that covers both formulations. First due to the incompressibility of the fluid one has

$$div(v) = 0 \quad \text{in} \quad S.$$

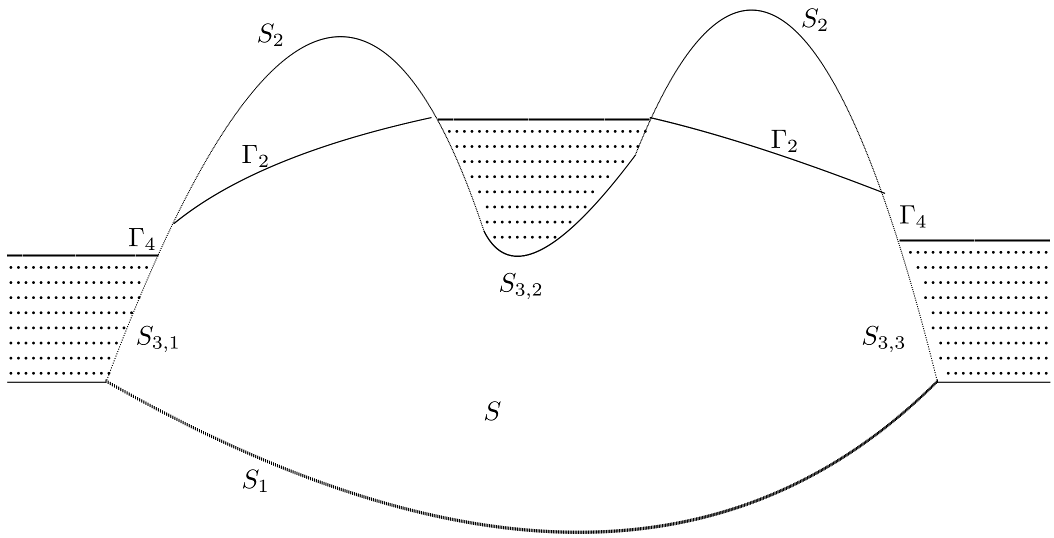

Figure 1

Thus for each sufficiently smooth function ξ and assuming v and S smooth enough, one has

$$0 = \int_S div(v).\xi dx = -\int_S v.\nabla\xi dx + \int_{\partial S} v.\nu\xi d\sigma(x)$$

where ν is the outward unit normal vector to ∂S. This reads by (1.1.1)

$$\int_S \mathcal{A}(x, \nabla u).\nabla\xi dx = \int_{\partial S} -v.\nu\xi d\sigma(x).$$

Assuming that the exterior atmospheric pressure is normalized to 0 and extending p by 0 to $\Omega \setminus S$, we obtain

$$\int_\Omega (\mathcal{A}(x, \nabla u) - \chi(\Omega \setminus S)\mathcal{A}(x, e)).\nabla\xi dx = \int_{\partial S} -v.\nu\xi d\sigma(x). \qquad (1.1.3)$$

Note that we are looking for a $p \geq 0$ or equivalently

$$u \geq x_2 \quad \text{in} \quad \Omega.$$

Thus $\chi(\Omega \setminus S)$ is a function, that we denote by g, such that

$$0 \le g \le 1, \quad g.(u - x_2) = 0 \quad \text{in} \quad \Omega.$$

We assume that no fluid can flow through the part of ∂S that is contained in Ω i.e. the free boundary. This leads to

$$v.\nu = 0 \quad \text{on} \quad \partial S \cap \Omega.$$

It follows from (1.1.3) that for $\Gamma = \partial\Omega$

$$\int_\Omega (\mathcal{A}(x, \nabla u) - g\mathcal{A}(x, e)).\nabla \xi dx = \int_{\partial S \cap \Gamma} -v.\nu \xi d\sigma(x). \tag{1.1.4}$$

Now if we assume that Γ_1 is impervious and since there is overflow on Γ_4, we obtain for $\xi \ge 0$ on S_2

$$\int_\Omega (\mathcal{A}(x, \nabla u) - g\mathcal{A}(x, e)).\nabla \xi dx \le \int_{\Gamma_3} -v.\nu \xi d\sigma(x). \tag{1.1.5}$$

Finally, we denote by φ the exterior pressure on $S_2 \cup S_3$ which is equal to the atmospheric pressure on S_2, and is equal to the fluid pressure on S_3. A first model for the flow through the reservoirs bottoms' consists on assuming the continuity of the pressure i.e. that we have

$$u = \psi = \varphi + x_2 \quad \text{on } S_3. \tag{1.1.6}$$

Now assuming that $\xi = 0$ on S_3, we get the following weak formulation (see for example [2], [17], and [23])

$$(P_D) \begin{cases} \text{Find } (u, g) \in W^{1,q}(\Omega) \times L^\infty(\Omega) \text{ such that :} \\ (i) \quad u \ge x_2, \quad 0 \le g \le 1, \quad g(u - x_2) = 0 \ \text{ a.e. in } \Omega \\ (ii) \quad u = \psi \quad \text{ on } S_2 \cup S_3 \\ (iii) \quad \int_\Omega \big(\mathcal{A}(x, \nabla u) - g\mathcal{A}(x, e)\big).\nabla \xi dx \ \le \ 0 \\ \qquad \forall \xi \in W^{1,q}(\Omega), \quad \xi = 0 \text{ on } S_3, \quad \xi \ge 0 \text{ on } S_2. \end{cases}$$

A second model for the flow through S_3 consists on prescribing the flux instead of the pressure i.e.

$$-v.\nu = \beta(x, \varphi - p) \quad \text{on } S_3, \tag{1.1.7}$$

where $\beta(x, v)$ is a function that is nondecreasing with respect to v. In this case, we obtain from (1.1.5) and (1.1.7), for $\xi \ge 0$ on S_2 (see [22], [25] and [26]):

$$(P_L) \begin{cases} \text{Find } (u, g) \in W^{1,q}(\Omega) \times L^\infty(\Omega) \text{ such that :} \\ (i) \quad u \ge x_2, \quad 0 \le g \le 1, \quad g(u - x_2) = 0 \ \text{ a.e. in } \Omega \\ (ii) \quad u = \psi \quad \text{ on } S_2 \\ (iii) \quad \int_\Omega \big(\mathcal{A}(x, \nabla u) - g\mathcal{A}(x, e)\big).\nabla \xi dx \ \le \ \int_{S_3} \beta(x, \psi - u)\xi d\sigma(x) \\ \qquad \forall \xi \in W^{1,q}(\Omega), \quad \xi \ge 0 \text{ on } S_2. \end{cases}$$

Now we would like to replace the above boundary conditions by the following unified boundary condition

$$-v.\nu \in \mathcal{B}(x, \varphi - p) \quad \text{on } \Gamma, \tag{1.1.8}$$

where for a.e. $x \in \Gamma$, $\mathcal{B}(x, .)$ is a multi-valued monotone function.
Note that if $\mathcal{B}$ is given by

$$\mathcal{B}(x, .) = \begin{cases} \mathbb{R} \times \{0\} & \text{for a.e. } x \in S_1 \\ \{0\} \times \mathbb{R} & \text{for a.e. } x \in S_2 \cup S_3, \end{cases} \tag{1.1.9}$$

we obtain

$$v.\nu = 0 \quad \text{on } S_1 \quad \text{and} \quad p = \varphi \quad \text{on } S_2 \cup S_3,$$

which corresponds to (P_D).
If for a.e. $x \in S_3$, $\beta(x, .)$ is a continuous nondecreasing function, and $\mathcal{B}$ is given by

$$\mathcal{B}(x, .) = \begin{cases} \mathbb{R} \times \{0\} & \text{for a.e. } x \in S_1 \\ \{0\} \times \mathbb{R} & \text{for a.e. } x \in S_2 \\ \beta(x, .) & \text{for a.e. } x \in S_3, \end{cases} \tag{1.1.10}$$

we obtain

$$v.\nu = 0 \quad \text{on } S_1, \quad p = \varphi \quad \text{on } S_2 \quad \text{and} \quad v.\nu = -\beta(x, \varphi - p) \quad \text{on } S_3,$$

which corresponds to (P_L).

For $\mathcal{B}$, we assume that

$$\text{for a.e. } x \in \Gamma,\ \mathcal{B}(x, .) \text{ is a maximal monotone graph of } \mathbb{R}^2 \tag{1.1.11}$$

$$\text{for a.e. } x \in \Gamma,\ 0 \in \mathcal{B}(x, 0) \tag{1.1.12}$$

$$\text{for a.e. } x \in \Gamma,\ D(\mathcal{B}(x, .)) = [a(x), b(x)],\ -\infty \leq a(x) \leq 0 \leq b(x) \leq +\infty. \tag{1.1.13}$$

Taking into account the assumptions (1.1.11)-(1.1.13), there exist for a.e. $x \in \Gamma$, two maximal monotone graphs $\mathcal{B}_1(x, .)$ and $\mathcal{B}_2(x, .)$ in $\mathbb{R}^2$ such that

$$\begin{cases} D(\mathcal{B}_2(x, .)) = \mathbb{R} \\ \mathcal{B}_2(x, .) = \mathcal{B}(x, .) \quad \text{in } (a(x), b(x)) \\ \mathcal{B}_2(x, s) = \{(s, \mathcal{B}^0(x, a(x)))\}, \quad \forall s \leq a(x) \quad \text{if } a(x) > -\infty \\ \mathcal{B}_2(x, s) = \{(s, \mathcal{B}^0(x, b(x)))\}, \quad \forall s \geq b(x) \quad \text{if } b(x) < \infty \\ D(\mathcal{B}_1(x, .)) = D(\mathcal{B}(x, .)) \\ \mathcal{B}_1(x, .) \equiv 0 \quad \text{in } (a(x), b(x)) \\ \mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2 \quad \text{in } [a(x), b(x)], \end{cases} \tag{1.1.14}$$

where for a.e. $x \in \Gamma$, $\mathcal{B}^0(x, .)$ is the minimal section of $\mathcal{B}(x, .)$ i.e. for each $s \in D(\mathcal{B}(x, .))$, $|\mathcal{B}^0(x, s)| = \min\limits_{\gamma \in \mathcal{B}(x,s)} |\gamma|$.

Moreover, we assume that:

$$\forall R>0\ \exists C_R>0:\quad \text{for a.e. } \ x\in\Gamma\ \forall s\in[-R,R]\cap[a(x),b(x)]\quad \mathcal{B}_2(x,s)\subset\{s\}\times[-C_R,C_R]. \tag{1.1.15}$$

Taking into account (1.1.5) and (1.1.8), we obtain the following unified weak formulation (see [4] and [39])

$$(P_U)\begin{cases} \text{Find } (u,g,\gamma)\in W^{1,q}(\Omega)\times L^\infty(\Omega)\times L^{q'}(\Gamma) \text{ such that :}\\ (i)\quad \psi(x)-u(x)\in D(\mathcal{B}(x,.)) \text{ for a.e. } x\in\Gamma\\ (ii)\quad u\geq x_2,\quad 0\leq g\leq 1,\quad g(u-x_2)=0 \ \text{ a.e. in } \Omega\\ (iii)\quad \gamma(x)\in\mathcal{B}(x,\psi(x)-u(x)) \text{ for a.e. } x\in\Gamma\\ \qquad \text{and } \gamma(x)\leq 0 \text{ for a.e. } x\in\Gamma \text{ such that } \psi(x)=x_2\\ (iv)\quad \displaystyle\int_\Omega \big(\mathcal{A}(x,\nabla u)-g\mathcal{A}(x,e)\big).\nabla(\xi-u)dx\ \geq \int_\Gamma \gamma.(\xi-u)d\sigma(x)\\ \forall\xi\in\mathbb{K}=\{\xi\in W^{1,q}(\Omega)/\,a(x)\leq\psi(x)-\xi(x)\leq c(x) \text{ for a.e. } x\in\Gamma\,\}, \end{cases}$$

where for a.e. $x\in\Gamma$, $c(x)=b(x)$ if $\varphi(x)>0$ and $c(x)=+\infty$ if $\varphi(x)=0$.

1.2 Existence of a Solution

In this section, we establish the existence of a solution of the problem (P_U).

Theorem 1.2.1. *Assume that φ is a nonnegative Lipschitz continuous function, $\mathcal{A}$ satisfies (1.1.2) and $\mathcal{B}$ satisfies (1.1.11)-(1.1.15). Then there exists a solution (u,g,γ) to the Problem (P_U).*

For $\epsilon>0$, we introduce the following approximated problem:

$$(P_\epsilon)\begin{cases} \text{Find } u_\epsilon\in V=\{\xi\in W^{1,q}(\Omega)/\,\xi_{|\Gamma}\in L^{q'}(\Gamma)\,\} \text{ such that :}\\ \quad\displaystyle\int_\Omega \epsilon(|u_\epsilon|^{q-2}u_\epsilon-|x_2|^{q-2}x_2).\xi+\big(\mathcal{A}(x,\nabla u_\epsilon)-G_\epsilon(u_\epsilon)\mathcal{A}(x,e)\big).\nabla\xi dx\\ \displaystyle +\int_\Gamma \epsilon(|u_\epsilon|^{q'-2}u_\epsilon-|x_2|^{q'-2}x_2).\xi d\sigma(x)\\ \displaystyle =\int_\Gamma (\mathcal{B}_1^\epsilon(x,\psi-u_\epsilon)+\mathcal{B}_2^\epsilon(x,\psi-u_\epsilon)).\xi d\sigma(x)\qquad\forall\xi\in V, \end{cases}$$

where $G_\epsilon: L^q(\Omega)\to L^\infty(\Omega)$ is defined for a.e. $x\in\Omega$ by $G_\epsilon(v(x))=1-H_\epsilon(v(x)-x_2)$ and $H_\epsilon(s)=\min\big(\dfrac{s^+}{\epsilon},1\big)$. $\mathcal{B}_i^\epsilon$ $(i=1,2)$, denotes the Yoshida approximation of $\mathcal{B}_i$. Note that $\mathcal{B}_i^\epsilon$ is nondecreasing and uniformly Lipschitz continuous with respect to the second variable (see [15], [16]), with Lipschitz constant equal to $1/\epsilon$. Taking into account (1.1.12), we deduce that $\mathcal{B}_i^\epsilon(x,0)=0$ for a.e. $x\in\Gamma$. By the monotonicity of $\mathcal{B}_i^\epsilon(x,.)$, this leads to

$$\mathcal{B}_i^\epsilon(x,u).u \geq 0 \quad \text{for a.e. } x \in \Gamma,\ \forall u \in \mathbb{R}. \tag{1.2.1}$$

We shall equip V with the norm $||u|| = |u|_{1,q} + |u|_{q',\Gamma}$.

Remark 1.2.1. *The approximation of (P_U) by a similar problem to (P_ϵ) was first introduced in [2] and [17] to solve the problem with Dirichlet condition and linear Darcy's law. It was extended in [22] to address the problem for leaky condition. Finally it was used in the form given here in [39] and in [4] for $q = 2$.*

We first establish the existence of a solution to (P_ϵ).

Theorem 1.2.2. *Assume that φ is a nonnegative Lipschitz continuous function, that $\mathcal{A}$ satisfies (1.1.2) and $\mathcal{B}$ satisfies (1.1.11)-(1.1.15). Then there exists a solution u_ϵ to (P_ϵ).*

Proof. The proof is based on the Schauder fixed point theorem. First we consider for $u \in V$ the operator $Au : V \to \mathbb{R}$ defined by

$$\begin{aligned} \xi \to < Au, \xi > &= \int_\Omega \epsilon |u|^{q-2}u.\xi + \mathcal{A}(x,\nabla u).\nabla \xi dx + \int_\Gamma \epsilon |u|^{q'-2}u.\xi d\sigma(x) \\ &\quad - \int_\Gamma (\mathcal{B}_1^\epsilon(x,\psi - u) + \mathcal{B}_2^\epsilon(x,\psi - u)).\xi d\sigma(x). \end{aligned}$$

Clearly A defines a continuous operator from V to V'. Moreover A is monotone due to the monotonicity of $\mathcal{A}(x,.)$, $\mathcal{B}_i^\epsilon(x,.)$ and the function $u \to |u|^{r-2}u$ for $r > 1$.
We claim that A is also coercive. Indeed, we have for each $u \in V$

$$< Au, u >= \int_\Omega \epsilon |u|^q + \mathcal{A}(x,\nabla u).\nabla u dx + \int_\Gamma \epsilon |u|^{q'} d\sigma(x) - \sum_{i=1}^{i=2} \int_\Gamma \mathcal{B}_i^\epsilon(x,\psi - u).u d\sigma(x). \tag{1.2.2}$$

Using (1.2.1) and the Lipschitz continuity of $\mathcal{B}_i^\epsilon(x,.)$, we have

$$\mathcal{B}_i^\epsilon(x,\psi - u).u \leq \mathcal{B}_i^\epsilon(x,\psi - u).\psi \leq \frac{1}{\epsilon}|\psi - u|.|\psi| \leq \frac{1}{\epsilon}(|\psi|^2 + |u|.|\psi|)$$

which leads by Hölder's inequality, for some positive constants c_0, c_1, to

$$\int_\Gamma \mathcal{B}_i^\epsilon(x,\psi - u).u d\sigma(x) \leq c_0 + c_1 |u|_{q',\Gamma}. \tag{1.2.3}$$

Using (1.1.2)$iii)$ and (1.2.2)-(1.2.3), we get for a positive constant c_2

$$< Au, u >\geq c_2(|u|_{1,q}^q + |u|_{q',\Gamma}^{q'}) - 2c_1 |u|_{q',\Gamma} - 2c_0 \quad \forall u \in V. \tag{1.2.4}$$

Since $q, q' > 1$, we obtain $\displaystyle\lim_{||u|| \to +\infty} \frac{< Au, u >}{||u||} = +\infty$.

Now for $v \in L^q(\Omega)$, we consider the mapping $F_v : V \to \mathbb{R}$ defined by

$$F_v(\xi) = \int_\Omega \epsilon |x_2|^{q-2} x_2.\xi dx + \int_\Omega G_\epsilon(v)\mathcal{A}(x,e).\nabla\xi dx + \int_\Gamma \epsilon |x_2|^{q'-2} x_2.\xi d\sigma(x).$$

Given that F_v is a continuous linear form on V, and A is continuous and coercive, there exists, for each $v \in L^q(\Omega)$, [35] a unique solution u_ϵ for the variational problem:

$$\begin{cases} u_\epsilon \in V, \\ < Au_\epsilon, w >=< F_v, w >, \quad \forall w \in V. \end{cases} \tag{1.2.5}$$

This defines a mapping $\mathcal{F}_\epsilon : L^q(\Omega) \to V$, $v \to u_\epsilon$, which satisfies

$$\exists R_\epsilon > 0 / \, ||\mathcal{F}_\epsilon(v)|| \leq R_\epsilon \ \forall v \in L^q(\Omega), \tag{1.2.6}$$

$$\mathcal{F}_\epsilon(\overline{B}_{R_\epsilon}) \subset \overline{B}_{R_\epsilon}, \tag{1.2.7}$$

$$\mathcal{F}_\epsilon : L^q(\Omega) \to L^q(\Omega) \text{ is continuous,} \tag{1.2.8}$$

where $\overline{B}_{R_\epsilon}$ is the closed ball of $L^q(\Omega)$ of center 0 and radius R_ϵ.
Indeed, using u_ϵ as a test function for (1.2.5) and taking into account (1.2.4), we get

$$c_2(|u_\epsilon|^q_{1,q} + |u_\epsilon|^{q'}_{q',\Gamma}) - 2c_1|u_\epsilon|_{q',\Gamma} - 2c_0 \leq c_3(|u_\epsilon|_{1,q} + |u_\epsilon|_{q',\Gamma})$$

which can be written

$$|u_\epsilon|^q_{1,q} + |u_\epsilon|^{q'}_{q',\Gamma} \leq c'_3(|u_\epsilon|_{1,q} + |u_\epsilon|_{q',\Gamma}) + c'_0. \tag{1.2.9}$$

We discuss three cases :

$\underline{|u_\epsilon|_{1,q} \leq 1}$: In this case, we deduce from (1.2.9) that $|u_\epsilon|^{q'}_{q',\Gamma} \leq c'_3|u_\epsilon|_{q',\Gamma} + c'_0 + c'_3$, which leads to $|u_\epsilon|_{q',\Gamma} \leq c_4$ for some constant c_4. Thus $|u_\epsilon|_{1,q} + |u_\epsilon|_{q',\Gamma} \leq c_4 + 1$.

$\underline{|u_\epsilon|_{q',\Gamma} \leq 1}$: In this case, we deduce from (1.2.9) that $|u_\epsilon|^q_{1,q} \leq c'_3|u_\epsilon|_{1,q} + c'_0 + c'_3$, which leads to $|u_\epsilon|_{1,q} \leq c_5$ for some constant c_5. Thus $|u_\epsilon|_{1,q} + |u_\epsilon|_{q',\Gamma} \leq c_5 + 1$.

$\underline{|u_\epsilon|_{1,q}, |u_\epsilon|^{q'}_{q',\Gamma} > 1}$: Let $r = \min(q, q')$. Then we have

$$\begin{aligned}(|u_\epsilon|_{1,q} + |u_\epsilon|_{q',\Gamma})^r &\leq 2^{r-1}(|u_\epsilon|^r_{1,q} + |u_\epsilon|^r_{q',\Gamma}) \leq 2^{r-1}(|u_\epsilon|^q_{1,q} + |u_\epsilon|^{q'}_{q',\Gamma}) \\ &\leq c'_3 2^{r-1}(|u_\epsilon|_{1,q} + |u_\epsilon|_{q',\Gamma}) + c'_0 2^{r-1}\end{aligned}$$

which leads to $|u|_{1,q} + |u_\epsilon|_{q',\Gamma} \leq c_6$ for some constant c_6.

Finally, we have proved (1.2.6). Now (1.2.7) is a consequence of (1.2.6) since $|u_\epsilon|_{q,\Omega} \leq ||u_\epsilon||$.

To prove (1.2.8), let $(v_k)_k$ be a sequence of $L^q(\Omega)$ which converges to v in $L^q(\Omega)$. We denote $\mathcal{F}_\epsilon(v_k)$ by u^k_ϵ. Since $u^k_\epsilon - u_\epsilon$ is a suitable test function for (1.2.5), we obtain by subtracting the equations written for u^k_ϵ and u_ϵ respectively

$$
\begin{aligned}
&\int_\Omega \big(\mathcal{A}(x,\nabla u_\epsilon^k)-\mathcal{A}(x,\nabla u_\epsilon)\big).\nabla(u_\epsilon^k-u_\epsilon)dx\\
&\quad+\int_\Omega \epsilon(|u_\epsilon^k|^{q-2}u_\epsilon^k-|u_\epsilon|^{q-2}u_\epsilon).(u_\epsilon^k-u_\epsilon)dx\\
&\quad+\int_\Gamma \epsilon(|u_\epsilon^k|^{q'-2}u_\epsilon^k-|u_\epsilon|^{q'-2}u_\epsilon).(u_\epsilon^k-u_\epsilon)d\sigma(x)\\
&\quad=\int_\Omega \big(G_\epsilon(v_k)-G_\epsilon(v)\big)\mathcal{A}(x,e).\nabla(u_\epsilon^k-u_\epsilon)dx\\
&\quad+\sum_{i=1}^{i=2}\int_\Gamma (\mathcal{B}_i^\epsilon(x,\psi-u_\epsilon^k)-\mathcal{B}_i^\epsilon(x,\psi-u_\epsilon)).(u_\epsilon^k-u_\epsilon)d\sigma(x)
\end{aligned}
$$

which leads by the monotonicity of $\mathcal{A}(x,.)$, $s\to|s|^{q'-2}s$, and $\mathcal{B}_i^\epsilon(x,.)$ to

$$
\int_\Omega \epsilon(|u_\epsilon^k|^{q-2}u_\epsilon^k-|u_\epsilon|^{q-2}u_\epsilon).(u_\epsilon^k-u_\epsilon)dx\le\int_\Omega \big(G_\epsilon(v_k)-G_\epsilon(v)\big)\mathcal{A}(x,e).\nabla(u_\epsilon^k-u_\epsilon)dx. \tag{1.2.10}
$$

Using (1.1.2)iii), the fact that G_ϵ is Lipschitz continuous with Lipschitz constant equal to $\frac{1}{\epsilon}$ and the Hölder inequality, we obtain since $q'>1$ and $|G_\epsilon(v_k)-G_\epsilon(v)|\le 1$

$$
\begin{aligned}
&\int_\Omega \big(G_\epsilon(v_k)-G_\epsilon(v)\big)\mathcal{A}(x,e).\nabla(u_\epsilon^k-u_\epsilon)dx\\
&\quad\le M\Big(\int_\Omega |G_\epsilon(v_k)-G_\epsilon(v)|^{q'}dx\Big)^{1/q'}.\Big(\int_\Omega|\nabla(u_\epsilon^k-u_\epsilon)|^q dx\Big)^{1/q}.\\
&\quad\le M||u_\epsilon^k-u_\epsilon||\Big(\int_\Omega\frac{1}{\epsilon}|v_k-v|dx\Big)^{1/q'}\\
&\quad\le \frac{M|\Omega|^{1/q'^2}}{\epsilon^{1/q'}}||u_\epsilon^k-u_\epsilon||\Big(\int_\Omega|v_k-v|^q dx\Big)^{1/qq'}.
\end{aligned} \tag{1.2.11}
$$

Combining (1.2.6), (1.2.10)-(1.2.11), we obtain for some constant $C(\epsilon)$

$$
\int_\Omega(|u_\epsilon^k|^{q-2}u_\epsilon^k-|u_\epsilon|^{q-2}u_\epsilon).(u_\epsilon^k-u_\epsilon)dx\le C(\epsilon)|v_k-v|_{q,\Omega}^{1/q'}
$$

which leads to

$$
\lim_{k\to\infty}\int_\Omega(|u_\epsilon^k|^{q-2}u_\epsilon^k-|u_\epsilon|^{q-2}u_\epsilon).(u_\epsilon^k-u_\epsilon)dx=0. \tag{1.2.12}
$$

Using the following inequalities for some $\mu>0$:

$$
\begin{array}{llll}
i) & if\quad q\ge 2, & \forall x,y\in\mathbb{R}^2 & \mu|x-y|^q\le(|x|^{q-2}x-|y|^{q-2}y).(x-y),\\
ii) & if\quad 1<q<2, & \forall x,y\in\mathbb{R}^2 & \mu|x-y|^2\le(|x|+|y|)^{2-q}.(|x|^{q-2}x-|y|^{q-2}y).(x-y),
\end{array}
$$

we easily obtain from (1.2.12) that $u_\epsilon^k \to u_\epsilon$ in $L^q(\Omega)$. Hence the continuity of $\mathcal{F}_\epsilon$ is established. In particular $\mathcal{F}_\epsilon : \overline{B}(0, R_\epsilon) \to \overline{B}(0, R_\epsilon)$ is continuous. Moreover by (1.2.6), $\mathcal{F}_\epsilon(\overline{B}(0, R_\epsilon))$ is relatively compact in $L^q(\Omega)$. Thus we can apply the Schauder fixed point theorem to obtain a fixed point for $\mathcal{F}_\epsilon$ which is a solution of (P_ϵ).

Now we have the following estimates:

Lemma 1.2.1. *Let H and ϵ_0 be two positive constants such that $H > \epsilon_0 + \max_{x\in\overline{\Omega}} \psi(x)$. Then we have for all $\epsilon \in (0, \epsilon_0)$:*

$$x_2 \le u_\epsilon \le H \qquad a.e.\ in\ \Omega. \tag{1.2.13}$$

Proof. *i*) Using $(u_\epsilon - x_2)^-$ as a test function for (P_ϵ) and taking into account that $G_\epsilon(u_\epsilon) = 1$ a.e. in $[u_\epsilon \le x_2]$, we obtain

$$\begin{aligned}
&\int_\Omega \mathcal{A}(x, \nabla u_\epsilon).\nabla(u_\epsilon - x_2)^- dx + \int_\Omega \epsilon(|u_\epsilon|^{q-2}u_\epsilon - |x_2|^{q-2}x_2).(u_\epsilon - x_2)^- dx \\
&\quad + \int_\Gamma \epsilon(|u_\epsilon|^{q'-2}u_\epsilon - |x_2|^{q'-2}x_2).(u_\epsilon - x_2)^- d\sigma(x) \\
&\quad = \int_\Omega \mathcal{A}(x, e).\nabla(u_\epsilon - x_2)^- dx + \sum_{i=1}^{i=2} \int_\Gamma \mathcal{B}_i^\epsilon(x, \psi - u_\epsilon).(u_\epsilon - x_2)^- d\sigma(x).
\end{aligned} \tag{1.2.14}$$

Using the fact that $\psi \ge x_2$, the monotonicity of $\mathcal{B}_i^\epsilon(x, .)$ and (1.2.1), one has for $i = 1, 2$

$$\int_\Gamma \mathcal{B}_i^\epsilon(x, \psi - u_\epsilon).(u_\epsilon - x_2)^- d\sigma(x) \ge \int_{\Gamma\cap[u_\epsilon \le x_2]} \mathcal{B}_i^\epsilon(x, x_2 - u_\epsilon).(x_2 - u_\epsilon)d\sigma(x) \ge 0. \tag{1.2.15}$$

Combining (1.2.14) and (1.2.15), we get

$$\begin{aligned}
&\int_{\Omega\cap[u_\epsilon \le x_2]} (\mathcal{A}(x, \nabla u_\epsilon) - \mathcal{A}(x, \nabla x_2)).\nabla(u_\epsilon - x_2)dx \\
&\quad + \int_{\Omega\cap[u_\epsilon \le x_2]} \epsilon(|u_\epsilon|^{q-2}u_\epsilon - |x_2|^{q-2}x_2).(u_\epsilon - x_2)dx \\
&\quad + \int_{\Gamma\cap[u_\epsilon \le x_2]} \epsilon(|u_\epsilon|^{q'-2}u_\epsilon - |x_2|^{q'-2}x_2).(u_\epsilon - x_2)d\sigma(x) \le 0.
\end{aligned} \tag{1.2.16}$$

Using the monotonicity of $\mathcal{A}(x, .)$ and $u \to |u|^{r-2}u$ for $r = q, q'$, and (1.2.16), we obtain $u_\epsilon \ge x_2$ a.e. in Ω.

ii) Note that for $\epsilon \in (0, \epsilon_0)$ and for $u_\epsilon(x) \ge H$, one has $u_\epsilon(x) \ge H \ge \epsilon_0 + \psi \ge \epsilon + x_2$, and therefore $G_\epsilon(u_\epsilon(x)) = 0$. It follows that

$$G_\epsilon(u_\epsilon(x))\mathcal{A}(x, e).\nabla(u_\epsilon - H)^+ = 0 \quad \text{ for a.e. } x \in \Omega. \tag{1.2.17}$$

Using (1.2.1) and the monotonicity of $\mathcal{B}_i^\epsilon(x,.)$, one has

$$\mathcal{B}_i^\epsilon(x,\psi-u_\epsilon)(u_\epsilon-H)^+ \le \mathcal{B}_i^\epsilon(x,H-u_\epsilon)(u_\epsilon-H)^+ \le 0 \quad \text{for a.e. } x\in\Gamma. \tag{1.2.18}$$

Using $(u_\epsilon-H)^+$ as a test function for (P_ϵ) and taking into account (1.2.17)-(1.2.18), we obtain

$$\begin{aligned}&\int_\Omega \mathcal{A}(x,\nabla u_\epsilon).\nabla(u_\epsilon-H)^+dx+\int_\Omega \epsilon(|u_\epsilon|^{q-2}u_\epsilon-|x_2|^{q-2}x_2).(u_\epsilon-H)^+dx\\ &\quad+\int_\Gamma \epsilon(|u_\epsilon|^{q'-2}u_\epsilon-|x_2|^{q'-2}x_2).(u_\epsilon-H)^+d\sigma(x)\le 0\end{aligned}$$

which can be written since $H\ge x_2$ for all $x\in\Omega$

$$\begin{aligned}&\int_\Omega \mathcal{A}(x,\nabla(u_\epsilon-H)).\nabla(u_\epsilon-H)^+dx+\int_\Omega \epsilon(|u_\epsilon|^{q-2}u_\epsilon-|H|^{q-2}H).(u_\epsilon-H)^+dx\\ &\quad+\int_\Gamma \epsilon(|u_\epsilon|^{q'-2}u_\epsilon-|H|^{q'-2}H).(u_\epsilon-H)^+d\sigma(x)\le 0.\end{aligned}$$

Using the monotonicity of $\mathcal{A}(x,.)$ and $u\to|u|^{q'-2}u$, we get

$$\int_\Omega \epsilon(|u_\epsilon|^{q-2}u_\epsilon-|H|^{q-2}H).(u_\epsilon-H)^+dx\le 0.$$

This clearly leads to $u_\epsilon\le H$ a.e. in Ω. □

Here we give an estimate for ∇u_ϵ.

Lemma 1.2.2. *Under the assumptions of Lemma 1.2.1, we have for some positive constant C independent of ϵ*

$$\forall\epsilon\in(0,\epsilon_0)\qquad \int_\Omega |\nabla u_\epsilon|^q dx\le C. \tag{1.2.19}$$

Proof. Using $u_\epsilon-\psi$ as a test function for (P_ϵ) and taking into account (1.2.1), we obtain

$$\begin{aligned}&\int_\Omega \mathcal{A}(x,\nabla u_\epsilon).\nabla(u_\epsilon-\psi)dx+\int_\Omega \epsilon(|u_\epsilon|^{q-2}u_\epsilon-|x_2|^{q-2}x_2).(u_\epsilon-\psi)dx\\ &\quad+\int_\Gamma \epsilon(|u_\epsilon|^{q'-2}u_\epsilon-|x_2|^{q'-2}x_2).(u_\epsilon-\psi)d\sigma(x)\\ &\quad\le\int_\Omega G_\epsilon(u_\epsilon)\mathcal{A}(x,e).\nabla(u_\epsilon-\psi)dx\end{aligned}$$

which can be written, for a positive constant C_1 independent of ϵ, by using (1.1.2), (1.2.13) and the fact that $|G_\epsilon(u_\epsilon)|\le 1$

$$\int_\Omega |\nabla u_\epsilon|^q dx\le C_1+C_1\int_\Omega|\nabla u_\epsilon|^{q-1}dx+\int_\Omega|\nabla u_\epsilon|dx$$

Using Young's inequality $ab \leq \dfrac{a^q}{q} + \dfrac{b^{q'}}{q'}$ for appropriate a and b, one gets for another positive constant independent of ϵ still denoted by C_1

$$\int_\Omega |\nabla u_\epsilon|^q dx \leq C_1 + \frac{1}{2}\int_\Omega |\nabla u_\epsilon|^q dx$$

which is (1.2.19) with $C = 2C_1$. □

Proof of Theorem 1.1. The proof consists in passing to the limit as $\epsilon \to 0$ in (P_ϵ). First we have $0 \leq G_\epsilon(u_\epsilon) \leq 1$, u_ϵ is bounded in $W^{1,q}(\Omega)$ by (1.2.13) and (1.2.19). Also by (1.1.2) and (1.2.19), $\mathcal{A}(x, \nabla u_\epsilon)$ is bounded in $\mathbb{L}^{q'}(\Omega)$. It follows by the Relich theorem, the complete continuity of the trace operator, and the reflexivity of the Lebesgue space $L^r(\Omega)$ for $r > 1$, that there exists a subsequence (u_{ϵ_k}) of (u_ϵ), functions $u \in W^{1,q}(\Omega)$, $g \in L^{q'}(\Omega)$, and $\mathcal{A}_0 \in \mathbb{L}^{q'}(\Omega)$ such that

$$
\begin{aligned}
G_{\epsilon_k}(u_{\epsilon_k}) &\rightharpoonup g && \text{in} \quad L^{q'}(\Omega) && (1.2.20)\\
u_{\epsilon_k} &\rightharpoonup u && \text{in} \quad W^{1,q}(\Omega) && (1.2.21)\\
u_{\epsilon_k} &\to u && \text{in} \quad L^q(\Omega) \quad \text{and} \quad \text{a.e. in } \Omega && (1.2.22)\\
u_{\epsilon_k} &\to u && \text{in} \quad L^q(\Gamma) \quad \text{and} \quad \text{a.e. in } \Gamma && (1.2.23)\\
\mathcal{A}(x, \nabla u_{\epsilon_k}) &\rightharpoonup \mathcal{A}_0 && \text{in} \quad \mathbb{L}^{q'}(\Omega). && (1.2.24)
\end{aligned}
$$

Moreover by (1.2.13) and the monotonicity of $\mathcal{B}_2^{\epsilon_k}(x, .)$, we have for some positive constant H_1 and for a.e. $x \in \Gamma$

$$
\begin{aligned}
&\mathcal{B}_2^{\epsilon_k}(x, -H_1) \leq \mathcal{B}_2^{\epsilon_k}(x, \psi - H) \leq \mathcal{B}_2^{\epsilon_k}(x, \psi - u_{\epsilon_k})\\
&\mathcal{B}_2^{\epsilon_k}(x, \psi - u_{\epsilon_k}) \leq \mathcal{B}_2^{\epsilon_k}(x, \psi - x_2) = \mathcal{B}_2^{\epsilon_k}(x, \varphi) \leq \mathcal{B}_2^{\epsilon_k}(x, H_1).
\end{aligned}
$$

Using (1.1.15), we deduce that we have for some positive constant C_{H_1}

$$|\mathcal{B}_2^{\epsilon_k}(x, \psi - u_{\epsilon_k})| \leq \max(|\mathcal{B}_2^0(x, -H_1)|, |\mathcal{B}_2^0(x, H_1)|) \leq C_{H_1}$$

from which follows that $(\mathcal{B}_2^{\epsilon_k}(x, \psi - u_{\epsilon_k}))$ is bounded in $L^\infty(\Gamma)$. Therefore, there exists a subsequence of (u_{ϵ_k}) still denoted by (u_{ϵ_k}) and an element γ of $L^{q'}(\Gamma)$ such that

$$\mathcal{B}_2^{\epsilon_k}(x, \psi - u_{\epsilon_k}) \rightharpoonup \gamma \quad \text{in } L^{q'}(\Gamma). \qquad (1.2.25)$$

We shall prove that (u, g, γ) is a solution of (P_U).

Since the sets $\{v \in W^{1,q}(\Omega) \,/\, v \geq x_2 \text{ a.e. in } \Omega\}$ and $\{v \in L^{q'}(\Omega) \,/\, 0 \leq v \leq 1 \text{ a.e. in } \Omega\}$ are weakly closed in $W^{1,q}(\Omega)$ and $L^{q'}(\Omega)$ respectively, and contain respectively u_{ϵ_k} and $G_{\epsilon_k}(u_{\epsilon_k})$, we obtain

$$u \geq x_2 \quad \text{and} \quad 0 \leq g \leq 1 \text{ a.e. in } \Omega. \qquad (1.2.26)$$

Moreover we have for all $s \geq 0$, $0 \leq \big(1-\min\big(\frac{s}{\epsilon},1\big)\big)s = \big(1-\frac{s}{\epsilon}\big)s\chi([0 \leq s \leq \epsilon]) \leq \epsilon$. Therefore

$$0 \leq \int_\Omega G_{\epsilon_k}(u_{\epsilon_k}).(u_{\epsilon_k} - x_2)dx \leq \epsilon_k|\Omega|$$

which leads by letting $\epsilon_k \to 0$ and taking into account (1.2.20) and (1.2.22), to

$$\int_\Omega g.(u - x_2)dx = 0.$$

By (1.2.26), we obtain

$$g.(u - x_2) = 0 \quad \text{a.e. in } \Omega. \tag{1.2.27}$$

Now since we have for a.e. $x \in \Gamma$ such that $\psi = x_2$, $\mathcal{B}_2^{\epsilon_k}(x, \psi - u_{\epsilon_k}) = \mathcal{B}_2^{\epsilon_k}(x, x_2 - u_{\epsilon_k}) \leq 0$, we deduce that

$$\gamma(x) \leq 0 \quad \text{for a.e. } x \in \Gamma \text{ such that } \psi = x_2. \tag{1.2.28}$$

Moreover, using (1.2.23) and (1.2.25), we obtain [15], Lemma 1.3, page 42

$$\gamma(x) \in \mathcal{B}_2(x, \psi(x) - u(x)) \quad \text{for a.e. } x \in \Gamma. \tag{1.2.29}$$

Using $\psi - u_{\epsilon_k}$ as a test function for (P_{ϵ_k}) and taking into account (1.2.1), (1.2.13) and (1.2.19), we get for some constant C independent of ϵ_k

$$0 \leq \int_\Gamma \mathcal{B}_1^{\epsilon_k}(x, \psi - u_{\epsilon_k}).(\psi - u_{\epsilon_k})d\sigma(x) \leq C$$

which can be written, since for a.e. $x \in \Gamma$ and for all $u \in D(\mathcal{B})$

$$\mathcal{B}_1^{\epsilon_k}(x, u) = \frac{1}{\epsilon_k}\big((u-b)^+ - (a-u)^+\big)$$

$$0 \leq \int_\Gamma \big((\psi - u_{\epsilon_k} - b)^+ - (a - \psi + u_{\epsilon_k})^+\big).(\psi - u_{\epsilon_k})d\sigma(x) \leq \epsilon_k C.$$

Letting $\epsilon_k \to 0$, we obtain

$$\int_\Gamma \big((\psi - u - b)^+ - (a - \psi + u)^+\big).(\psi - u)d\sigma(x) = 0.$$

Since $a \leq 0 \leq b$ a.e. in Γ, one has $(\psi - u - b)^+.(\psi - u) \geq 0$ and $-(a - \psi + u)^+.(\psi - u) \geq 0$ for a.e. $x \in \Gamma$. It follows that $\big((\psi - u - b)^+ - (a - \psi + u)^+\big).(\psi - u) = 0$ for a.e. $x \in \Gamma$, which leads to $a \leq \psi - u \leq b$ a.e. in Γ. Hence $\psi - u \in D(\mathcal{B}(x, .))$ for a.e. $x \in \Gamma$. We deduce then from (1.2.29) and the definition of $\mathcal{B}_2$ that $\gamma(x) \in \mathcal{B}(x, \psi(x) - u(x))$ for a.e. $x \in \Gamma$.

So far, we have proved $(P_U)i)$, $ii)$ and $iii)$. It remains to show $(P_U)iv)$. First remark that because of (1.2.13), any element of $W^{1,q}(\Omega)$ is a test function for (P_{ϵ_k}). Next let $\xi \in \mathbb{K}$ and note that

$$\mathcal{B}_1^{\epsilon_k}(x, \psi - u_{\epsilon_k}).(\xi - u_{\epsilon_k}) \geq 0 \quad \text{a.e. in } \Gamma. \tag{1.2.30}$$

Indeed one has first

$$-(a-\psi+u_{\epsilon_k})^+.(\xi-u_{\epsilon_k}) = (a-\psi+u_{\epsilon_k})^+.(u_{\epsilon_k}-\xi) \geq (a-\psi+u_{\epsilon_k})^+.(u_{\epsilon_k}-\psi+a) \geq 0 \quad \text{a.e. in } \Gamma.$$

To show that $(\psi - u_{\epsilon_k} - b)^+.(\xi - u_{\epsilon_k}) \geq 0$ a.e. in Γ, we consider two cases:

$*\underline{\psi > x_2}$: In this case $c = b$ and $\psi - \xi \leq b$. Therefore

$$(\psi - u_{\epsilon_k} - b)^+.(\xi - u_{\epsilon_k}) \geq (\psi - u_{\epsilon_k} - b)^+.(\psi - b - u_{\epsilon_k}) \geq 0 \quad \text{a.e. in } \Gamma.$$

$*\underline{\psi = x_2}$: In this case $(\psi - u_{\epsilon_k} - b)^+ = (x_2 - u_{\epsilon_k} - b)^+ = 0$ since $u_{\epsilon_k} - x_2 \geq 0$ and $b \geq 0$ a.e. in Γ.

Taking $\xi - u_{\epsilon_k}$ as a test function for (P_{ϵ_k}) and using (1.2.30), we get

$$\begin{aligned}
&\int_\Omega \big(\mathcal{A}(x, \nabla u_{\epsilon_k}) - G_{\epsilon_k}(u_{\epsilon_k})\mathcal{A}(x, e)\big).\nabla(\xi - u_{\epsilon_k})dx \\
&\quad + \int_\Omega \epsilon_k(|u_{\epsilon_k}|^{q-2}u_{\epsilon_k} - |x_2|^{q-2}x_2).(\xi - u_{\epsilon_k})dx \\
&\quad + \int_\Gamma \epsilon(|u_{\epsilon_k}|^{q'-2}u_{\epsilon_k} - |x_2|^{q'-2}x_2).(\xi - u_{\epsilon_k})d\sigma(x) \\
&\quad \geq \int_\Gamma \mathcal{B}_2^{\epsilon_k}(x, \psi - u_{\epsilon_k}).(\xi - u_{\epsilon_k})d\sigma(x).
\end{aligned} \tag{1.2.31}$$

It remains to verify $(P_U)iv)$ by passing to the limit in (1.2.31), which we will be able to do after proving the following Lemma:

Lemma 1.2.3. *We have*

$$\int_\Omega \mathcal{A}(x, \nabla u).\nabla\xi dx = \int_\Omega \mathcal{A}_0(x).\nabla\xi dx, \quad \forall \xi \in W^{1,q}(\Omega). \tag{1.2.32}$$

$$\lim_{k\to\infty} \int_\Omega \mathcal{A}(x, \nabla u_{\epsilon_k}).\nabla u_{\epsilon_k} dx = \int_\Omega \mathcal{A}(x, \nabla u).\nabla u dx. \tag{1.2.33}$$

Proof. Choosing $\xi = u$ in (1.2.31) and taking into account that u_{ϵ_k} is uniformly bounded, one gets

$$\begin{aligned}
\int_\Omega \mathcal{A}(x, \nabla u_{\epsilon_k}).\nabla u_{\epsilon_k} dx \quad &\leq \quad \int_\Omega \mathcal{A}(x, \nabla u_{\epsilon_k}).\nabla u dx + \int_\Omega G_{\epsilon_k}(u_{\epsilon_k})\mathcal{A}(x, e).\nabla(u_{\epsilon_k} - u)dx \\
&\quad + \int_\Gamma \mathcal{B}_2^{\epsilon_k}(x, \psi - u_{\epsilon_k}).(u_{\epsilon_k} - u)d\sigma(x) + C\epsilon_k.
\end{aligned} \tag{1.2.34}$$

Using (1.2.23) and (1.2.25), we obtain

$$\lim_{k\to\infty} \int_\Gamma \mathcal{B}_2^{\epsilon_k}(x, \psi - u_{\epsilon_k}).(u_{\epsilon_k} - u)d\sigma(x) = 0. \tag{1.2.35}$$

By (1.2.24), we have

$$\lim_{k\to\infty}\int_\Omega \mathcal{A}(x,\nabla u_{\epsilon_k}).\nabla u dx = \int_\Omega \mathcal{A}_0(x).\nabla u dx. \tag{1.2.36}$$

Note that

$$\int_\Omega G_{\epsilon_k}(u_{\epsilon_k})\mathcal{A}(x,e).\nabla(u_{\epsilon_k}-u)dx = \int_\Omega G_{\epsilon_k}(u_{\epsilon_k})\mathcal{A}(x,e).\nabla(u_{\epsilon_k}-x_2)dx \\ -\int_\Omega G_{\epsilon_k}(u_{\epsilon_k})\mathcal{A}(x,e).\nabla(u-x_2)dx \tag{1.2.37}$$

Using (1.2.20) and taking into account (1.2.27), we have

$$\lim_{k\to\infty}\int_\Omega G_{\epsilon_k}(u_{\epsilon_k})\mathcal{A}(x,e).\nabla(u-x_2)dx = \int_\Omega g\mathcal{A}(x,e).\nabla(u-x_2)dx = 0. \tag{1.2.38}$$

For the second integral in the right hand side of (1.2.37), we rewrite it as

$$\int_\Omega G_{\epsilon_k}(u_{\epsilon_k})\mathcal{A}(x,e).\nabla(u_{\epsilon_k}-x_2)dx = \int_\Omega \mathcal{A}(x,e).\nabla v_k dx = 0$$

where $v_k = \displaystyle\int_0^{u_{\epsilon_k}-x_2}(1-H_{\epsilon_k}(s))ds$.
Since we have $\nabla v_k = (1-H_{\epsilon_k}(u_{\epsilon_k}-x_2)).\nabla(u_{\epsilon_k}-x_2)$, $|v_k(x)|\le\epsilon_k$ for a.e. $x\in\Omega$, and $|v_k|_{1,q}$ is bounded in $W^{1,q}(\Omega)$, we deduce that $v_k \rightharpoonup 0$ weakly in $W^{1,q}(\Omega)$. Therefore

$$\lim_{k\to\infty}\int_\Omega G_{\epsilon_k}(u_{\epsilon_k})\mathcal{A}(x,e).\nabla(u_{\epsilon_k}-x_2)dx = 0. \tag{1.2.39}$$

Using (1.2.37)-(1.2.39), we get

$$\lim_{k\to\infty}\int_\Omega G_{\epsilon_k}(u_{\epsilon_k})\mathcal{A}(x,e).\nabla(u_{\epsilon_k}-u)dx = 0. \tag{1.2.40}$$

Combining (1.2.34)-(1.2.36) and (1.2.40), we get

$$\limsup_{k\to\infty}\int_\Omega \mathcal{A}(x,\nabla u_{\epsilon_k}).\nabla u_{\epsilon_k} dx \le \int_\Omega \mathcal{A}_0(x).\nabla u dx. \tag{1.2.41}$$

Let now $v\in W^{1,q}(\Omega)$. By (1.1.2), we have

$$\int_\Omega \big(\mathcal{A}(x,\nabla u_{\epsilon_k})-\mathcal{A}(x,\nabla v)\big).\nabla(u_{\epsilon_k}-v)dx \ge 0, \quad \forall k$$

or

$$\int_\Omega \mathcal{A}(x,\nabla u_{\epsilon_k}).\nabla u_{\epsilon_k} dx - \int_\Omega \mathcal{A}(x,\nabla u_{\epsilon_k}).\nabla v dx - \int_\Omega \mathcal{A}(x,\nabla v).\nabla(u_{\epsilon_k}-v)dx \ge 0, \quad \forall k. \tag{1.2.42}$$

Passing to the limsup in (1.2.42) and taking into account (1.2.21), (1.2.24) and (1.2.41), we obtain

$$\int_{\Omega} \mathcal{A}_0(x).\nabla u dx - \int_{\Omega} \mathcal{A}_0(x).\nabla v dx - \int_{\Omega} \mathcal{A}(x, \nabla v).\nabla(u-v) dx \geq 0$$

or

$$\int_{\Omega} \mathcal{A}_0(x).\nabla(u-v) dx \geq \int_{\Omega} \mathcal{A}(x, \nabla v).\nabla(u-v) dx. \tag{1.2.43}$$

Choosing $v = u \pm t\xi$, with $t \in [0,1]$ and $\xi \in W^{1,q}(\Omega)$, in (1.2.43), we obtain

$$\int_{\Omega} \mathcal{A}_0(x).\nabla \xi dx = \int_{\Omega} \mathcal{A}(x, \nabla u \pm t\nabla \xi).\nabla \xi dx.$$

Letting $t \to 0$ and using (1.1.2)ii) and the Lebesgue theorem, we obtain (1.2.32).

Using $\xi = u$ in (1.2.32), and taking into account (1.2.41), we obtain

$$\limsup_{k\to\infty} \int_{\Omega} \mathcal{A}(x, \nabla u_{\epsilon_k}).\nabla u_{\epsilon_k} dx \leq \int_{\Omega} \mathcal{A}(x, \nabla u).\nabla u dx. \tag{1.2.44}$$

Now rewriting (1.2.42) for $v = u$, and passing to the liminf, we obtain

$$\liminf_{k\to\infty} \int_{\Omega} \mathcal{A}(x, \nabla u_{\epsilon_k}).\nabla u_{\epsilon_k} dx \geq \int_{\Omega} \mathcal{A}_0(x).\nabla u dx = \int_{\Omega} \mathcal{A}(x, \nabla u).\nabla u dx. \tag{1.2.45}$$

Combining (1.2.44) and (1.2.45), we get (1.2.33). □

1.3 Regularity and Monotonicity of the Solutions

Throughout this section, we shall denote a solution of (P_U) by (u, g, γ). We show that u is bounded and locally Hölder continuous in Ω. Under suitable assumptions on $\mathcal{A}$, we also give a monotonicity property for g which together with the continuity of u, allows to define the free boundary as an x_1x_2-graph.

Proposition 1.3.1. *We have for some positive constant* h_0

$$u \leq h_0 \quad a.e. \ in \ \Omega. \tag{1.3.1}$$

Proof. Let h be such that $h > \max_{x\in\overline{\Omega}} \psi(x)$. Note that $\xi = u - (u-h)^+$ is a test function for (P_U). Indeed

$$\begin{array}{ll} \text{If } u \leq h, \text{ then} & \psi - \xi = \psi - u + (u-h)^+ = \psi - u \in [a,b] \subset [a,c] \\ \text{If } u > h, \text{ then} & \psi - \xi = \psi - h \geq \psi - u \geq a \quad \text{and} \quad \psi - h \leq 0 \leq c. \end{array}$$

So we have

$$\int_\Omega \big(\mathcal{A}(x,\nabla u) - g\mathcal{A}(x,e)\big).\nabla(u-h)^+dx \;\leq\; \int_\Gamma \gamma.(u-h)^+d\sigma(x). \tag{1.3.2}$$

Since $h \geq \psi(x) \geq x_2$, we have by $(P_U)ii)$ $g.\nabla(u-h)^+ = 0$ a.e. in Ω. For the same reason, we get by (1.1.12), $(P_U)iii)$ and the monotonicity of $\mathcal{B}(x,.)$ that $\gamma.(u-h)^+ \leq 0$ a.e. in Γ. It follows from (1.3.2) and (1.1.2)$iii)$ that

$$\int_\Omega \lambda|\nabla(u-h)^+|^q dx \;\leq\; 0$$

which leads to $\nabla(u-h)^+ = 0$ a.e. in Ω. Therefore $(u-h)^+ = C$ for some positive constant C. Hence $u \leq h + C = h_0$ a.e. in Ω. □

Proposition 1.3.2. *We have*

$$div\big(\mathcal{A}(x,\nabla u) - g\mathcal{A}(x,e)\big) = 0 \quad \textit{in} \quad \mathcal{D}'(\Omega). \tag{1.3.3}$$

$$\textit{If} \;\; div\big(\mathcal{A}(x,e)\big) \geq 0, \quad \textit{then} \quad div\big(\mathcal{A}(x,\nabla u)\big) = div\big(g\mathcal{A}(x,e)\big) \geq 0 \quad \textit{in} \quad \mathcal{D}'(\Omega). \tag{1.3.4}$$

Proof. $i)$ (1.3.3) follows immediately by taking $u \pm \xi$ as a test function for (P_U), where $\xi \in \mathcal{D}(\Omega)$.

$ii)$ Let $\xi \in \mathcal{D}(\Omega)$, $\xi \geq 0$ and $\epsilon > 0$. Using $u \pm \min\big(\frac{u-x_2}{\epsilon},1\big)\xi$ as test functions for (P_U), and taking into account $(P_U)ii)$, we get

$$\int_\Omega \mathcal{A}(x,\nabla u).\nabla\Big(\min\big(\frac{u-x_2}{\epsilon},1\big)\xi\Big)dx = 0. \tag{1.3.5}$$

Since $div\big(\mathcal{A}(x,e)\big) \geq 0$, we have

$$\int_\Omega \mathcal{A}(x,e).\nabla\Big(\big(1-\min\big(\frac{u-x_2}{\epsilon},1\big)\big)\xi\Big)dx \leq 0. \tag{1.3.6}$$

Adding (1.3.5) and (1.3.6), and using the fact that $\nabla x_2 = e$, we obtain

$$\begin{aligned}
&\int_\Omega \min\big(\frac{u-x_2}{\epsilon},1\big)\big(\mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla x_2)\big).\nabla\xi dx \\
&\quad + \frac{1}{\epsilon}\int_{[u-x_2\leq\epsilon]} \xi\big(\mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla x_2)\big).\big(\nabla u - \nabla x_2\big)dx \\
&\quad \leq -\int_\Omega \mathcal{A}(x,e).\nabla\xi.
\end{aligned} \tag{1.3.7}$$

Letting $\epsilon \to 0$ in (1.3.7) and using the monotonicity of $\mathcal{A}(x,.)$, we get

$$\int_\Omega \mathcal{A}(x,\nabla u).\nabla\xi dx \leq 0.$$

□

Proposition 1.3.3. *We have $u \in C^{0,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0,1)$, and the set $[u > x_2]$ is open.*

Proof. This is a consequence of (1.3.1) and (1.3.3) [29]. $\square$

Remark 1.3.1. *i) Assume that*

$$\mathcal{A}(x,e) = k(x)e \quad \text{for a.e. } x \in \Omega \quad \text{with} \quad k : \Omega \longrightarrow I\!R \quad \text{and} \quad \frac{\partial k}{\partial x_2} \geq 0 \ \text{in} \ \mathcal{D}'(\Omega). \quad (1.3.8)$$

Then we obtain from (1.3.4) that gk is nondecreasing in x_2.

ii) We also deduce from (1.3.3) and $(P_U)ii)$ that $div(\mathcal{A}(x,\nabla u)) = 0$ in $\mathcal{D}'([u > x_2])$ i.e. u is $\mathcal{A}$-harmonic in $[u > x_2]$. Therefore if there exist nonnegative constants κ, σ and positive constants α_0, α_1 with $\sigma \leq 1$ and $\alpha_1 \geq \alpha_0$ such that for all $x, y \in \Omega$, $\zeta, \xi \in I\!R^2$

$$\sum_{i,j} \frac{\partial \mathcal{A}^i}{\partial \zeta_j}(x,\zeta)\xi_i\xi_j \geq \alpha_0(\kappa + |\zeta|^{q-2})|\xi|^2 \quad (1.3.9)$$

$$\Big|\frac{\partial \mathcal{A}^i}{\partial \zeta_j}(x,\zeta)\Big| \leq \alpha_1(\kappa + |\zeta|^{q-2}) \quad (1.3.10)$$

$$|\mathcal{A}(x,\zeta) - \mathcal{A}(y,\zeta)| \leq \alpha_1(1 + |\zeta|^{q-1})(|x-y|^\sigma), \quad (1.3.11)$$

then we have [28], $u \in C^{1,\delta}_{loc}([u > x_2])$ for some $\delta \in (0,1)$.

Assumptions (1.3.9)-(1.3.11) are satisfied for example if $\mathcal{A}(x,\zeta) = |a(x)\zeta.\zeta|^{\frac{q-2}{2}}a(x)\zeta$ with $a(x)$ a bounded 2×2 matrix satisfying $a \in C^{0,\sigma}_{loc}(\Omega)$ for some $\sigma \in (0,1)$. In particular if $q = 2$, we have $\mathcal{A}(x,\zeta) = a(x)\zeta$ and (1.3.9)-(1.3.11) are satisfied obviously.

In the rest of the Chapter, we shall assume that (1.3.8)-(1.3.11) are satisfied except when $\mathcal{A}(x,\zeta) = a(x)\zeta$. Moreover we assume that Ω is vertically convex i.e.

$$\forall (x_1,x_2),(x_1,x_2') \in \Omega, \qquad \{x_1\} \times [x_2,x_2'] \subset \Omega.$$

We also define the functions s_- and s_+ for $x_1 \in \pi_{x_1}(\Omega)$ by:

$$s_-(x_1) \ = \ \inf\{x_2 : (x_1,x_2) \in \Omega\}, \qquad s_+(x_1) \ = \ \sup\{x_2 : (x_1,x_2) \in \Omega\}$$

and assume that s_- (resp. s_+) is continuous except on a finite set S_- (resp.S_+).

Then we have:

Theorem 1.3.1. *If $x_0 = (x_{01}, x_{02}) \in [u > x_2]$, then there exists $\epsilon > 0$ such that (see Figure 2):*

$$u(x_1,x_2) > x_2 \qquad \forall (x_1,x_2) \in C_\epsilon = \Big\{ (x_1,x_2) \in \Omega \,/\, |x_1 - x_{01}| < \epsilon\,,\ x_2 < x_{02} + \epsilon \Big\}.$$

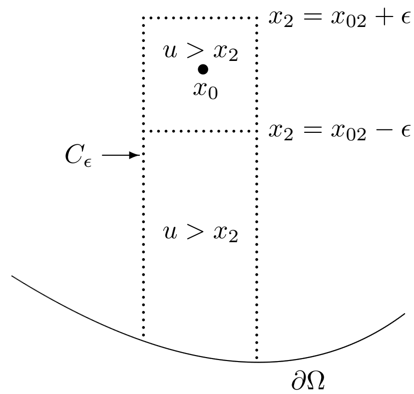

Figure 2

We need the following strong comparison principle proved in [27]

Lemma 1.3.1. *Let D be a domain of $\mathbb{R}^2$ and let $u_1, u_2 \in C^1(D)$ such that*

$$\begin{cases} div(\mathcal{A}(x, \nabla u_1)) \geq div(\mathcal{A}(x, \nabla u_2)) \quad in \;\; \mathcal{D}'(D) \\ u_1 \leq u_2 \quad in \;\; D \quad and \;\; S = \{x \in D / \, \nabla u_1(x) = \nabla u_2(x) = 0\,\} = \emptyset. \end{cases}$$

Then we have

$$either \;\; u_1 \equiv u_2 \;\; in \; D \qquad or \qquad u_1 > u_2 \;\; in \; D.$$

Proof of Theorem 1.3.1. By the continuity of u, there exists $\epsilon > 0$ such that

$$Q_\epsilon = \{(x_1, x_2) \in \Omega \, / \; |x_1 - x_{01}| < \epsilon, \; |x_2 - x_{02}| < \epsilon\} \subset [u > x_2].$$

From $(P_U)ii)$, we deduce that $g = 0$ and then $gk = 0$ a.e. in Q_ϵ. But since gk is non-decreasing in x_2 and $gk \geq 0$ a.e. in Ω, we obtain $gk = 0$ a.e. in C_ϵ. Using (1.3.3), we obtain

$$div\big(\mathcal{A}(x, \nabla u)\big) \; = \; 0 \qquad \text{in } \mathcal{D}'(C_\epsilon).$$

Since $u \geq x_2$ in C_ϵ, $div(\mathcal{A}(x, \nabla x_2)) = \dfrac{\partial k}{\partial x_2} \geq 0$ in $\mathcal{D}'(C_\epsilon)$, $u > x_2$ in Q_ϵ and $\nabla x_2 = e \neq 0$, we deduce from Lemma 1.3.1 that $u(x) > x_2 \; \forall x \in C_\epsilon$. □

Corollary 1.3.1. *If $u(x_{01}, x_{02}) = x_{02}$, then $u(x_{01}, x_2) = x_2 \quad \forall x_2 \in [x_{02}, s_+(x_{01}))$.*

Remark 1.3.2. *Theorem 1.3.1 means that if a point (x_{01}, x_{02}) is wet, then so are all the points below. This is due to gravity and most likely to the fact that by (1.3.8), an important component of the permeability is nondecreasing with respect to x_2.*

Now we are able to define a function Φ that represents the free boundary:

$$\forall x_1 \in \pi_{x_1}(\Omega), \quad \Phi(x_1) = \begin{cases} \sup \left\{ x_2 \ : \ (x_1, x_2) \in [u > x_2] \right\} & \text{if this set is not empty} \\ s_-(x_1) & \text{otherwise,} \end{cases}$$

where π_{x_1} is the projection on the x_1-axis.

Then we have:

Proposition 1.3.4. *Φ is lower semi-continuous (l.s.c) on $\pi_{x_1}(\Omega)$ except perhaps on S_-. Moreover*

$$[u > x_2] = [x_2 < \Phi(x_1)]. \tag{1.3.12}$$

Proof. Let $x_{01} \in \pi_{x_1}(\Omega) \setminus S_-$. Since s_- is continuous on $\pi_{x_1}(\Omega) \setminus S_-$ and $\Phi(x_1) \geq s_-(x_1)$, it is clear that Φ is l.s.c at x_{01} if $\Phi(x_{01}) = s_-(x_{01})$.
Now assume that $\Phi(x_{01}) > s_-(x_{01})$ and let $\epsilon > 0$ small enough. There exists x_{02} such that $x_0 = (x_{01}, x_{02}) \in [u > x_2]$ and $\Phi(x_{01}) > x_{02} > \Phi(x_{01}) - \epsilon > s_-(x_{01})$. By continuity of u, there exits $\eta > 0$ small enough such that $u(x) > x_2$ in $B_\eta(x_0) \subset \Omega$. Using Corollary 1.3.1, we obtain $u(x) > x_2$ in $\big((x_{01}-\eta, x_{01}+\eta)\times(-\infty, x_{02})\big)\cap\Omega$. This leads to $\Phi(x_1) \geq x_{02} > \Phi(x_{01})-\epsilon$ for all $x_1 \in (x_{01} - \eta, x_{01} + \eta)$.

Let $(x_{01}, x_{02}) \in [u > x_2]$. By Theorem 1.3.1, there exists $\epsilon > 0$ small enough such that $u(x_{01}, x_2) > x_2$ for all $x_2 \in (s_-(x_{01}), x_{02} + \epsilon)$. In particular $\Phi(x_{01}) \geq x_{02} + \epsilon > x_{02}$ and $(x_{01}, x_{02}) \in [x_2 < \Phi(x_1)]$.

Conversely, let $(x_{01}, x_{02}) \in [x_2 < \Phi(x_1)]$. If $u(x_{01}, x_{02}) = x_{02}$, then by Corollary 1.3.1 we would have $u(x_{01}, x_2) = x_2$ for all $x_2 \in [x_{02}, s_+(x_{01}))$ and therefore $\Phi(x_{01}) \leq x_{02}$, which contradicts the assumption. $\square$

2 The Dam Problem with Dirichlet Boundary Condition

In this section, we assume that $\mathcal{B}$ is given by (1.1.9). In this case we obtain the problem

$$(P_D)\begin{cases} \text{Find } (u,g) \in W^{1,q}(\Omega) \times L^\infty(\Omega) \text{ such that :} \\ i) \quad u = \psi \text{ on } S_2 \cup S_3 \\ ii) \quad u \geq x_2, \quad 0 \leq g \leq 1, \quad g(u - x_2) = 0 \ \text{ a.e. in } \Omega \\ iii) \quad \displaystyle\int_\Omega \big(\mathcal{A}(x,\nabla u) - g\mathcal{A}(x,e)\big).\nabla\xi dx \ \leq \ 0 \\ \forall \xi \in W^{1,q}(\Omega) \ \text{ such that } \ \xi = 0 \text{ on } S_3, \quad \text{and } \ \xi \geq 0 \text{ on } S_2. \end{cases}$$

2.1 Some Properties of the Solutions

Throughout this section, we shall denote a solution of (P_D) by (u,g). First we have the following regularity result:

Proposition 2.1.1. *$u \in C^{0,\alpha}_{loc}(\Omega \cup S_2 \cup S_3)$ for some $\alpha \in (0,1)$.*

Proof. This is a consequence of (1.3.1), (1.3.3) and $(P_D)i)$ (see [29]). □

Corollary 2.1.1. *The dam is saturated below S_3 i.e. we have*

$$u(x_1,x_2) > x_2 \quad \forall (x_1,x_2) \in \Omega, \ x_1 \in \pi_{x_1}(S_3).$$

Proof. Let $x_0 = (x_{01}, x_{02}) \in S_{3,i}$ for some $i \in \{1, ..., N\}$. Since $u(x_{01}, x_{02}) = \psi(x_{01}, x_{02}) > x_{02}$, we deduce from Proposition 2.1.1 that for some $\epsilon > 0$ small enough one has $u(x) > x_2$ in $B_\epsilon(x_0) \cap \Omega$. Using Theorem 1.3.1, we deduce that $u(x) > x_2$ below $B_\epsilon(x_0) \cap \Omega$. □

Remark 2.1.1. *By Corollary 2.1.1, we have*

$$\forall x_1 \in Int(\pi_{x_1}(S_3)) \quad \Phi(x_1) = s_+(x_1).$$

It follows that Φ is continuous on $Int(\pi_{x_1}(S_3)) \setminus S_+$.

The following theorem will be used several times in this section.

Theorem 2.1.1. *Let C_h be a connected component of $[u > x_2] \cap [x_2 > h]$ and $Z_h = \Omega \cap \big(\pi_{x_1}(C_h) \times (h, +\infty)\big)$. Assume that $\overline{Z_h} \cap S_3 = \emptyset$. Then we have*

$$\int_{Z_h} \big(\mathcal{A}(x,\nabla u) - g\mathcal{A}(x,e)\big).e dx \leq 0. \tag{2.1.1}$$

Proof. Let $(a_1,a_2)=\pi_{x_1}(C_h)$ and let for $\delta>0$ small enough, $\alpha_\delta\in\mathcal{D}((a_1,a_2))$ be a function such that $0\le\alpha_\delta(x_1)\le 1$ and $\alpha_\delta=1$ in $(a_1+\delta,a_2-\delta)$.
First we have

$$\int_{Z_h}\big(\mathcal{A}(x,\nabla u)-g\mathcal{A}(x,e)\big).edx=\int_{Z_h}\big(\mathcal{A}(x,\nabla u)-g\mathcal{A}(x,e)\big).\nabla(\alpha_\delta(x_2-h))dx$$
$$+\int_{Z_h}\big(\mathcal{A}(x,\nabla u)-g\mathcal{A}(x,e)\big).\nabla((1-\alpha_\delta)(x_2-h))dx. \tag{2.1.2}$$

Since $\overline{Z_h}\cap S_3=\emptyset$, $\chi(Z_h)\alpha_\delta(x_2-h)$ is a test function for (P_D) and we have

$$\int_{Z_h}\big(\mathcal{A}(x,\nabla u)-g\mathcal{A}(x,e)\big).\nabla(\alpha_\delta(x_2-h))dx\le 0. \tag{2.1.3}$$

Set $\zeta_\delta=(1-\alpha_\delta)(x_2-h)$ and remark that for $\epsilon>0$, $\pm\chi(Z_h).\Big(\dfrac{u-x_2}{\epsilon}\wedge\zeta_\delta\Big)$ are test functions for (P_D). So we have by taking into account $(P_D)ii)$

$$\int_{Z_h}\mathcal{A}(x,\nabla u).\nabla\Big(\frac{u-x_2}{\epsilon}\wedge\zeta_\delta\Big)dx=0 \tag{2.1.4}$$

Using the monotonicity of $\mathcal{A}$, we get from (2.1.4)

$$\int_{Z_h\cap[u-x_2\ge\epsilon\zeta_\delta]}\big(\mathcal{A}(x,\nabla u)-\mathcal{A}(x,e)\big).\nabla\zeta_\delta dx\le-\int_{Z_h}\mathcal{A}(x,e).\nabla\Big(\frac{u-x_2}{\epsilon}\wedge\zeta_\delta\Big)dx. \tag{2.1.5}$$

Note that

$$\begin{aligned}\int_{Z_h}\chi([u>x_2])\mathcal{A}(x,e).\nabla\zeta_\delta dx &=\int_{Z_h}\chi([u>x_2])\mathcal{A}(x,e).\nabla\Big(\zeta_\delta-\frac{u-x_2}{\epsilon}\Big)^+dx\\ &\quad+\int_{Z_h}\mathcal{A}(x,e).\nabla\Big(\frac{u-x_2}{\epsilon}\wedge\zeta_\delta\Big)dx.\end{aligned} \tag{2.1.6}$$

Since k is nondecreasing in x_2, we deduce by the second mean value theorem, that for a.e. $x_1\in(a_1,a_2)$, there exists $h^*(x_1)\in[h,\Phi(x_1)]$ such that

$$\begin{aligned}\int_{Z_h}\chi([u>x_2])\mathcal{A}(x,e).\nabla\Big(\zeta_\delta-\frac{u-x_2}{\epsilon}\Big)^+dx&=\int_{a_1}^{a_2}\Big(\int_h^{\Phi(x_1)}k(x)\Big(\zeta_\delta-\frac{u-x_2}{\epsilon}\Big)^+_{x_2}dx_2\Big)dx_1\\ &=\int_{a_1}^{a_2}k(x_1,\Phi(x_1)_-)\Big(\int_{h^*(x_1)}^{\Phi(x_1)}\Big(\zeta_\delta-\frac{u-x_2}{\epsilon}\Big)^+_{x_2}(x_1,x_2)dx_2\Big)dx_1\\ &\le\int_{a_1}^{a_2}k(x_1,\Phi(x_1)_-)\zeta_\delta(x_1,\Phi(x_1))dx_1\end{aligned} \tag{2.1.7}$$

where for a.e. $x_1\in(a_1,a_2)$, $k(x_1,\Phi(x_1)_-)$ is the left limit of $k(x_1,.)$ at $\Phi(x_1)$.

Adding (2.1.5), (2.1.6) and using (2.1.7), we get

$$\int_{Z_h\cap[u-x_2\geq\epsilon\zeta_\delta]}\big(\mathcal{A}(x,\nabla u)-\mathcal{A}(x,e)\big).\nabla\zeta_\delta dx+\int_{Z_h}\chi([u>x_2])\mathcal{A}(x,e).\nabla\zeta_\delta dx$$
$$\leq\int_{a_1}^{a_2}k(x_1,\Phi(x_1)_-)\zeta_\delta(x_1,\Phi(x_1))dx_1$$

which leads by letting $\epsilon\to 0$ to

$$\int_{Z_h}\big(\mathcal{A}(x,\nabla u)-\chi([u=x_2])\mathcal{A}(x,e)\big).\nabla\zeta_\delta dx\leq\int_{a_1}^{a_2}k(x_1,\Phi(x_1)_-)\zeta_\delta(x_1,\Phi(x_1))dx_1. \quad (2.1.8)$$

Using (2.1.2)-(2.1.3) and (2.1.8), we obtain

$$\int_{Z_h}\big(\mathcal{A}(x,\nabla u)-g\mathcal{A}(x,e)\big).edx\leq\int_{Z_h}(\chi([u=x_2])-g)k(x)(1-\alpha_\delta)$$
$$+\int_{a_1}^{a_2}k(x_1,\Phi(x_1)_-)\zeta_\delta(x_1,\Phi(x_1))dx_1.$$

Letting δ go to 0, we get (2.1.1). □

Remark 2.1.2. *Let $Z_h=\big((a_1,a_2)\times(h,+\infty)\big)\cap\Omega$. If $\overline{Z_h}\cap S_3=\emptyset$ and for $i=1,2$, we have $u(a_i,x_2)=x_2\ \forall x_2\geq h$, then the inequality (2.1.1) holds for the domain Z_h also .*

From now on, we assume that there is no impervious part above Ω. Then we have:

Theorem 2.1.2. *Let $x_0=(x_{01},x_{02})\in\Omega$ and $B_r=B_r(x_0)\subset\Omega$. If $u=x_2$ in B_r, then we have(see Figure 3):*

$$g\ =1\quad\ a.e.\ in\ D_r\ =\ \Big\{\ (x_1,x_2)\in\Omega\ /\ |x_1-x_{01}|<r\quad and\quad x_{02}<x_2\Big\}\cup B_r. \quad (2.1.9)$$

Proof. It is clear by Theorem 1.3.1 that we have $u=x_2$ in D_r, and therefore $\pi_{x_1}(B_r)\subset\pi_{x_1}(S_2)$. Applying Theorem 2.1.1 with domains $Z_h\subset D_r$ as in Remark 2.1.2, we obtain

$$0\leq\int_{Z_h}k(x)(1-g)dx\ =\ \int_{Z_h}\big(\mathcal{A}(x,\nabla u)-g\mathcal{A}(x,e)\big).edx\ \leq\ 0.$$

This leads to $g=1$ a.e. in Z_h. Thus $g=1$ a.e. in D_r. □

Now we prove a non-oscillation result.

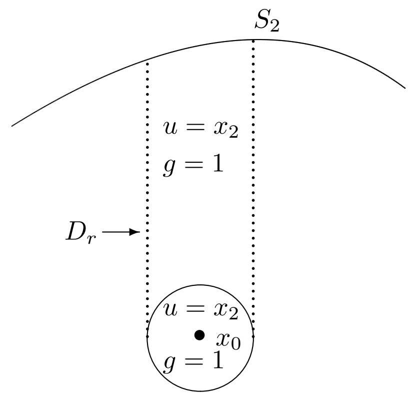

Figure 3

Theorem 2.1.3. *Let $x_0 = (x_{01}, x_{02}) \in \Omega$ such that $B_r = B_r(x_0) \subset \Omega$. Then the following situations (see Figure 4) are impossible:*

$$i) \begin{cases} u(x_1, x_2) \; = \; x_2 & \forall (x_1, x_2) \in B_r \cap [x_1 = x_{01}] \\ u(x_1, x_2) \; > \; x_2 & \forall (x_1, x_2) \in B_r, \quad x_1 \neq x_{01}. \end{cases}$$

$$ii) \begin{cases} u(x_1, x_2) \; > \; x_2 & \forall (x_1, x_2) \in B_r \cap [x_1 < x_{01}] \\ u(x_1, x_2) \; = \; x_2 & \forall (x_1, x_2) \in B_r \cap [x_1 \geq x_{01}]. \end{cases}$$

$$iii) \begin{cases} u(x_1, x_2) \; = \; x_2 & \forall (x_1, x_2) \in B_r \cap [x_1 \leq x_{01}] \\ u(x_1, x_2) \; > \; x_2 & \forall (x_1, x_2) \in B_r \cap [x_1 > x_{01}]. \end{cases}$$

Proof. *i*) From the assumption and $(P_D)ii)$, we have $g = 0$ a.e. in B_r and by (1.3.3) this leads to $div(\mathcal{A}(x, \nabla u)) = 0 \leq div(\mathcal{A}(x, \nabla x_2))$ in $\mathcal{D}'(B_r)$. Since neither $u > x_2$ in B_r nor $u \equiv x_2$ in B_r, we get a contradiction with the strong maximum principle(Lemma 1.3.1).

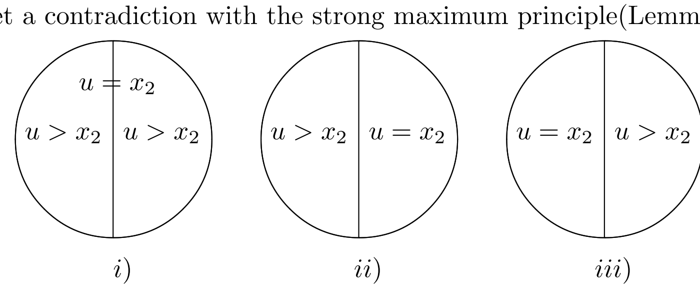

Figure 4

ii) From the assumption, $(P_D)ii)$, and (2.1.9), we have $g\mathcal{A}(x, e) = \chi(B_r \cap [x_1 > x_{01}])k(x)e$. Then by using (1.3.3) and (1.3.8), we obtain in $\mathcal{D}'(B_r)$

$$div(\mathcal{A}(x, \nabla u)) = div(\chi(B_r \cap [x_1 > x_{01}])k(x)e) \leq div(k(x)e) = div(\mathcal{A}(x, \nabla x_2)).$$

Hence we get a contradiction with Lemma 1.3.1, as in the previous case.

iii) Similar to *ii*). ☐

2.2 Continuity of the Free Boundary

For the rest of this section, we assume that $\mathcal{A}$ is strictly monotone i.e.

$$\big(\mathcal{A}(x,\xi)-\mathcal{A}(x,\zeta)\big).(\xi-\zeta)>0 \qquad \forall\xi,\,\zeta\in I\!R^2,\quad \xi\neq\zeta \quad \text{ a.e. } x\in\Omega. \tag{2.2.1}$$

The main result of this section is the continuity of the function Φ.

Theorem 2.2.1. *Φ is continuous at each point $x_{01}\in Int(\pi_{x_1}(S_2))$ such that $(x_{01},\Phi(x_{01}))\in\Omega$.*

Proof. Let $x_{01}\in Int(\pi_{x_1}(S_2))$ such that $x_0=(x_{01},\Phi(x_{01}))\in\Omega$ and let $\epsilon>0$. Using the continuity of u, there exists a ball $B_{\epsilon'}(x_0)$ $(0<\epsilon'<\epsilon)$ such that

$$\pi_{x_1}(B_{\epsilon'}(x_0))\subset S_2 \;\text{ and }\; u(x)\le x_2+\epsilon \quad \forall x\in B_{\epsilon'}(x_0). \tag{2.2.2}$$

By Theorem 2.1.3, we have for example

$$\exists\underline{x}=(\underline{x}_1,\underline{x}_2)\in B_{\epsilon'}(x_0) \quad \text{such that} \quad \underline{x}_1<x_{01} \quad \text{and} \quad u(\underline{x})=\underline{x}_2. \tag{2.2.3}$$

Then we set (see Figure 5) $h=\max(\underline{x}_2,\Phi(x_{01}))$, $Z=\big((\underline{x}_1,x_{01})\times(h,+\infty)\big)\cap\Omega$, $v=(\epsilon+h-x_2)^++x_2$ and $\xi=(u-v)^+$. Using (2.2.2)-(2.2.3), the fact that $u(x_0)=x_{02}$, and Corollary 1.3.1, it is clear that $\xi=0$ on ∂Z and therefore $\pm\xi$ are test functions for (P_D). So we have

$$\int_Z \big(\mathcal{A}(x,\nabla u)-g\mathcal{A}(x,e)\big).\nabla(u-v)^+dx=0. \tag{2.2.4}$$

A simple calculation shows that

$$\int_Z \big(\mathcal{A}(x,\nabla v)-\chi([v=x_2])\mathcal{A}(x,e)\big).\nabla(u-v)^+dx=0. \tag{2.2.5}$$

Since by (2.2.2) $\partial Z\cap S_3=\emptyset$, we have by Theorem 2.1.1 and Remark 2.1.2

$$\int_{Z\cap[v=x_2]} \big(\mathcal{A}(x,\nabla u)-g\mathcal{A}(x,e)\big).edx\le 0. \tag{2.2.6}$$

Subtracting (2.2.5) from (2.2.4) and adding (2.2.6) to the result, we obtain

$$\int_{Z\cap[v>x_2]} \big(\mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla v)\big).\nabla(u-v)^+dx+\int_{Z\cap[v=x_2]}\mathcal{A}(x,\nabla u).\nabla u-g\mathcal{A}(x,e).edx\le 0. \tag{2.2.7}$$

Note that

$$\begin{aligned}\int_{Z\cap[v=x_2]} \mathcal{A}(x,\nabla u).\nabla u - g\mathcal{A}(x,e).edx &= \int_{Z\cap[u>v=x_2]} \mathcal{A}(x,\nabla u).\nabla u \\ &+ \int_{Z\cap[u=v=x_2]} (1-g)k(x)dx \geq 0.\end{aligned}$$

It follows then from (2.2.7) that

$$\int_{Z\cap[v>x_2]} \big(\mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla v)\big).\nabla(u-v)^+dx \leq 0$$

which leads by (2.2.1) to $\nabla(u-v)^+ = 0$ a.e. in $Z\cap[v>x_2]$. By (2.2.2), we deduce that $(u-v)^+ = 0$ in $Z\cap[v>x_2]$. In particular, we obtain $u(x_1,h+\epsilon) = h+\epsilon$ $\forall x_1 \in (\underline{x}_1, x_{01})$ which leads by Corollary 1.3.1 to $u = x_2$ in $Z\cap[x_2\geq h+\epsilon]$.

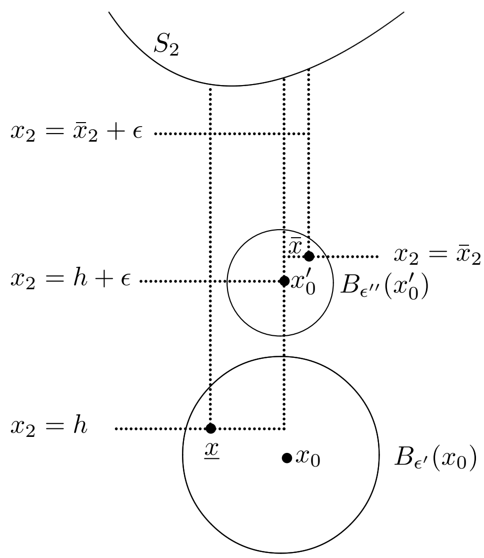

Figure 5

Let $x_0' = (x_{01}, h+\epsilon)$. Since $u(x_0') = h+\epsilon$, we deduce from the continuity of u that there exists a ball $B_{\epsilon''}(x_0')$ $(0<\epsilon''<\epsilon')$ such that $u(x)\leq x_2+\epsilon$ for all $x\in B_{\epsilon''}(x_0')$. Taking into account this result and Theorem 2.1.3, there exists $\bar{x} = (\bar{x}_1,\bar{x}_2) \in B_{\epsilon''}(x_0')$ such that: $x_{01} < \bar{x}_1$, $h+\epsilon\leq\bar{x}_2$ and $u(\bar{x}) = \bar{x}_2$. Set $Z' = \big((x_{01},\bar{x}_1)\times(\bar{x}_2,+\infty)\big)\cap\Omega$, $w = (\epsilon+\bar{x}_2-x_2)^+ + x_2$ and $\xi = (u-w)^+$. Then one can argue as in the previous step, to conclude that $(u-w)^+ = 0$ in $Z'\cap[w>x_2]$ and then $u = x_2$ in $Z'\cap[x_2\geq\bar{x}_2+\epsilon]$.

Finally, we have proved that $u = x_2$ in Z'', where $Z'' = \big((\underline{x}_1, \bar{x}_1) \times (\bar{x}_2 + \epsilon], +\infty)\big) \cap \Omega$. This leads to $\Phi(x_1) \leq \bar{x}_2 + \epsilon \leq \Phi(x_{01}) + 3\epsilon \ \forall x_1 \in (\underline{x}_1, \bar{x}_1)$. Hence Φ is upper semi-continuous at x_{01}. Taking into account Proposition 1.3.4, we obtain the continuity of Φ at x_{01} and the theorem is proved. $\square$

Remark 2.2.1. *For each $x_1 \in \pi_{x_1}(S_2 \cap S_3)$ such that $\{x_1\} \times (s_-(x_1), s_+(x_1)) \subset \Omega$ we have $\Phi(x_1) = s_+(x_1)$. Indeed otherwise we have $\Phi(x_{01}) < s_+(x_{01})$ for some point $x_{01} \in \pi_{x_1}(S_2 \cap S_3)$, with $\{x_{01}\} \times (s_-(x_{01}), s_+(x_{01})) \subset \Omega$. For clarity, we assume that the connected component of $\overline{S_2}$ (resp. $\overline{S_3}$) which contains $(x_{01}, s_+(x_{01}))$ is located to the left (resp. right) of the line $x_1 = x_{01}$. So there exists $\eta > 0$ such that $(x_{01} - \eta, x_{01}) \subset Int(\pi_{x_1}(S_2))$ and $(x_{01}, x_{01} + \eta) \subset Int(\pi_{x_1}(S_3))$. Now from Corollary 2.1.1, we have $u > x_2$ in $Z_+ = \big((x_{01}, x_{01} + \eta) \times (\Phi(x_{01}), +\infty)\big) \cap \Omega$. Arguing as in the proof of Theorem 2.2.1, one can show that for some $\epsilon > 0$ small enough we have $u = x_2$ in $Z_- = \big((x_{01} - \epsilon, x_{01}) \times (\Phi(x_{01}) + \epsilon, +\infty)\big) \cap \Omega$. Thus we get a contradiction with Theorem 2.1.3 iii).*

Remark 2.2.2. *Φ is continuous at each point $x_{01} \in Int(\pi_{x_1}(S_2)) \setminus S_-$ such that $\Phi(x_{01}) = s_-(x_{01})$. Indeed in this case, one has for each $\epsilon > 0$ small enough, $(x_{01}, \Phi(x_{01}) + \epsilon) \in \Omega$ and $u(x_{01}, \Phi(x_{01}) + \epsilon) = \Phi(x_{01}) + \epsilon$. Therefore one can adapt the proof of Theorem 2.2.1 to get $u = x_2$ in $\big((x_{01} - \epsilon', x_{01} + \epsilon') \times (\Phi(x_{01}) + 3\epsilon, +\infty)\big) \cap \Omega$ (for some $\epsilon' > 0$) which means the upper semi-continuity of Φ at x_{01}.*

As a consequence of the continuity of the function Φ, we obtain the expression of g.

Corollary 2.2.1. *We have*

$$g = \chi([u = x_2]). \tag{2.2.8}$$

Proof. First by (1.3.12) and $(P_D)ii)$, we have

$$g = 0 \quad \text{a.e. in } [x_2 < \Phi(x_1)]. \tag{2.2.9}$$

Now let $x_0 = (x_{01}, x_{02}) \in [x_2 > \Phi(x_1)]$. We have necessarily $s_+(x_{01}) > \Phi(x_{01})$. Moreover from Remarks 2.1.1 and 2.2.1, we deduce that $x_{01} \in Int(\pi_{x_1}(S_2))$. By Theorem 2.2.1 and Remark 2.2.2, Φ is continuous in $Int(\pi_{x_1}(S_2)) \setminus S_-$. Assume that $x_{01} \notin S_-$. By continuity, there exists a ball $B_r(x_0)$ such that $B_r(x_0) \subset [x_2 > \Phi(x_1)]$. From (2.1.9), we have $g = 1$ a.e. in $B_r(x_0)$. It follows that

$$g = 1 \quad \text{a.e. in } [x_2 > \Phi(x_1)]. \tag{2.2.10}$$

Finally because Φ is continuous except on a finite set, the set $[x_2 = \Phi(x_1)]$ is of Lebegue's measure zero. Thus we get by (2.2.9)-(2.2.10)

$$g \ = \ \chi([x_2 > \Phi(x_1)])$$

which is (2.2.8). $\square$

2.3 Existence and Uniqueness of Minimal and Maximal Solutions

In this section, we show the existence and uniqueness of two solutions which minimize (resp. maximize) a functional. Moreover one is minimal and the other one is maximal in the usual sense among all solutions. First we establish a key result.

Theorem 2.3.1. *Let (u_1, g_1) and (u_2, g_2) be two solutions of (P_D).*
Set $u_m = \min(u_1, u_2)$, $u_M = \max(u_1, u_2)$, $g_m = \min(g_1, g_2)$, and $g_M = \max(g_1, g_2)$. Then we have for $i = 1, 2$ and for all $\zeta \in W^{1,q}(\Omega)$

$$i) \quad \int_\Omega \big((\mathcal{A}(x, \nabla u_i) - \mathcal{A}(x, \nabla u_m)) - (g_i - g_M)\mathcal{A}(x, e)\big).\nabla\zeta dx = 0$$

$$ii) \quad \int_\Omega \big((\mathcal{A}(x, \nabla u_i) - \mathcal{A}(x, \nabla u_M)) - (g_i - g_m)\mathcal{A}(x, e)\big).\nabla\zeta dx = 0.$$

Proof. *i)* Let $\zeta \in C^1(\bar{\Omega})$, $\zeta \geq 0$. For $\delta, \epsilon > 0$, we consider $\alpha_\delta(x) = \Big(1 - \dfrac{d(x, A_m)}{\delta}\Big)^+$ and $\xi = \min\Big(\alpha_\delta\zeta, \dfrac{u_i - u_m}{\epsilon}\Big)$, where $A_m = [u_m > x_2]$. We have

$$\begin{aligned}
&\int_\Omega ((\mathcal{A}(x, \nabla u_i) - \mathcal{A}(x, \nabla u_m)) - (g_i - g_M)\mathcal{A}(x, e)).\nabla\zeta dx \\
&= \int_\Omega ((\mathcal{A}(x, \nabla u_i) - \mathcal{A}(x, \nabla u_m)) - (g_i - g_M)\mathcal{A}(x, e)).\nabla(\alpha_\delta\zeta) dx \\
&+ \int_\Omega ((\mathcal{A}(x, \nabla u_i) - \mathcal{A}(x, \nabla u_m)) - (g_i - g_M)\mathcal{A}(x, e)).\nabla\big((1 - \alpha_\delta)\zeta\big) dx.
\end{aligned} \tag{2.3.1}$$

Since $(1 - \alpha_\delta)\zeta$ is a test function for (P_D), we have

$$\int_\Omega \big(\mathcal{A}(x, \nabla u_i) - g_i\mathcal{A}(x, e)\big).\nabla\big((1 - \alpha_\delta)\zeta\big) dx \leq 0 \tag{2.3.2}$$

Given that $(1 - \alpha_\delta)\zeta = 0$ on A_m and by (2.2.8) $g_M = 1$ a.e. in $[u_m = x_2]$, we obtain

$$\int_\Omega \big(\mathcal{A}(x, \nabla u_m) - g_M\mathcal{A}(x, e)\big).\nabla\big((1 - \alpha_\delta)\zeta\big) dx = \int_{[u_m = x_2]} k(x)(1 - g_M)\big((1 - \alpha_\delta)\zeta\big)_{x_2} dx = 0 \tag{2.3.3}$$

Subtracting (2.3.3) from (2.3.2), we get

$$\int_\Omega ((\mathcal{A}(x, \nabla u_i) - \mathcal{A}(x, \nabla u_m)) - (g_i - g_M)\mathcal{A}(x, e)).\nabla\big((1 - \alpha_\delta)\zeta\big) dx \leq 0 \tag{2.3.4}$$

Now clearly $\pm\xi$ are test functions for (P_D). So we have for $i,\ j = 1, 2$ with $i \neq j$

$$\int_\Omega ((\mathcal{A}(x, \nabla u_i) - \mathcal{A}(x, \nabla u_j)) - (g_i - g_j)\mathcal{A}(x, e)).\nabla\xi dx = 0 \tag{2.3.5}$$

Since $\xi = 0$ on the set $[u_i = u_m]$, we have to integrate only on the set $[u_i - u_m > 0]$ where $u_m = u_j$. So (2.3.5) becomes

$$\int_\Omega ((\mathcal{A}(x,\nabla u_i) - \mathcal{A}(x,\nabla u_m)) - (g_i - g_M)\mathcal{A}(x,e)).\nabla\xi dx = 0$$

which can be written by the monotonicity of $\mathcal{A}$

$$\begin{aligned}\int_{[u_i - u_m \geq \epsilon\zeta]} (\mathcal{A}(x,\nabla u_i) - \mathcal{A}(x,\nabla u_m)).\nabla(\alpha_\delta\zeta)dx - \int_\Omega k(x)(g_i - g_M).(\alpha_\delta\zeta)_{x_2}dx \\ \leq \int_\Omega k(x)(g_M - g_i)\Big(\alpha_\delta\zeta - \frac{u_i - u_m}{\epsilon}\Big)^+_{x_2} dx. \qquad (2.3.6)\end{aligned}$$

Using (2.2.8), we have

$$\begin{aligned}\int_\Omega k(x)(g_M - g_i)\Big(\alpha_\delta\zeta - \frac{u_i - u_m}{\epsilon}\Big)^+_{x_2} dx &= \int_{[u_i > u_m = x_2]} k(x)\Big(\alpha_\delta\zeta - \frac{u_i - u_m}{\epsilon}\Big)^+_{x_2} dx \\ &= \int_{D_i} dx_1 \int_{\Phi_m(x_1)}^{\Phi_i(x_1)} k(x)\Big(\alpha_\delta\zeta - \frac{u_i - u_m}{\epsilon}\Big)^+_{x_2} dx_2. \qquad (2.3.7)\end{aligned}$$

with

$$D_i = \Big\{ x_1 \in \pi_{x_1}\big(\Omega\big) \,/\, \Phi_m(x_1) < \Phi_i(x_1) \Big\} \quad i = 1,2 \qquad \text{and} \qquad \Phi_m = \min(\Phi_1, \Phi_2).$$

Since k is nondecreasing in x_2, we deduce by the second mean-value theorem that for a.e. $x_1 \in D_i$, there exists $\Phi_*(x_1) \in [\Phi_m(x_1), \Phi_i(x_1)]$ such that

$$\int_{\Phi_m(x_1)}^{\Phi_i(x_1)} k(x)\Big(\alpha_\delta\zeta - \frac{u_i - u_m}{\epsilon}\Big)^+_{x_2} dx_2 = k(x_1, \Phi_i(x_1)_-) \int_{\Phi_*(x_1)}^{\Phi_i(x_1)} \Big(\alpha_\delta\zeta - \frac{u_i - u_m}{\epsilon}\Big)^+_{x_2} dx_2 \qquad (2.3.8)$$

Then by (2.3.6)-(2.3.8), we get

$$\begin{aligned}\int_{[u_i - u_m \geq \epsilon\zeta]} (\mathcal{A}(x,\nabla u_i) - \mathcal{A}(x,\nabla u_m)).\nabla(\alpha_\delta\zeta)dx - \int_\Omega k(x)(g_i - g_M).(\alpha_\delta\zeta)_{x_2}dx \\ \leq \int_{D_i} k(x_1, \Phi_i(x_1)_-)(\alpha_\delta\zeta)(x_1, \Phi_i(x_1))dx_1\end{aligned}$$

which leads by letting ϵ go to zero to

$$\begin{aligned}&\int_\Omega (\mathcal{A}(x,\nabla u_i) - \mathcal{A}(x,\nabla u_m) - (g_i - g_M)\mathcal{A}(x,e)).\nabla(\alpha_\delta\zeta)dx \\ &\leq \int_{D_i} k(x_1, \Phi_i(x_1)_-)(\alpha_\delta\zeta)(x_1, \Phi_i(x_1))dx_1 \leq \int_{D_i} (k\alpha_\delta\zeta)(x_1, \Phi_i(x_1))dx_1. \qquad (2.3.9)\end{aligned}$$

Using (2.3.1), (2.3.4) and (2.3.9), we get

$$\int_\Omega (\mathcal{A}(x,\nabla u_i)-\mathcal{A}(x,\nabla u_m)-(g_i-g_M)\mathcal{A}(x,e)).\nabla\zeta dx \le \int_{D_i} \big(k\alpha_\delta\zeta\big)(x_1,\Phi_i(x_1))dx_1. \quad (2.3.10)$$

Now since for each $x_{01}\in D_i$, we have $(x_{01},\Phi_i(x_{01}))\notin \bar{A}_m$, we deduce that $\alpha_\delta(x_{01},\Phi_i(x_{01}))$ converges to 0 when δ goes to 0. Using the Lebesgue theorem, we obtain by letting $\delta\to 0$ in (2.3.10)

$$\int_\Omega (\mathcal{A}(x,\nabla u_i)-\mathcal{A}(x,\nabla u_m)-(g_i-g_M)\mathcal{A}(x,e)).\nabla\zeta dx \le 0.$$

Remarking that the last inequality holds also for $M-\zeta$ with $M=\max_{\bar\Omega}\zeta$, we get

$$\int_\Omega (\mathcal{A}(x,\nabla u_i)-\mathcal{A}(x,\nabla u_m)-(g_i-g_M)\mathcal{A}(x,e)).\nabla\zeta dx = 0 \qquad \forall\zeta\in C^1(\bar\Omega),\ \zeta\ge 0.$$

Since $C^1(\bar\Omega)$ is dense in $W^{1,q}(\Omega)$ and since each function $\zeta\in W^{1,q}(\Omega)$ can be written as $\zeta=\zeta^+-\zeta^-$, we get

$$\int_\Omega (\mathcal{A}(x,\nabla u_i)-\mathcal{A}(x,\nabla u_m)-(g_i-g_M)\mathcal{A}(x,e)).\nabla\zeta dx = 0 \qquad \forall\zeta\in W^{1,q}(\Omega).$$

ii) It is enough to establish the result for $i=1$ since it is similar for $i=2$. We have for $\zeta\in W^{1,q}(\Omega)$

$$\begin{aligned}
&\int_\Omega (\mathcal{A}(x,\nabla u_1)-\mathcal{A}(x,\nabla u_M)-(g_1-g_m)\mathcal{A}(x,e)).\nabla\zeta dx\\
&\quad=\int_{[u_2\ge u_1]} (\mathcal{A}(x,\nabla u_1)-\mathcal{A}(x,\nabla u_2)-(g_1-g_2)\mathcal{A}(x,e)).\nabla\zeta dx\\
&\quad=-\int_{[u_2\ge u_1]} (\mathcal{A}(x,\nabla u_2)-\mathcal{A}(x,\nabla u_1)-(g_2-g_1)\mathcal{A}(x,e)).\nabla\zeta dx\\
&\quad=-\int_\Omega (\mathcal{A}(x,\nabla u_2)-\mathcal{A}(x,\nabla u_m)-(g_2-g_M)\mathcal{A}(x,e)).\nabla\zeta dx = 0.
\end{aligned}$$

□

As a consequence of Theorem 2.3.1, we obtain:

Corollary 2.3.1. *Let* (u_1,g_1) *and* (u_2,g_2) *be two solutions of* (P_D)*. Then* $(\min(u_1,u_2),\max(g_1,g_2))$ *and* $(\max(u_1,u_2),\min(g_1,g_2))$ *are also solutions of* (P_D)*.*

Proof. We will only prove that $(\min(u_1,u_2),\max(g_1,g_2))$ is a solution for (P_D). The proof that $(\max(u_1,u_2),\min(g_1,g_2))$ is a solution, is similar.
Indeed first it is clear that we have $(u_m,g_M)\in W^{1,q}(\Omega)\times L^\infty(\Omega)$, $u_m=\min(u_1,u_2)\geq x_2$ and $0\leq g_M=\max(g_1,g_2)\leq 1$ a.e. in Ω. Moreover if $u_m>x_2$, then $u_1>x_2$ and $u_2>x_2$, which leads to $g_1=g_2=0$ and therefore $g_M=0$.
Since $u_1=u_2=\psi$ on $S_2\cup S_3$, we have $u_m=\psi$ on $S_2\cup S_3$.
Finally let $\zeta\in W^{1,q}(\Omega)$ such that $\zeta=0$ on S_3 and $\zeta\geq 0$ on S_2. Then we have by Theorem 2.3.1 and since (u_i,g_i) is a solution of (P_D)

$$\int_\Omega(\mathcal{A}(x,\nabla u_m)-g_M\mathcal{A}(x,e)).\nabla\zeta dx=\int_\Omega(\mathcal{A}(x,\nabla u_i)-g_i\mathcal{A}(x,e)).\nabla\zeta dx\leq 0.$$

□

Consider now the set of all solutions of (P_D)

$$\mathcal{S}_D=\{\,(u,g)\in W^{1,q}(\Omega)\times L^\infty(\Omega)\,/\,(u,g)\text{ is a solution of }(P_D)\,\}.$$

We define the following mapping $\mathcal{I}_D$ on $\mathcal{S}_D$ by

$$\forall(u,g)\in\mathcal{S}_D,\qquad \mathcal{I}_D(u,g)=\int_\Omega\mathcal{A}(x,\nabla u).\nabla u dx-\frac{1}{q}\int_\Omega g\mathcal{A}(x,e).edx.$$

The main result of this section is the following theorem:

Theorem 2.3.2. *There exist a unique minimal solution (u_m,g_M) and a unique maximal solution (u_M,g_m) in $\mathcal{S}_D$ in the following sense:*

$$\mathcal{I}_D(u_m,g_M)=\min_{(u,g)\in\mathcal{S}_D}\mathcal{I}_D(u,g),\qquad \mathcal{I}_D(u_M,g_m)=\max_{(u,g)\in\mathcal{S}_D}\mathcal{I}_D(u,g)$$
$$\forall(u,g)\in\mathcal{S}_D\qquad u_m\leq u\leq u_M,\qquad g_m\leq g\leq g_M\quad in\quad\Omega.$$

We first prove a monotonicity result for $\mathcal{I}_D$.

Lemma 2.3.1. *$\mathcal{I}_D$ is strictly monotone i.e. for each (u_1,g_1), $(u_2,g_2)\in\mathcal{S}_D$:*

i) $u_1\leq u_2,\quad g_2\leq g_1\quad in\quad\Omega\qquad\Longrightarrow\qquad\mathcal{I}_D(u_1,g_1)\leq\mathcal{I}_D(u_2,g_2).$
ii) $u_1\leq u_2,\quad g_2\leq g_1\quad in\quad\Omega\quad and\quad u_1\neq u_2\qquad\Longrightarrow\qquad\mathcal{I}_D(u_1,g_1)<\mathcal{I}_D(u_2,g_2).$

Proof. *i)* Let (u_1,g_1), $(u_2,g_2)\in\mathcal{S}_D$ such that $u_1\leq u_2$ and $g_2\leq g_1$ a.e. in Ω. Then we have

$$\mathcal{I}_D(u_1,g_1)-\mathcal{I}_D(u_2,g_2)=\int_\Omega\big(\mathcal{A}(x,\nabla u_1).\nabla u_1-g_1\mathcal{A}(x,e).e\big)dx$$
$$-\int_\Omega\big(\mathcal{A}(x,\nabla u_2).\nabla u_2-g_2\mathcal{A}(x,e).e\big)dx+\frac{1}{q'}\int_\Omega(g_1-g_2)\mathcal{A}(x,e).edx$$

$$
\begin{aligned}
&= \int_\Omega \big(\mathcal{A}(x,\nabla u_1) - g_1\mathcal{A}(x,e)\big).\nabla u_1 dx - \int_\Omega \big(\mathcal{A}(x,\nabla u_2) - g_2\mathcal{A}(x,e)\big).\nabla u_2 dx \\
&+\frac{1}{q'}\int_\Omega (g_1-g_2)\mathcal{A}(x,e).edx \\
&= \int_\Omega \big(\mathcal{A}(x,\nabla u_1) - g_1\mathcal{A}(x,e)\big).\nabla (u_1-\psi) dx - \int_\Omega \big(\mathcal{A}(x,\nabla u_2) - g_2\mathcal{A}(x,e)\big).\nabla (u_2-\psi) dx \\
&+\frac{1}{q'}\int_\Omega (g_1-g_2)\mathcal{A}(x,e).edx + \int_\Omega \big((\mathcal{A}(x,\nabla u_1)-\mathcal{A}(x,\nabla u_2)) - (g_1-g_2)\mathcal{A}(x,e)\big).\nabla\psi dx.
\end{aligned}
$$

Using Theorem 2.3.1 $i)$ and the fact that $u_i - \psi$ $(i=1,2)$ are test functions for (P_D), we get

$$\mathcal{I}_D(u_1,g_1) - \mathcal{I}_D(u_2,g_2) = \frac{1}{q'}\int_\Omega (g_1-g_2)k(x)dx \le 0$$

which proves $i)$.

$ii)$ Assume that $u_1 \le u_2$ and $g_2 \le g_1$ a.e. in Ω and $\mathcal{I}_D(u_1,g_1) = \mathcal{I}_D(u_2,g_2)$. Then $\int_\Omega (g_1-g_2)k(x)dx = 0$ and therefore $g_1 = g_2$ a.e. in Ω. Using Theorem 2.3.1 $i)$ for $\xi = u_1 - u_2$, we get

$$\int_\Omega \big(\mathcal{A}(x,\nabla u_1) - \mathcal{A}(x,\nabla u_2)\big).\nabla(u_1-u_2)dx = 0$$

which leads by (2.2.1) to $\nabla(u_1-u_2) = 0$ a.e. in Ω. But since $u_1 - u_2 = 0$ on $S_2 \cup S_3$, we obtain $u_1 = u_2$ and $ii)$ is proved. □

Proof of Theorem 2.3.2. First remark that for each $(u,g) \in \mathcal{S}_D$, we have

$$\mathcal{I}_D(u_1,g_1) \ge -\frac{1}{q}\int_\Omega k(x)gdx \ge -\frac{|\Omega|}{q}M.$$

We deduce that there exists a minimizing sequence $(u_k,g_k)_{k\in I\!N}$ for $\mathcal{I}_D$ i.e.

$$\forall k \in I\!N \quad (u_k,g_k)\in\mathcal{S}_D \quad \text{and} \quad \lim_{k\to+\infty}\mathcal{I}_D(u_k,g_k) = m = \inf_{(u,g)\in\mathcal{S}_D}\mathcal{I}_D(u,g). \qquad (2.3.11)$$

Now we define another sequence $(v_k,f_k)_{k\in I\!N}$ by

$$\begin{cases} (v_0,f_0) = (u_0,g_0) \\ (v_{k+1},f_{k+1}) = \big(\min(u_{k+1},v_k), \max(g_{k+1},f_k)\big) \qquad \forall k\in I\!N. \end{cases}$$

By Corollary 2.3.1 and Lemma 2.3.1, it is clear that for all $k\in I\!N$, $(v_k,f_k)\in\mathcal{S}_D$ and we have

$$\begin{cases} \forall k\in I\!N \qquad v_k \le u_k, \qquad g_k \le f_k \\ \forall k \in I\!N \qquad m \le \mathcal{I}_D(v_k,f_k) \le \mathcal{I}_D(u_k,g_k). \end{cases}$$

This clearly leads to

$$m = \lim_{k\to+\infty} \mathcal{I}_D(v_k, f_k). \tag{2.3.12}$$

From the definition of $(v_k, f_k)_{k\in\mathbb{N}}$, we deduce that

$$\forall k \in \mathbb{N} \qquad v_{k+1} \le v_k \quad \text{and} \quad f_k \le f_{k+1} \quad \text{a.e. in } \Omega.$$

But since v_k and f_k are uniformly bounded ($x_2 \le v_k \le h_0$ by Proposition 1.3.1) and $0 \le f_k \le 1$ a.e. in Ω, we obtain by Beppo-Levi's theorem that there exists $(v, f) \in L^q(\Omega)\times L^{q'}(\Omega)$ such that

$$\begin{cases} v_k \longrightarrow v & \text{in} \quad L^q(\Omega) \qquad \text{and a.e. in } \Omega \\ f_k \longrightarrow f & \text{in} \quad L^{q'}(\Omega) \qquad \text{and a.e. in } \Omega \end{cases} \tag{2.3.13}$$

Now since $\pm(v_k - \psi)$ are test functions for (P_D), we obtain by using (1.1.2)$iii)$, $(P_D)ii)$ and the Hölder inequality

$$\begin{aligned} \int_\Omega \mathcal{A}(x, \nabla v_k).\nabla v_k dx &= \int_\Omega \mathcal{A}(x, \nabla v_k).\nabla \psi dx - \int_\Omega g\mathcal{A}(x, e).\nabla\varphi dx \\ &\le C\Big(\int_\Omega |\nabla v_k|^q dx\Big)^{1/q'} + C \end{aligned}$$

where C is some positive constant. By (1.1.2) $iii)$, we deduce that $(v_k)_k$ is bounded in $W^{1,q}(\Omega)$. So we have up to a subsequence

$$v_{k_p} \rightharpoonup v \quad \text{in} \quad W^{1,q}(\Omega) \tag{2.3.14}$$

$$v_{k_p} \longrightarrow v \quad \text{in} \quad L^q(S_3) \quad \text{and a.e. in } \ S_3. \tag{2.3.15}$$

From the continuity of the trace operator and (2.3.15), we have $v = \psi$ on $S_2 \cup S_3$. From (2.3.13), we obtain that

$$v \ge x_2, \qquad 0 \le f \le 1, \qquad f(v - x_2) = 0 \quad \text{a.e. in } \Omega.$$

We would like to prove that $(v, f) \in \mathcal{S}_D$. It suffices to verify that it satisfies $(P_D)iii)$.
From the fact that (v_k) is bounded in $W^{1,q}(\Omega)$, we deduce that up to a subsequence still denoted by (v_{k_p}), one has

$$\mathcal{A}(x, \nabla v_{k_p}) \rightharpoonup \mathcal{A}_0 \qquad \text{in} \quad \mathbb{L}^{q'}(\Omega). \tag{2.3.16}$$

Let $p, s \in \mathbb{N}$ such that $p \le s$. We have by Theorem 2.3.1 $i)$

$$\int_\Omega \{\big(\mathcal{A}(x, \nabla v_{k_p}) - \mathcal{A}(x, \nabla v_{k_s})\big) - (f_{k_p} - f_{k_s})\mathcal{A}(x, e)\}.\nabla v_{k_p} dx = 0$$

from which we deduce by letting respectively $s \to +\infty$ and $p \to +\infty$, and using (2.3.14) and (2.3.16)

$$\lim_{p\to+\infty}\int_\Omega \mathcal{A}(x,\nabla v_{k_p}).\nabla v_{k_p}dx=\int_\Omega \mathcal{A}_0.\nabla v dx. \tag{2.3.17}$$

Using the monotonicity of $\mathcal{A}$, (2.3.14), (2.3.16) and (2.3.17) , we easily obtain

$$\mathcal{A}(x,\nabla v_{k_p})\rightharpoonup \mathcal{A}(x,\nabla v) \qquad \text{in} \qquad I\!\!L^{q'}(\Omega). \tag{2.3.18}$$

Finally, let $\xi\in W^{1,q}(\Omega)$ such that $\xi\geq 0$ on S_2 and $\xi=0$ on S_3. For any $p\in I\!\!N$, we have

$$\int_\Omega \big(\mathcal{A}(x,\nabla v_{k_p})-f_{k_p}\mathcal{A}(x,e)\big).\nabla\xi dx\leq 0. \tag{2.3.19}$$

Using (2.3.13), (2.3.18), we get by letting $p\to+\infty$ in (2.3.19)

$$\int_\Omega \big(\mathcal{A}(x,\nabla v)-f\mathcal{A}(x,e)\big).\nabla\xi dx\leq 0.$$

Thus (v,f) is a solution of (P_D).
Now, using (2.3.12)-(2.3.13) and (2.3.16)-(2.3.18), we obtain

$$\begin{aligned} m=\lim_{p\to+\infty}\mathcal{I}_D(v_{k_p},f_{k_p}) &= \lim_{p\to+\infty}\int_\Omega \mathcal{A}(x,\nabla v_{k_p}).\nabla v_{k_p}dx-\frac{1}{q}\int_\Omega f_{k_p}\mathcal{A}(x,e).edx\\ &=\int_\Omega \mathcal{A}(x,\nabla v).\nabla v dx-\frac{1}{q}\int_\Omega f\mathcal{A}(x,e).edx=\mathcal{I}_D(v,f).\end{aligned}$$

Let $(u,g)\in\mathcal{S}_D$. Since $\big(\min(u,v),\max(g,f)\big)\in\mathcal{S}_D$, we deduce that:
$\mathcal{I}_D(v,f)=\mathcal{I}_D\big(\min(u,v),\max(g,f)\big)$ which leads by Lemma 2.3.1 to:
$(v,f)=\big(\min(u,v),\max(g,f)\big)$ i.e. $v\leq u$ and $g\leq f$ a.e. in Ω. The uniqueness of (v,f) is clear. This achieves the proof of the first part of Theorem 2.3.2.

Let us prove the second part of the theorem. First remark that

$$\begin{aligned}\mathcal{I}_D(u,g) &= \int_\Omega \mathcal{A}(x,\nabla u).\nabla u dx-\frac{1}{q}\int_\Omega gk(x)dx\\ &\leq \int_\Omega \mathcal{A}(x,\nabla u).\nabla u dx\leq M\int_\Omega |\nabla u|^q dx.\end{aligned}$$

Moreover

$$\int_\Omega \big(\mathcal{A}(x,\nabla u)-g\mathcal{A}(x,e)\big).\nabla(u-\psi)dx=0$$

which leads by (1.1.2)iii) to

$$\begin{aligned}\lambda\int_\Omega |\nabla u|^q dx &\leq \int_\Omega \mathcal{A}(x,\nabla u).\nabla u dx\\ &= \int_\Omega \mathcal{A}(x,\nabla u).\nabla\psi dx-\int_\Omega g\mathcal{A}(x,e).\nabla\varphi dx\end{aligned}$$

$$\leq \quad M\Big(\int_\Omega |\nabla u|^q dx\Big)^{1/q'} . \Big(\int_\Omega |\nabla \psi|^q dx\Big)^{1/q} + c$$

and then

$$\int_\Omega |\nabla u|^q dx \leq C \qquad \text{for some positive constant } C.$$

Thus $\mathcal{I}_D(u,g)$ is bounded for all $(u,g) \in \mathcal{S}_D$. Let then $(u_k, g_k)_{k\in I\!N}$ be a sequence in $\mathcal{S}_D$ such that

$$\lim_{k\to+\infty} \mathcal{I}_D(u_k, g_k) = \sup_{(u,g)\in\mathcal{S}_D} \mathcal{I}_D(u,g).$$

We consider the following sequence $(w_k, h_k)_{k\in I\!N}$ defined by

$$\begin{cases}(w_0, h_0) = (u_0, g_0) \\ (w_{k+1}, h_{k+1}) = \big(\max(u_{k+1}, w_k), \min(g_{k+1}, h_k)\big) \qquad \forall k \in I\!N.\end{cases}$$

By Corollary 2.3.1, we have $(w_k, h_k) \in \mathcal{S}_D$ for each $k \in I\!N$. By Lemma 2.3.1, we obtain

$$\forall k \in I\!N \qquad \mathcal{I}_D(u_k, g_k) \leq \mathcal{I}_D(w_k, h_k) \leq \sup_{(u,g)\in\mathcal{S}_D} \mathcal{I}_D(u,g).$$

Therefore

$$\sup_{(u,g)\in\mathcal{S}_D} \mathcal{I}_D(u,g) = \lim_{k\to+\infty} \mathcal{I}_D(w_k, h_k). \tag{2.3.20}$$

We have also

$$\forall k \in I\!N \qquad w_k \leq w_{k+1} \quad \text{and} \quad h_{k+1} \leq h_k \quad \text{a.e. in } \Omega.$$

Using the monotonicity of $(w_k, h_k)_{k\in I\!N}$, (2.3.20) and arguing as above, we prove that for a subsequence $(w_{k_p}, h_{k_p})_{p\in I\!N}$, we have

$$\begin{aligned}
&w_{k_p} \rightharpoonup w \quad \text{in} \quad W^{1,q}(\Omega) \\
&w_{k_p} \longrightarrow w \quad \text{in} \quad L^q(\Omega) \quad \text{and a.e. in } \Omega \\
&w_{k_p} \longrightarrow w \quad \text{in} \quad L^q(S_3) \quad \text{and a.e. in } S_3 \\
&\mathcal{A}(x, \nabla w_{k_p}) \rightharpoonup \mathcal{A}(x, \nabla w) \quad \text{in} \quad I\!L^{q'}(\Omega) \\
&h_{k_p} \longrightarrow h \quad \text{in} \quad L^{q'}(\Omega) \quad \text{and a.e. in } \Omega.
\end{aligned}$$

Thus we obtain that (w,h) is a solution of (P_D) which satisfies $\mathcal{I}_D(w,h) = \sup_{(u,g)\in\mathcal{S}_D} \mathcal{I}_D(u,g)$. We also prove, as in the case of minimal solution, that for all $(u,g) \in \mathcal{S}_D$: $u \leq w$, $h \leq g$ a.e. in Ω. $\square$

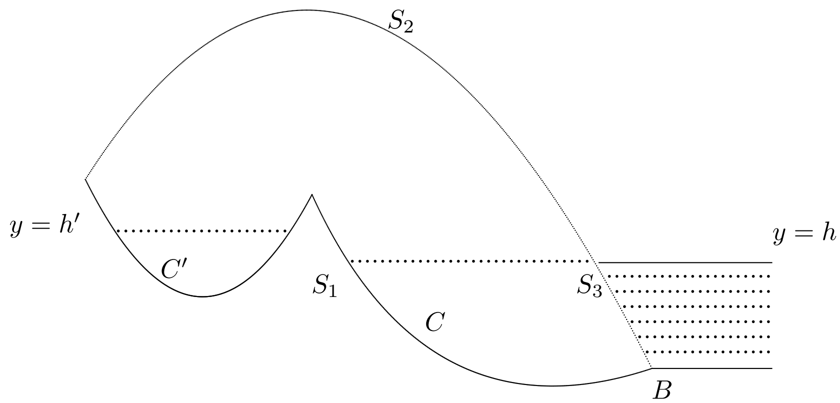

Figure 6

2.4 Reservoirs-Connected Solution

Assume that we are in the situation of Figure 6 with C and C' denoting the regions shown in the figure. Then it is not difficult to verify that

$$(u,g) = \begin{cases} (h,0) & \text{in } C \\ (x_2,1) & \text{otherwise} \end{cases}$$

and

$$(u,g) = \begin{cases} (h,0) & \text{in } C \\ (h',0) & \text{in } C' \\ (x_2,1) & \text{elsewhere,} \end{cases}$$

are solutions of (P_D). Moreover we can obtain more solutions just by replacing h' by any $0 < k < h'$. This example is an extension of an example given in [24] in the case of linear Darcy's law. It shows that in general the solution of the problem (P_D) is not unique. The first solution in the previous example is such that the only connected component of $[u > x_2]$ is connected to the unique reservoir. It seems that it is the only solution that is relevant from the physical point of view. This type of solution was introduced in [21] under the name of S_3-connected solution. Here we call it reservoirs-connected solution. Hence we have the definition:

Definition 2.4.1. *A solution (u,g) of (P_D) is called a reservoirs connected solution if for each connected component C of $[u > x_2]$, we have $\bar{C} \cap S_3 \neq \emptyset$.*

Remark 2.4.1. *If C_i is the connected component of $[u > x_2]$ that contains $S_{3,i}$ on its boundary, then by continuity and thanks to Remark 2.1.1, C_i contains the strip of Ω below $S_{3,i}$.*

The following theorem characterizes the connected components of $[u > x_2]$ which are not related to S_3.

Theorem 2.4.1. *Let (u, g) be a solution of (P_D) and C a connected component of $[u > x_2]$ such that $\bar{C} \cap S_3 = \emptyset$. If we set $h_c = \sup\{ x_2 \,/\, (x_1, x_2) \in C \}$. Then we have*

$$\begin{cases} C = \{ (x_1, x_2) \in \Omega \,/\, x_1 \in \pi_{x_1}(C)\,,\ x_2 < h_c \ \}, \\ u = x_2 + (h_c - x_2)^+.\chi(C), \quad g \ = \ 1 - \chi(C) \quad in \quad Z = \Omega \cap (\pi_{x_1}(C) \times \mathbb{R}) \end{cases}$$

Proof. By assumption and Theorem 1.3.1, we have $\pi_{x_1}(C) \subset \pi_{x_1}(S_2)$. Then $\pm\chi(Z)(u - x_2) = \pm\chi(C)(u - x_2)$ are test functions of (P_D) and we have

$$\int_Z (\mathcal{A}(x, \nabla u) - g\mathcal{A}(x, e)).\nabla(u - x_2) = 0. \tag{2.4.1}$$

Applying Theorem 2.1.1 to Z, we obtain

$$\int_Z (\mathcal{A}(x, \nabla u) - g\mathcal{A}(x, e)).e \leq 0. \tag{2.4.2}$$

Adding (2.4.1) and (2.4.2), we get

$$\int_Z (\mathcal{A}(x, \nabla u).\nabla u - kg) \leq 0$$

which leads by (1.1.2)iii) to

$$\int_{Z \cap [u > x_2]} \lambda |\nabla u|^q + \int_{Z \cap [u = x_2]} k(1 - g) \leq 0.$$

It follows that $\nabla u = 0$ a.e. in $Z \cap [u > x_2] = C$ and $g = 1$ a.e. in $Z \cap [u = x_2] = Z \setminus C$. Hence we obtain $u = x_2 + (h_c - x_2)^+.\chi(C)$ and $g = 1 - \chi(C)$ a.e. in Z. □

The result of Theorem 2.4.1 leads to the following definition (see [21] and also [23] for an extension):

Definition 2.4.2. *We call a pool in Ω a pair of functions defined in Ω by $(p, \chi) = ((h - x_2)^+, 1)\chi(C)$, where C is a connected component of $\Omega \cap [x_2 < h]$.*

Remark 2.4.2. *Thanks to this definition, Theorem 2.4.1 becomes:*
For each solution (u, g) of (P_D) and each connected component C of $[u > x_2]$ such that $\bar{C} \cap S_3 = \emptyset$, $(u - x_2, 1 - g)$ agrees with a pool in the strip $\Omega \cap (\pi_{x_1}(C) \times \mathbb{R})$.

Now we have:

Theorem 2.4.2. *Each solution* (u, g) *of* (P_D) *can be written as*

$$u = u_r + \sum_{i \in I} p_i \quad \text{and} \quad g = g_r - \sum_{i \in I} \chi_i$$

where (u_r, g_r) *is a reservoirs-connected solution and* (p_i, χ_i) *are pools.*

Proof. Let $(C_i)_{i \in I}$ be the family of all connected components of $[u > x_2]$ such that $C_i \cap S_3 = \emptyset$. Set

$$(u', g') = (u, g) - \sum_{i \in I} \big(\chi(C_i)(u - x_2), -\chi(C_i)\big).$$

Since each connected component C of $[u' > x_2]$ is such that $C \cap S_3 \neq \emptyset$, it follows that if (u', g') is a solution of (P_D), it will be a reservoirs-connected solution. Let us then verify that (u', g') is a solution of (P_D).

i) $(u', g') \in W^{1,q}(\Omega) \times L^\infty(\Omega)$:

Since we have

$$(u - x_2)\chi(C_i) \in W^{1,q}(\Omega), \text{ and } \nabla(\chi(C_i)(u - x_2)) = \chi(C_i)\nabla(u - x_2) \quad \forall i \in I \quad [21],$$

we deduce that

$$\nabla u' = \nabla u - \sum_{i \in I} \chi(C_i)\nabla(u - x_2) \in L^q(\Omega).$$

Moreover since $0 \leq g' \leq 1$ a.e. in Ω, we have $g' \in L^\infty(\Omega)$.

ii) Clearly we have $u' \geq x_2$ and $g'(u' - x_2) = 0$ a.e. in Ω.

iii) Let $\xi \in W^{1,q}(\Omega)$ such that $\xi \geq 0$ on S_2 and $\xi = 0$ on S_3. Using Theorem 2.4.1, we have

$$\int_\Omega (\mathcal{A}(x, \nabla u') - g'\mathcal{A}(x, e)).\nabla\xi = \int_\Omega (\mathcal{A}(x, \nabla u) - g\mathcal{A}(x, e)).\nabla\xi \leq 0.$$

Thus the Theorem is proved. □

We deduce immediately from Theorem 2.4.2:

Corollary 2.4.1. *The minimal solution* (u_m, g_M) *is a reservoirs-connected solution.*

2.5 Uniqueness of the Reservoirs-Connected Solution

In this section, we address the question of uniqueness of the reservoirs-connected solution.

2.5.1 The case of Linear Darcy's Law

Here we assume that:

$$\mathcal{A}(x,\xi) = a(x).\xi \quad \forall \xi \in I\!R^2, \text{ a.e. } x \in \Omega,\ a(x) = (a_{ij}(x)) \text{ is a } 2 \times 2 \text{ matrix.} \tag{2.5.1}$$

$$\exists \lambda, M > 0 : \quad \lambda|\xi|^2 \leq a(x)\xi.\xi \leq M|\xi|^2 \quad \forall \xi \in I\!R^2, \text{ a.e. } x \in \Omega. \tag{2.5.2}$$

$$a_{12}(x) = 0 \quad \text{ for a.e. } x \in \Omega. \tag{2.5.3}$$

$$\tfrac{\partial a_{22}}{\partial x_2} \geq 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{2.5.4}$$

Theorem 2.5.1. *Assume that (2.5.1)-(2.5.4) are satisfied. Then there is one and only one reservoirs-connected solution.*

Proof. Let (u, g) be a reservoirs-connected solution of (P_D). Let C_i (resp. $C_{m,i}$) be the connected component of $[u > x_2]$ (resp. $[u_m > x_2]$) such that $\overline{C_i} \cap S_{3,i} \neq \emptyset$ (resp. $\overline{C_{m,i}} \cap S_{3,i} \neq \emptyset$). Using Remark 2.1.1, we know that C_i and $C_{m,i}$ contain $S_{3,i}$ on their boundaries as well as the strip below it.
Consider $\zeta \in \mathcal{D}(C_{m,i} \cup B_r(x_i))$, where $x_i \in S_{3,i}$ and r is a small positive number. Then from Theorem 2.3.1 *i*), we have since $g = g_M = 0$ in $C_{m,i}$

$$\int_{C_{m,i}} a(x)\nabla(u - u_m).\nabla\zeta dx = 0. \tag{2.5.5}$$

Define w in $\Omega_i = C_{m,i} \cup B_r(x_i)$ by $w = \chi(C_{m,i})(u - u_m)$. Since $u = u_m = \psi$ on S_3, it is clear that $w \in H^1(\Omega_i)$ and if one extends a by I_2 into $B_r(x_i) \setminus C_{m,i}$, we obtain from (2.5.5)

$$\int_{\Omega_i} a(x)\nabla w.\nabla\zeta dx = 0 \qquad \forall \zeta \in \mathcal{D}(\Omega_i). \tag{2.5.6}$$

Using the strict ellipticity of a, the fact that $w \geq 0$ in Ω_i, $w = 0$ in $B_r(x_i) \setminus C_{m,i}$ and the strong maximum principle, we get from (2.5.6) that $w = 0$ in Ω_i which leads to $u = u_m$ in $C_{m,i}$. Now we prove that $C_i = C_{m,i}$. Indeed since $C_{m,i}$ is a nonempty open set in the connected set C_i, it suffices to prove that $C_{m,i}$ is also closed relative to C_i. Indeed let $(x_k)_k$ be a sequence of points in $C_{m,i}$ which converges to an element x in C_i. By continuity of u and u_m, we obtain $u(x) = u_m(x)$. Since $u(x) > x_2$, we obtain $u_m(x) > x_2$ which means that $x \in C_{m,i}$.
Hence $u = u_m$ in Ω and from Corollary 2.2.1, we get $g = g_M$ in Ω. □

2.5.2 The case of a Nonlinear Darcy's Law

In this section, we prove the uniqueness of the reservoirs-connected solution for a Darcy's law corresponding to:

$$\mathcal{A}(x,\xi) = |a(x)\xi.\xi|^{\frac{q-2}{2}} a(x)\xi, q > 1, q \neq 2 \text{ and } a(x) = (a_{ij}(x)) \text{ is a } 2 \times 2 \text{ matrix.} \tag{2.5.7}$$

Moreover, we assume that:

$$a(x) \text{ is symmetric and belongs to } C^{0,1}(\Omega). \tag{2.5.8}$$

$$\exists \lambda, M > 0: \ \lambda|\xi|^2 \leq a(x)\xi.\xi \leq M|\xi|^2 \ \forall \xi \in I\!R^2, \text{ a.e. } x \in \Omega. \tag{2.5.9}$$

$$\text{For each } i \in \{1,...,N\}, \varphi = h_i - x_2 \text{ on } S_{3,i}. \tag{2.5.10}$$

$$\exists r_i > 0, \exists x_i \in S_{3,i}, \exists \alpha_i \in (0,1) : S_{3,i}^{r_i} = S_{3,i} \cap B_r(x_i) \text{ is } C^{1,\alpha_i}. \tag{2.5.11}$$

$$a_{12}(x) = 0 \quad \text{for a.e. } x \in \Omega. \tag{2.5.12}$$

$$\frac{\partial a_{22}^{q/2}}{\partial x_2} \geq 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{2.5.13}$$

Then we have:

Theorem 2.5.2. *Assume that (2.5.7)-(2.5.13) are satisfied. Then there is one and only one reservoirs-connected solution.*

To prove Theorem 2.5.2, we need three lemmas. We shall denote by (u, g) a reservoirs-connected solution of (P_D) and for each $i \in \{1, ..., N\}$, we shall denote by C_i (resp. $C_{m,i}$) the connected component of $[u > x_2]$ (resp. $[u_m > x_2]$) which contains $S_{3,i}$ on its boundary.

Lemma 2.5.1. *For each $i \in \{1, \ldots, N\}$, we have the following alternatives:*

$$\begin{array}{l} i) \text{ either } \exists x_i' \in S_{3,i}^{r_i}, \quad \exists r_i' \in (0, r_i) : \quad \forall x \in \overline{B(x_i', r_i') \cap \Omega} \qquad \nabla u(x) \neq 0, \\ ii) \text{ or } u = h_i \text{ in } C_i. \end{array}$$

Proof. First note that u satisfies

$$\begin{cases} div(\mathcal{A}(x, \nabla u)) = 0 & \text{in} \quad B(x_i, r_i) \cap \Omega \\ u = \psi = h_i & \text{on} \quad B(x_i, r_i) \cap \partial\Omega. \end{cases}$$

We deduce that for all $r \in (0, r_i)$, we have $u \in C^{1,\alpha_i}(\overline{B(x_i, r) \cap \Omega})$ [34]. So either $i)$ is true or we must have $\nabla u(x) = 0 \ \forall x \in S_{3,i}^{r_i}$. Assume that we are in the second case and set

$$\begin{cases} w_0(x) = u(x) - h_i & \text{for } x \in B(x_i, r_i) \cap \bar{\Omega} \\ w_0(x) = 0 & \text{for } x \in B(x_i, r_i) \setminus \Omega. \end{cases}$$

Since $u - h_i = 0$ on $S_{3,i}^{r_i}$, we have $w_0 \in W^{1,q}(B(x_i, r_i))$. Moreover because $u \in C^{1,\alpha_i}(B(x_i, r_i) \cap \overline{\Omega})$ and $\nabla u = 0$ on $S_{3,i}^{r_i}$, we deduce that $div(\mathcal{A}(x, \nabla w_0)) = 0$ in $\mathcal{D}'(B(x_i, r_i))$.
Now since $\nabla w_0 = 0$ in $B(x_i, r_i) \setminus \Omega$ and since the zeros of the gradient of a nonconstant $\mathcal{A}$-Harmonic function, under the conditions (2.5.7)-(2.5.9) are isolated [5], we conclude that $w_0 = 0$ in $B(x_i, r_i) \cap \Omega$ i.e. $u = h_i$ in $B(x_i, r_i) \cap \Omega$.
Arguing as before, $u - h_i$ is an $\mathcal{A}$-Harmonic function in C_i such that $u - h_i = 0$ and $\nabla(u - h_i) = 0$ in $B(x_i, r_i) \cap C_i$. We conclude that $u - h_i = 0$ in C_i. □

Lemma 2.5.2. *If u and u_m are not both constant in C_i and $C_{m,i}$ respectively, then there exists $x_i' \in B(x_i, r_i) \cap S_{3,i}$, $r_i' \in (0, r_i)$, $0 < \lambda_0$, $\lambda_1 < +\infty$ such that*

$$\forall x \in \overline{B(x_i', r_i') \cap \Omega}, \qquad \lambda_0 \leq \lambda(x) \leq \lambda_1, \tag{2.5.14}$$

where $\lambda(x) = \displaystyle\int_0^1 |\nabla w_t(x)|^{q-2} dt$ *and* $w_t = tu + (1-t)u_m$.

Proof. We will consider only the case where u is not constant in C_i. So the situation $i)$ of Lemma 2.5.1 holds. Since u and u_m are of class C^1 in $\overline{B(x_i, r_i) \cap \Omega}$, there exists x_i', $r_i' \in (0, r_i)$, $c_i, c_i' > 0$ such that

$$c_i \leq |\nabla u(x)| \leq c_i' \quad \forall x \in K_i = \overline{B(x_i', r_i') \cap \Omega}. \tag{2.5.15}$$

$$|\nabla u_m(x)| \leq c_m \quad \forall x \in K_i. \tag{2.5.16}$$

We distinguish two cases:

<u>$Case\, 1: \;\; q > 2$</u>:

Using (2.5.15)-(2.5.16), we obtain

$$\lambda(x) = \int_0^1 |\nabla w_t(x)| dt \leq (c_m + c_i')^{(q-2)} = \lambda_1 \quad \forall x \in K_i. \tag{2.5.17}$$

Now clearly $\lambda(x)$ is continuous on K_i. Let us denote by λ_0 the minimum value of $\lambda(x)$ on K_i. There exists $x_* \in K_i$ such that $\lambda_0 = \lambda(x_*)$. We claim that $\lambda_0 > 0$. Indeed otherwise we will have since $q > 2$, $\nabla w_t(x_*) = 0$ for all $t \in [0,1]$. This leads to $\nabla u(x_*) = \nabla u_m(x_*) = 0$ which is impossible.

<u>$Case\, 2: \;\; 1 < q < 2$</u>:

Using (2.5.15)-(2.5.16), we obtain

$$\lambda(x) = \int_0^1 |\nabla w_t(x)| dt \geq (c_m + c_i')^{(q-2)} = \lambda_0 \quad \forall x \in K_i.$$

We would like to show that $\lambda(x) \leq \lambda_1 < \infty$ in K_i.
If $\nabla w_t(x)$ does not vanish for each $(t, x) \in [0,1] \times K_i$, then $|\nabla w_t(x)|^{q-2}$ is continuous in $[0,1] \times K_i$ and therefore $\lambda(x)$ is continuous in K_i. If we denote by λ_1 the maximum value of $\lambda(x)$ on K_i. Then we have $\lambda(x) \leq \lambda_1 < \infty$ in K_i.

If $\nabla w_t(x) = 0$ for some $(t, x) \in [0,1] \times K_i$, then $t \in [0, 1)$. Otherwise we will have $\nabla u(x) = 0$. Moreover if there exists two values $t_1 \neq t_2 \in [0,1]$ such that $\nabla w_{t_i}(x) = 0$, then $\nabla u(x) = \nabla u_m(x) = 0$. Therefore for each $x \in K_i$ there exists at most one value $t(x) \in [0, 1)$ such that $\nabla w_{t(x)}(x) = 0$. In this case, we have $\nabla u_m(x) = \frac{-t(x)}{1-t(x)} \nabla u(x)$ and

$$\lambda(x) \quad = \quad \frac{|\nabla u(x)|^{q-2}}{(1-t(x))^{q-2}} \int_0^1 |t - t(x)|^{q-2} dt$$

$$= \frac{(1-t(x))^{2-q}}{q-1}\big[(1-t(x))^{q-1}+t^{q-1}(x)\big]|\nabla u(x)|^{q-2} \leq \frac{2c_i^{q-2}}{q-1}.$$

□

Lemma 2.5.3. *For each $i \in \{1,\dots,N\}$, there exists $x_i' \in B(x_i,r_i) \cap S_{3,i}$, $r_i' \in (0,r_i)$ such that*

$$u = u_m \qquad \text{in } \; B(x_i',r_i') \cap \Omega.$$

Proof. If u is constant in C_i and u_m is constant in $C_{m,i}$, then the result is trivial by Lemma 2.5.1.
In the following, we assume that either u is not constant in C_i or u_m is not constant in $C_{m,i}$. By Lemma 2.5.2, we know that there exists $x_i' \in S_{3,i}^{r_i}$, $r_i' \in (0,r_i)$, and λ_0, $\lambda_1 > 0$ such that

$$\forall x \in \overline{B(x_i',r_i') \cap \Omega} \qquad \lambda_0 \leq \lambda(x) \leq \lambda_1 < +\infty. \tag{2.5.18}$$

Since $B(x_i',r_i') \cap \Omega \subset C_i \cap C_{m,i}$ and $g = g_M = 0$ a.e. in $C_i \cap C_{m,i}$, we obtain from Theorem 2.3.1 i)

$$\int_{B(x_i',r_i')\cap\Omega} (\mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla u_m)).\nabla\zeta dx = 0 \qquad \forall \zeta \in \mathcal{D}(B(x_i',r_i')). \tag{2.5.19}$$

Note that for each $x \in B(x_i',r_i') \cap \Omega$, we have

$$\mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla u_m)) = \int_0^1 \frac{d}{dt}\mathcal{A}(x,\nabla w_t(x))dt. \tag{2.5.20}$$

Writing $w = u - u_m$, $A(x) = (A_{ij}(x))$, with $A_{ij}(x) = \int_0^1 \frac{\partial}{\partial \zeta_j}\mathcal{A}_i(x,\nabla w_t(x))dt$, we deduce form (2.5.19)-(2.5.20) that

$$\int_{B(x_i',r_i')\cap\Omega} A(x)(\nabla w).\nabla\zeta dx = 0 \qquad \forall \zeta \in \mathcal{D}(B(x_i',r_i')). \tag{2.5.21}$$

If we denote $\frac{\partial}{\partial \zeta_j}\mathcal{A}_i(x,\nabla w_t(x))$ by $A_{ij}(t,x)$ and set $A(t,x) = (A_{ij}(t,x))$, then a simple calculation shows that

$$\begin{aligned} A_{ij}(t,x) &= a_{ij}|a(x)\nabla w_t.\nabla w_t|^{\frac{q-2}{2}} + (q-2)\Big(\sum_{k=1}^{k=2} a_{ik}w_{tx_k}\Big)\Big(\sum_{k=1}^{k=2} a_{jk}w_{tx_k}\Big)|a(x)\nabla w_t.\nabla w_t|^{\frac{q-4}{2}} \\ &= |a(x)\nabla w_t.\nabla w_t|^{\frac{q-2}{2}}\Big(a_{ij} + (q-2)\frac{\Big(\sum_{k=1}^{k=2} a_{ik}w_{tx_k}\Big)\Big(\sum_{k=1}^{k=2} a_{jk}w_{tx_k}\Big)}{|a(x)\nabla w_t.\nabla w_t|}\Big). \end{aligned}$$

This means that

$$A(t,x) = |a(x)\nabla w_t.\nabla w_t|^{\frac{q-2}{2}}\Big(a(x) + (q-2)\frac{(a(x)\nabla w_t)\otimes(a(x)\nabla w_t)}{|a(x)\nabla w_t.\nabla w_t|}\Big)$$

where for $h=(h_1,h_2), k=(k_1,k_2)\in\mathbb{R}^2$, $h\otimes k$ denotes the matrix (h_ik_j). Moreover since a is symmetric, we can write

$$A(t,x)=|a(x)\nabla w_t.\nabla w_t|^{\frac{q-2}{2}}\sqrt{a(x)}\Big(I_2+(q-2)\frac{(\sqrt{a(x)}\nabla w_t)\otimes(\sqrt{a(x)}\nabla w_t)}{|a(x)\nabla w_t.\nabla w_t|}\Big)\sqrt{a(x)}$$

where $\sqrt{a(x)}$ is the symmetric definite positive matrix satisfying $\sqrt{a(x)}\sqrt{a(x)}=a(x)$.
Let $y=(y_1,y_2)\in\mathbb{R}^2$. We have

$$A(t,x).y.y=|a(x)\nabla w_t.\nabla w_t|^{\frac{q-2}{2}}B(t,x).(\sqrt{a(x)}y).(\sqrt{a(x)}y) \tag{2.5.22}$$

with

$$B(t,x)=I_2+(q-2)\frac{(\sqrt{a(x)}\nabla w_t)\otimes(\sqrt{a(x)}\nabla w_t)}{|a(x)\nabla w_t.\nabla w_t|}.$$

We claim that

$$\min(1,q-1)|y|^2\le B(t,x).y.y\le\max(1,q-1)|y|^2. \tag{2.5.23}$$

Indeed if $\sqrt{a(x)}=(b_{ij})$, then (2.5.23) is an immediate consequence of

$$0\le(\sqrt{a(x)}\nabla w_t)\otimes(\sqrt{a(x)}\nabla w_t).y.y=\Big(y_1\sum_{k=1}^{k=2}b_{1k}w_{tx_k}+y_2\sum_{k=1}^{k=2}b_{2k}w_{tx_k}\Big)^2\le|y|^2|\sqrt{a(x)}\nabla w_t|^2.$$

It follows from (2.5.22)-(2.5.23) that $\forall t\in[0,1]$, $\forall x\in\overline{B(x_i',r_i')\cap\Omega}$

$$\min(1,q-1)\lambda|a(x)\nabla w_t.\nabla w_t|^{\frac{q-2}{2}}|y|^2\le A(t,x).y.y\le\max(1,q-1)M|a(x)\nabla w_t.\nabla w_t|^{\frac{q-2}{2}}|y|^2$$

which leads for some positive constant C_1,C_2 depending only on q , λ and M, to

$$C_1\lambda(x)|y|^2\le A(x).y.y\le C_2\lambda(x)|y|^2\quad\forall x\in\overline{B(x_i',r_i')\cap\Omega}.$$

Using (2.5.18), we obtain for some other positive constants $\widetilde{\lambda}_0$, $\widetilde{\lambda}_1$ depending only on q, λ, M, λ_0 and λ_1,

$$\widetilde{\lambda}_0|y|^2\le A(x).y.y\le\widetilde{\lambda}_1|y|^2\quad\forall x\in\overline{B(x_i',r_i')\cap\Omega}. \tag{2.5.24}$$

Finally, we extend w by 0 to $B(x_i',r_i')\setminus\Omega$. Since $w=0$ on $B(x_i',r_i')\cap S_{3,i}$, the obtained function belongs to $W^{1,q}(B(x_i',r_i'))$. Moreover, we extend $A(x)$ by $\widetilde{\lambda}_0I_2$ to $B(x_i',r_i')\setminus\Omega$. Thanks to (2.5.24), the obtained matrix remains bounded and strictly elliptic in $B(x_i',r_i')$. Now thanks to (2.5.21), w satisfies $div\big(A(x)\nabla w\big)=0$ in $W^{-1,q'}(B(x_i',r_i'))$, $w\ge 0$, and $w=0$ in $B(x_i',r_i')\setminus\Omega$. We conclude by the strong maximum principle that $w=0$ in $B(x_i',r_i')$, which means that $u=u_m$ in $B(x_i',r_i')\cap\Omega$. □

Proof of Theorem 2.5.2. First note that since $u_m\le u$ in Ω, we have $C_{m,i}\subset C_i$. Moreover u and u_m are $\mathcal{A}-$Harmonic in $C_{m,i}$ and by Lemma 2.5.3, $u=u_m$ in $B(x_i',r_i')\cap\Omega$. It follows from [5] Theorem 4.1, that $u=u_m$ in $C_{m,i}$.
As in the linear case, one can prove that $C_i=C_{m,i}$. Thus $u=u_m$ in $C_{m,i}=C_i$ $\forall i\in\{1,...,N\}$ and $u=u_m$ in Ω. From Corollary 2.2.1, we deduce that $g=g_M$ in Ω. □

3 The Dam Problem with Leaky Boundary Condition

In this section, we assume that $\mathcal{B}$ is given by (1.1.10) and we study the problem:

$$(P_L)\begin{cases} \text{Find } (u,g)\in W^{1,q}(\Omega)\times L^\infty(\Omega) \text{ such that :}\\ (i)\quad u=\psi \text{ on } S_2\\ (ii)\quad u\geq x_2,\quad 0\leq g\leq 1,\quad g(u-x_2)=0 \ \text{ a.e. in } \Omega\\ (iv)\quad \displaystyle\int_\Omega \big(\mathcal{A}(x,\nabla u)-g\mathcal{A}(x,e)\big).\nabla\xi dx \leq \int_{S_3}\beta(x,\psi-u)\xi d\sigma(x)\\ \forall\xi\in W^{1,q}(\Omega) \ \text{ such that } \ \xi\geq 0 \text{ on } S_2. \end{cases}$$

The main difference between the model we are considering here and the one we studied in Section 2, is the fact that the region below a reservoir is not necessarily completely saturated if the flux through the bottom is not strong enough, which makes it possible to have a free boundary there. Moreover, as we shall prove it, the function g is not a characteristic function of the wet part of the dam.
For $\beta : S_3\times\mathbb{R}\longrightarrow\mathbb{R}$, we assume that

$$\begin{aligned} &\beta(0)=0\\ &\text{for a.e. } \quad x\in S_3,\quad \beta(x,.)\quad \text{is nondecreasing}\\ &\forall s\in\mathbb{R}\ \exists C_s>0: \quad \text{for a.e. } x\in S_3\ \ |\beta(x,s)|\leq C_s. \end{aligned}$$

3.1 Properties of the Solutions

Throughout this section, we shall denote a solution of (P_L) by (u,g). First we give a regularity result.

Proposition 3.1.1. *$u\in C^{0,\alpha}_{loc}(\Omega\cup S_2)$ for some $\alpha\in(0,1)$.*

Proof. This is a consequence of (1.3.1), (1.3.3) and $(P_L)i)$ (see [29]). □

From now on, we assume that the function $x\longmapsto\beta(x,\varphi(x))$ extends to S_2 so that

$$\beta(x,\varphi(x))=0 \qquad \text{a.e. } x\in S_2.$$

The following theorem will play the same role as Theorem 2.1.1 of the previous section.

Theorem 3.1.1. *Let C_h be a connected component of $[u>x_2]\cap[x_2>h]$ and $Z_h=\Omega\cap\big(\pi_{x_1}(C_h)\times(h,+\infty)\big)$. Then we have for each nonnegative function $f\in W^{1,q}(Z_h)$ depending only on x_2*

$$\int_{Z_h} \big(\mathcal{A}(x,\nabla u) - g\mathcal{A}(x,e)\big).fedx \le \int_{Z_h} \gamma \mathcal{A}(x,e).fedx \tag{3.1.1}$$

where $\gamma(x) = \dfrac{\beta(x,\varphi)}{k(x)\nu_2} = \dfrac{\beta((x_1,s_+(x_1)),\varphi(x_1,s_+(x_1)))}{k(x)\nu_2(x_1,s_+(x_1))}$.

Proof. Let $(a_1,a_2) = \pi_{x_1}(C_h)$ and let for $\delta > 0$ small enough, $\alpha_\delta \in \mathcal{D}((a_1,a_2))$ be a function such that $0 \le \alpha_\delta(x_1) \le 1$ and $\alpha_\delta = 1$ in $(a_1+\delta, a_2-\delta)$. First we have

$$\int_{Z_h} \big(\mathcal{A}(x,\nabla u) - g\mathcal{A}(x,e)\big).f(x_2)edx = \int_{Z_h} \big(\mathcal{A}(x,\nabla u) - g\mathcal{A}(x,e)\big).\nabla\Big(\alpha_\delta \int_h^{x_2} f(s)ds\Big)dx$$
$$+ \int_{Z_h} \big(\mathcal{A}(x,\nabla u) - g\mathcal{A}(x,e)\big).\nabla\Big((1-\alpha_\delta)\int_h^{x_2} f(s)ds\Big)dx. \tag{3.1.2}$$

Since $\chi(Z_h)\alpha_\delta \displaystyle\int_h^{x_2} f(s)ds$ is a test function for (P_L), we have with $S_3^{Z_h} = S_3 \cap \partial Z_h$

$$\int_{Z_h} \big(\mathcal{A}(x,\nabla u) - g\mathcal{A}(x,e)\big).\nabla\Big(\alpha_\delta \int_h^{x_2} f(s)ds\Big)dx \le \int_{S_3^{Z_h}} \beta(x,\psi-u).\Big(\alpha_\delta \int_h^{s_+(x_1)} f(s)ds\Big)d\sigma(x). \tag{3.1.3}$$

Set $\zeta_\delta = (1-\alpha_\delta)\displaystyle\int_h^{x_2} f(s)ds$ and remark that for $\epsilon > 0$, $\pm\chi(Z_h).\Big(\dfrac{u-x_2}{\epsilon} \wedge \zeta_\delta\Big)$ are test functions for (P_L). So we have by taking into account $(P_L)ii)$

$$\int_{Z_h} \mathcal{A}(x,\nabla u).\nabla\Big(\frac{u-x_2}{\epsilon} \wedge \zeta_\delta\Big)dx = \int_{S_3^{Z_h}} \beta(x,\psi-u).\Big(\frac{u-x_2}{\epsilon} \wedge \zeta_\delta\Big)d\sigma(x)$$

which leads by the monotonicity of $\mathcal{A}$ to

$$\int_{Z_h \cap [u-x_2 \ge \epsilon\zeta_\delta]} \big(\mathcal{A}(x,\nabla u) - \mathcal{A}(x,e)\big).\nabla\zeta_\delta dx \le \int_{S_3^{Z_h}} \beta(x,\psi-u).\Big(\frac{u-x_2}{\epsilon} \wedge \zeta_\delta\Big)d\sigma(x)$$
$$- \int_{Z_h} \mathcal{A}(x,e).\nabla\Big(\frac{u-x_2}{\epsilon} \wedge \zeta_\delta\Big)dx. \tag{3.1.4}$$

Note that

$$\int_{Z_h} \chi([u > x_2])\mathcal{A}(x,e).\nabla\zeta_\delta dx = \int_{Z_h} \chi([u > x_2])\mathcal{A}(x,e).\nabla\Big(\zeta_\delta - \frac{u-x_2}{\epsilon}\Big)^+ dx$$
$$+ \int_{Z_h} \mathcal{A}(x,e).\nabla\Big(\frac{u-x_2}{\epsilon} \wedge \zeta_\delta\Big)dx. \tag{3.1.5}$$

Since k is nondecreasing in x_2, we deduce by using the second mean value theorem, that for a.e. $x_1 \in (a_1,a_2)$, there exists $h^*(x_1) \in [h, \Phi(x_1)]$ such that

$$\int_{Z_h} \chi([u > x_2])\mathcal{A}(x,e).\nabla\Big(\zeta_\delta - \frac{u-x_2}{\epsilon}\Big)^+ dx = \int_{a_1}^{a_2} \Big(\int_h^{\Phi(x_1)} k(x)\Big(\zeta_\delta - \frac{u-x_2}{\epsilon}\Big)^+_{x_2} dx_2\Big)dx_1$$

$$
\begin{aligned}
&= \int_{a_1}^{a_2} k(x_1,\Phi(x_1)_-)\Big(\int_{h^*(x_1)}^{\Phi(x_1)} \Big(\zeta_\delta - \frac{u-x_2}{\epsilon}\Big)^+_{x_2}(x_1,x_2)dx_2\Big)dx_1 \\
&\le \int_{a_1}^{a_2} k(x_1,\Phi(x_1)_-)\zeta_\delta(x_1,\Phi(x_1))dx_1
\end{aligned} \tag{3.1.6}
$$

Adding 3.1.4) and (3.1.5) and taking into account (3.1.6), we get

$$
\begin{aligned}
&\int_{Z_h\cap[u-x_2\ge\epsilon\zeta_\delta]} \big(\mathcal{A}(x,\nabla u)-\mathcal{A}(x,e)\big).\nabla\zeta_\delta dx + \int_{Z_h}\chi([u>x_2])\mathcal{A}(x,e).\nabla\zeta_\delta dx \\
&\le \int_{S_3^{Z_h}} \beta(x,\psi-u).\Big(\frac{u-x_2}{\epsilon}\wedge\zeta_\delta\Big)d\sigma(x) + \int_{a_1}^{a_2} k(x_1,\Phi(x_1)_-)\zeta_\delta(x_1,\Phi(x_1))dx_1
\end{aligned}
$$

which leads by letting $\epsilon\to 0$ to

$$
\begin{aligned}
&\int_{Z_h} \big(\mathcal{A}(x,\nabla u)-\chi([u=x_2])\mathcal{A}(x,e)\big).\nabla\zeta_\delta dx \\
&\le \int_{S_3^{Z_h}\cap[u>x_2]} \beta(x,\psi-u)\zeta_\delta d\sigma(x) + \int_{a_1}^{a_2} k(x_1,\Phi(x_1)_-)\zeta_\delta(x_1,\Phi(x_1))dx_1.
\end{aligned} \tag{3.1.7}
$$

Using (3.1.2)-(3.1.3) and (3.1.7), we obtain

$$
\begin{aligned}
\int_{Z_h}\big(\mathcal{A}(x,\nabla u)-g\mathcal{A}(x,e)\big).f(x_2)edx &\le \int_{S_3^{Z_h}} \beta(x,\psi-u)\Big(\int_h^{s_+(x_1)} f(s)ds\Big)d\sigma(x) \\
&+ \int_{Z_h} (\chi([u=x_2])-g)k(x)(1-\alpha_\delta)g_0 \\
&+ \int_{a_1}^{a_2} k(x_1,\Phi(x_1)_-)\zeta_\delta(x_1,\Phi(x_1))dx_1.
\end{aligned} \tag{3.1.8}
$$

Letting δ go to 0 in (3.1.8), we get

$$
\begin{aligned}
\int_{Z_h}\big(\mathcal{A}(x,\nabla u)-g\mathcal{A}(x,e)\big).fedx &\le \int_{S_3^{Z_h}} \beta(x,\psi-u)\Big(\int_h^{s_+(x_1)} f(s)ds\Big)d\sigma(x) \\
&\le \int_{S_3^{Z_h}} \beta(x,\varphi)\Big(\int_h^{s_+(x_1)} f(s)ds\Big)d\sigma(x) \\
&= \int_{Z_h} \gamma\mathcal{A}(x,e).fedx
\end{aligned}
$$

and the lemma is proved. □

Remark 3.1.1. *If $Z_h = \big((a_1,a_2)\times(h,+\infty)\big)\cap\Omega$ and for $i=1,2$, we have $u(a_i,x_2)=x_2$ $\forall x_2\ge h$, then the inequality (3.1.1) holds for the domain Z_h also.*

As a consequence of Theorem 3.1.1, we obtain a characterization of the unsaturated set.

Theorem 3.1.2. *Let $x_0 = (x_{01}, x_{02})$ such that $B_r = B_r(x_0) \subset \Omega$. If $B_r \subset \Omega$ and $u = x_2$ in B_r, then we have*

$$g = 1 - \gamma \quad a.e. \ in \ D_r = \{\, (x_1, x_2) \in \Omega \ / \ |x_1 - x_{01}| < r \quad and \quad x_{02} < x_2 \,\} \cup B_r. \qquad (3.1.9)$$

Proof. By Corollary 1.3.1, we have $u = x_2$ in D_r. Moreover it is clear that it is enough to prove (3.1.9) in the following two cases: $\pi_{x_1}(B_r) \subset \pi_{x_1}(S_2)$ or $\pi_{x_1}(B_r) \subset \pi_{x_1}(S_3)$.

i) $\quad \pi_{x_1}(B_r) \subset \pi_{x_1}(S_2)$:

Applying Theorem 3.1.1 with $f = 1$ and domains $Z_h \subset D_r$ of type as in Remark 3.1.1, we obtain

$$0 \le \int_{Z_h} k(x)(1-g)dx \; = \; \int_{Z_h} \big(\mathcal{A}(x, \nabla u) - g\mathcal{A}(x, e)\big).edx \; \le \; 0$$

from which we deduce that $g = 1$ a.e. in Z_h. Thus $g = 1$ a.e. in D_r.

ii) $\quad \pi_{x_1}(B_r) \subset \pi_{x_1}(S_3)$:

Let $\xi \in W^{1,q}(D_r)$ such that $\xi = 0$ on $\partial D_r \cap \Omega$. Then $\pm\chi(D_r)\xi$ are test functions for (P_L) and we have

$$\int_{D_r} k(x)(1-g)\xi_{x_2} dx \; = \; \int_{\partial D_r \cap S_3} \beta(x, \varphi)\xi d\sigma(x) \qquad (3.1.10)$$

Using (1.3.3), we get in the distributional sense $(k(x)(1-g))_{x_2} = 0$ in D_r, which leads by applying Green's formula in (3.1.10) to $k(x)(1-g)\nu_2 \; = \; \beta(x, \varphi)$ a.e. on $\partial D_r \cap S_3$. Hence we get

$$g(x) \; = \; 1 - \frac{\beta(x, \varphi)}{k(x)\nu_2} = 1 - \gamma(x) \qquad \text{a.e. in } D_r.$$

$\square$

Now we have a non-oscillation result.

Theorem 3.1.3. *Let $x_0 = (x_{01}, x_{02}) \in \Omega$ such that $B_r = B_r(x_0) \subset \Omega$. Then the following situations are impossible*

$$i) \begin{cases} u(x_1, x_2) \; = \; x_2 & \forall (x_1, x_2) \in B_r \cap [x_1 = x_{01}] \\ u(x_1, x_2) \; > \; x_2 & \forall (x_1, x_2) \in B_r, \quad x_1 \neq x_{01}. \end{cases}$$

$$ii) \begin{cases} u(x_1, x_2) \; > \; x_2 & \forall (x_1, x_2) \in B_r \cap [x_1 < x_{01}] \\ u(x_1, x_2) \; = \; x_2 & \forall (x_1, x_2) \in B_r \cap [x_1 \ge x_{01}]. \end{cases}$$

$$iii) \begin{cases} u(x_1, x_2) \; = \; x_2 & \forall (x_1, x_2) \in B_r \cap [x_1 \le x_{01}] \\ u(x_1, x_2) \; > \; x_2 & \forall (x_1, x_2) \in B_r \cap [x_1 > x_{01}]. \end{cases}$$

Proof. $i)$ By the assumption and $(P)ii)$, we have $g = 0$ a.e. in B_r. By (1.3.3), this leads to $div(\mathcal{A}(x, \nabla u)) = 0 \leq div(\mathcal{A}(x, \nabla x_2))$ in $\mathcal{D}'(B_r)$. Using the strong maximum principle (Lemma 1.3.1) applied for $u_1 = x_2$ and $u_2 = u$, we deduce that either $u > x_2$ in B_r or $u \equiv x_2$ in B_r. This contradicts the assumption.

$ii)$ Let $\xi \in \mathcal{D}(B_r)$, $\xi \geq 0$. Setting $B_r^+ = B_r \cap [x_1 > x_{01}]$, we have by (3.1.9)

$$\begin{aligned}
\int_{B_r} \mathcal{A}(x, \nabla u).\nabla\xi dx &= \int_{B_r} g\mathcal{A}(x, e).\nabla\xi dx = \int_{B_r} k(x)g\xi_{x_2}dx \\
&= \int_{B_r^+} k(x)\Big(1 - \frac{\beta((x_1, s_+(x_1)), \varphi(x_1, s_+(x_1)))}{k(x)\nu_2(x_1, s_+(x_1))}\Big)\xi_{x_2}dx \\
&= \int_{B_r^+} k\xi_{x_2}dx = \int_{B_r} k\xi_{x_2}dx - \int_{B_r^-} k\xi_{x_2}dx \geq \int_{B_r} k\xi_{x_2}dx.
\end{aligned}$$

It follows that $div(\mathcal{A}(x, \nabla u)) \leq div(\mathcal{A}(x, \nabla x_2))$ in $\mathcal{D}'(B_r)$. Applying Lemma 1.3.1, we get a contradiction with the assumption $ii)$.

$iii)$ Similar to $ii)$. $\square$

Assuming that the dam is a rectangular domain supplied by a unique reservoir located on its top (see Figure 7) and that $\mathcal{A}(x, \xi) = \xi$, Carrillo and Chipot showed in [22] that the dam is unsaturated above the line $x_2 = \frac{L}{2}\sqrt{\frac{\beta}{1-\beta}}$ provided that $\beta(h) \leq \frac{4D^2}{4D^2+L^2}$.

In [36], the author showed that if $\beta(h) \geq 1$, then the dam is completely saturated. The following theorem is an extension of that simple result which gives a sufficient condition to get total saturation in a more general framework.

Theorem 3.1.4. *We have*

$$0 \leq g \leq \mu(x) = 1 - \min(1, \gamma) \ \ a.e. \ \ in \ \Omega. \tag{3.1.11}$$

We first prove a lemma.

Lemma 3.1.1. *Let $i \in \{1, ..., N\}$ and $Z = \big((a_1, a_2) \times (h, +\infty)\big) \cap \Omega$ with $\pi_{x_1}(Z) \subset \pi_{x_1}(S_{3,i})$ and $(a_1, a_2) \times \{h\} \subset \Omega$ (see Figure 8). Then we have*

$$\int_Z k(x)(g - \mu)^+\xi_{x_2}dx \leq 0 \qquad \forall\xi \in H^1(\mathbb{R}^2),\ \xi \geq 0, \ \ and \ \ \xi = 0 \ \ on \ \ \partial Z \cap \Omega. \tag{3.1.12}$$

Proof. Let ξ as in the Lemma. For $\epsilon > 0$, $\pm(H_\epsilon(u - x_2) - 1)\xi$ are test functions for (P_L) and then we have with $S_3^Z = S_3 \cap \overline{Z}$

$$\int_Z \big(\mathcal{A}(x, \nabla u) - g\mathcal{A}(x, e)\big).\nabla\big((H_\epsilon(u - x_2) - 1)\xi\big)dx = \int_{S_3^Z} \beta(x, \psi - u)(H_\epsilon(u - x_2) - 1)\xi d\sigma(x)$$

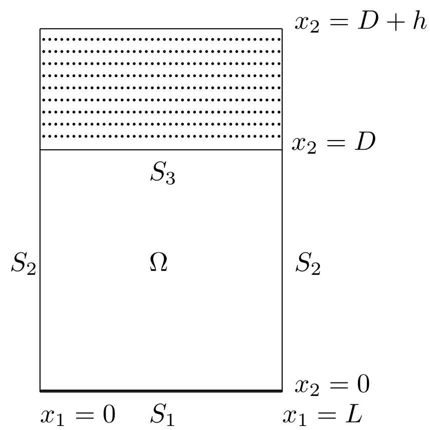

Figure 7

from which we deduce that

$$\begin{aligned}
&\int_Z gk\xi_{x_2}dx = \int_Z g\mathcal{A}(x,e).\nabla\xi dx \\
&= \int_Z (1 - H_\epsilon(u - x_2))\mathcal{A}(x,\nabla u).\nabla\xi dx - \int_Z H'_\epsilon(u - x_2)\xi\mathcal{A}(x,\nabla u).(\nabla u - \nabla x_2)dx \\
&+ \int_Z g\mathcal{A}(x,e).\nabla(H_\epsilon(u - x_2)\xi) + \int_{S_3^Z} \beta(x,\psi - u)(H_\epsilon(u - x_2) - 1)\xi d\sigma(x) \\
&\leq \int_Z (1 - H_\epsilon(u - x_2))\mathcal{A}(x,\nabla u).\nabla\xi dx - \int_Z H'_\epsilon(u - x_2)\xi\mathcal{A}(x,e).\nabla(u - x_2)dx \\
&+ \int_{S_3^Z} \beta(x,\psi - u)(H_\epsilon(u - x_2) - 1)\xi d\sigma(x) \\
&= \int_Z \big(1 - H_\epsilon(u - x_2)\big)\big(\mathcal{A}(x,\nabla u) - \mathcal{A}(x,e)\big).\nabla\xi dx + \int_Z (H_\epsilon(u - x_2) - 1)\xi k_{x_2}dx
\end{aligned}$$

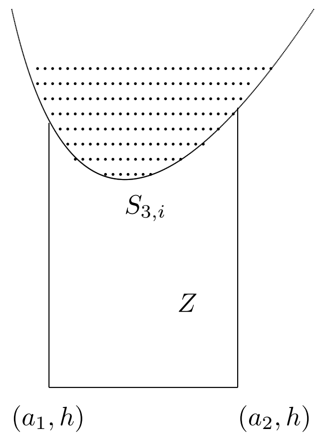

Figure 8

$$- \int_{S_3^Z} k\nu_2 (H_\epsilon(u - x_2) - 1)\xi + \int_{S_3^Z} \beta(x, \psi - u)(H_\epsilon(u - x_2) - 1)\xi d\sigma(x). \quad (3.1.13)$$

Now set $\alpha(x) = \min(1, \gamma(x))$. Then $\mu(x) = 1 - \alpha(x)$ and since $(\alpha k)_{x_2} \geq 0$, we have

$$\begin{aligned} \int_Z -\mu k \xi_{x_2} dx &= \int_Z k_{x_2} \xi dx - \int_{S_3^Z} k\nu_2 \xi d\sigma(x) + \int_{S_3^Z} \alpha k \nu_2 \xi d\sigma(x) - \int_Z \xi (\alpha k)_{x_2} dx \\ &\leq \int_Z k_{x_2} \xi dx + \int_{S_3^Z} (\alpha(x) - 1) k \nu_2 \xi d\sigma(x). \end{aligned} \quad (3.1.14)$$

Adding (3.1.13) and (3.1.14), we get

$$\begin{aligned} \int_Z (g - \mu) k \xi_{x_2} dx &\leq \int_Z (1 - H_\epsilon(u - x_2))\big(\mathcal{A}(x, \nabla u) - \mathcal{A}(x, e)\big).\nabla \xi dx \\ &+ \int_Z H_\epsilon(u - x_2) k_{x_2} \xi dx + \int_{S_3^Z} \big[(1 - H_\epsilon(u - x_2))\big(k\nu_2 - \beta(x, \psi - u)\big) + (\alpha - 1) k \nu_2\big] \xi d\sigma(x). \end{aligned} \quad (3.1.15)$$

The last integral in (3.1.15) can be written as

$$
\begin{aligned}
&\int_{S_3^Z}(1-H_\epsilon(u-x_2))(\beta(x,\varphi)-\beta(x,\psi-u))\xi d\sigma(x)\\
&+\int_{S_3^Z}\Big[(1-H_\epsilon(u-x_2))\big(1-\frac{\beta(x,\varphi)}{k\nu_2}\big)+(\alpha-1)\Big]k\nu_2\xi d\sigma(x)\\
&\le \int_{S_3^Z}(1-H_\epsilon(u-x_2))(\beta(x,\varphi)-\beta(x,\psi-u))\xi d\sigma(x)=J_1^\epsilon. \qquad (3.1.16)
\end{aligned}
$$

We get from (3.1.15)-(3.1.16)

$$
\int_Z(g-\mu)k\xi_{x_2}dx\le\int_Z(1-H_\epsilon(u-x_2))\big(\mathcal{A}(x,\nabla u)-\mathcal{A}(x,e)\big).\nabla\xi dx+\int_Z H_\epsilon(u-x_2)k_{x_2}\xi dx+J_1^\epsilon.
$$

Moreover we have

$$
\lim_{\epsilon\to 0}J_1^\epsilon=\int_{S_3^Z}\chi[u=x_2](\beta(x,\varphi)-\beta(x,\psi-x_2))\xi d\sigma(x)=0. \qquad (3.1.17)
$$

Therefore we get by letting $\epsilon\to 0$ and taking into account (3.1.17)

$$
\int_Z(g-\mu)k\xi_{x_2}dx\le\int_Z\chi([u>x_2])k_{x_2}\xi dx\quad \forall\xi\in H^1(\mathbb{R}^2),\ \xi=0 \text{ on } \partial Z\cap\Omega \text{ and } \xi\ge 0. \qquad (3.1.18)
$$

In what follows, we extend the functions k, $(g-\mu)$ and $\chi([u>x_2])k_{x_2}$ by 0 and we denote the extensions respectively by k, $(g-\mu)$ and θ. Note that (3.1.18) holds in particular for functions with compact support in $Z\cup S_3^Z$. Let then $\xi\in H^1(\mathbb{R}^2)$, $\xi\ge 0$ with ξ having a compact support in $Z\cup S_3^Z$. Set $\epsilon_0=d(supp\xi,\partial Z\cap\Omega)$ and let $\epsilon\in(0,\epsilon_0/2)$. For each $y\in B_\epsilon(0)$, the function $x\to\xi(x+y)$ is nonnegative, belongs to $H^1(\mathbb{R}^2)$ and has a compact support in $Z\cup S_3^Z$. Therefore it can be used in (3.1.18) to get

$$
\int_{\mathbb{R}^2}(g(x)-\mu(x))k(x)\xi_{x_2}(x+y)dx\le\int_{\mathbb{R}^2}\theta(x)\xi(x+y)dx
$$

from which we deduce that

$$
\int_{\mathbb{R}^2}\rho_\epsilon\Big(\int_{\mathbb{R}^2}(\bar g-\bar\mu)\xi_{x_2}(x+y)dx\Big)dy\le\int_{\mathbb{R}^2}\rho_\epsilon(y)\Big(\int_{\mathbb{R}^2}\theta(x)\xi(x+y)dx\Big)dy
$$

where $\bar g=gk$, $\bar\mu=\mu k$ and ρ_ϵ is a smooth function satisfying $\rho_\epsilon\ge 0$, $supp\rho_\epsilon\subset B_\epsilon(0)$ and $\int_{\mathbb{R}^2}\rho_\epsilon=1$. Writing $f_\epsilon=\rho_\epsilon * f$ for a function f, we get

$$
\int_{\mathbb{R}^2}(\bar g_\epsilon-\bar\mu_\epsilon)\xi_{x_2}dx\le\int_{\mathbb{R}^2}\theta_\epsilon\xi dx\qquad\forall\xi\in H^1(\mathbb{R}^2),\ \xi\ge 0,\ supp(\xi)\subset Z\cup S_3^Z.
$$

In particular, we obtain for the function $\xi = \min\big(1, \dfrac{(\bar{g}_\epsilon - \bar{\mu}_\epsilon)^+}{\delta}\big)\zeta$, with $\delta > 0$, $\zeta \in H^1(\mathbb{R}^2)$, $\zeta \geq 0$, $supp(\zeta) \subset Z \cup S_3^Z$

$$\int_{\mathbb{R}^2} (\bar{g}_\epsilon - \bar{\mu}_\epsilon)\Big(\min\big(1, \frac{(\bar{g}_\epsilon - \bar{\mu}_\epsilon)^+}{\delta}\big)\zeta\Big)_{x_2} dx \leq \int_{\mathbb{R}^2} \theta_\epsilon \min\big(1, \frac{(\bar{g}_\epsilon - \bar{\mu}_\epsilon)^+}{\delta}\big)\zeta dx.$$

As $\delta \to 0$, we get

$$* \quad \int_{\mathbb{R}^2} \min\big(1, \frac{(\bar{g}_\epsilon - \bar{\mu}_\epsilon)^+}{\delta}\big)(\bar{g}_\epsilon - \bar{\mu}_\epsilon)\zeta_{x_2} \longrightarrow \int_{\mathbb{R}^2} (\bar{g}_\epsilon - \bar{\mu}_\epsilon)^+ \zeta_{x_2}$$

$$* \quad \int_{\mathbb{R}^2} (\bar{g}_\epsilon - \bar{\mu}_\epsilon)\zeta\Big(\min\big(1, \frac{(\bar{g}_\epsilon - \bar{\mu}_\epsilon)^+}{\delta}\big)\Big)_{x_2} = \int_{\mathbb{R}^2 \cap [0 < \bar{g}_\epsilon - \bar{\mu}_\epsilon < \delta]} \frac{\zeta}{\delta}(\bar{g}_\epsilon - \bar{\mu}_\epsilon)(\bar{g}_\epsilon - \bar{\mu}_\epsilon)_{x_2}$$

$$= \int_{\mathbb{R}^2} \frac{\zeta}{2\delta}\Big(\big(\min(\delta, (\bar{g}_\epsilon - \bar{\mu}_\epsilon)^+)\big)^2\Big)_{x_2} = -\frac{1}{2\delta}\int_{\mathbb{R}^2} \big(\min(\delta, (\bar{g}_\epsilon - \bar{\mu}_\epsilon)^+\big)^2 \zeta_{x_2} \to 0$$

$$* \quad \int_{\mathbb{R}^2} \theta_\epsilon \min\big(1, \frac{(\bar{g}_\epsilon - \bar{\mu}_\epsilon)^+}{\delta}\big)\zeta \leq \int_{\mathbb{R}^2} \theta_\epsilon \zeta.$$

It follows that

$$\int_{\mathbb{R}^2} (\bar{g}_\epsilon - \bar{\mu}_\epsilon)^+ \zeta_{x_2} \leq \int_{\mathbb{R}^2} \theta_\epsilon \zeta$$

which leads by letting $\epsilon \to 0$, to

$$\int_Z (g - \mu)^+ k\zeta_{x_2} \leq \int_Z \chi([u > x_2])\zeta k_{x_2}.$$

Now it is not difficult to extend this inequality to functions as in the lemma. Hence if we write it for $\zeta = (1 - H_\epsilon(u - x_2))\xi$, with $\xi \in H^1(\mathbb{R}^2)$, $\xi = 0$ on $\partial Z \cap \Omega$, $\xi \geq 0$, we obtain since $g(u - x_2) = 0$

$$\int_Z (g - \mu)^+ k\xi_{x_2} \leq \int_Z \chi([u > x_2])(1 - H_\epsilon(u - x_2))k_{x_2}\xi.$$

Letting $\epsilon \to 0$, the lemma follows. □

Proof of Theorem 3.1.4. We give the proof only below S_3, since below S_2, one has $\mu = 1 \geq g$ a.e. So let $i \in \{1, ..., N\}$ and let Z be a domain below $S_{3,i}$ such that $Z = \big((a_1, a_2) \times (h, +\infty)\big) \cap \Omega$ and such that $(a_1, a_2) \times \{h\} \subset \Omega$. Using Lemma 3.1.1 for $\xi = \kappa(x_1).(x_2 - h)\chi(Z)$ with $\kappa(x_1) = (x_1 - a_1)(a_2 - x_1)$, we obtain

$$\int_Z k(x)(g - \mu)^+ \kappa(x_1)dx = 0$$

which leads to $(g - \mu)^+ = 0$ a.e. in Z and $g \leq \mu$ a.e. in Z. Hence the theorem is proved. □

Corollary 3.1.1. *Let T be a domain contained in subset of $S_3 \setminus S_+$. If $\beta(x, \varphi) \geq k(x)\nu_2$ a.e. on T, then the subset Z_T of Ω located below T is totally saturated.*

Proof. We deduce from Theorem 3.1.4 and the assumption, that $g = 0$ a.e. in Z_T. From (1.3.3) we then get $div(\mathcal{A}(x, \nabla u)) = 0$ in $\mathcal{D}'(Z_T)$. This leads by Lemma 1.3.1 to $u = x_2$ in Z_T or $u > x_2$ in Z_T. Suppose that $u = x_2$ in Z_T and let $\xi \in W^{1,q}(Z_T)$ with $\xi = 0$ on $\partial Z_T \cap \Omega$. Using $\pm\xi$ as test functions for (P_L) and the fact that $u = x_2$ and $g = 0$ in Z_T, we obtain

$$\int_{Z_T} k\xi_{x_2} = \int_T \beta(x, \varphi)\xi d\sigma(x)$$

which leads to

$$\int_{T \cup (\partial Z_T \cap S_1)} k\xi\nu_2 d\sigma(x) = \int_T \beta(x, \varphi)\xi d\sigma(x).$$

It follows that $\beta(x, \varphi) = 0$ on T and we get a contradiction with $\beta(x, \varphi) \geq k(x)\nu_2 > 0$. Hence $u > x_2$ in Z_T. □

3.2 Continuity of the Free Boundary

From now on, we assume that $\mathcal{A}$ is strictly monotone in the sense of (2.2.1). We first give a theorem dealing with the continuity of the free boundary below S_2.

Theorem 3.2.1. *Φ is continuous at each point $x_{01} \in Int(\pi_{x_1}(S_2))$ such that $(x_{01}, \Phi(x_{01})) \in \Omega$ (resp. $x_{01} \notin S_-$ and $\Phi(x_{01}) = s_-(x_{01})$).*

Proof. Since for test functions vanishing on S_3, the problem (P_L) behaves exactly as the problem (P_D), the proof is exactly the same as the one of Theorem 2.2.1 by taking into account Remark 2.2.2. □

In the following theorem, we prove the continuity of the free boundary below S_3. For this purpose, we need the following assumptions which will be assumed in the sequel:

$$\mathcal{A}(x, te) = t^{q-1}\mathcal{A}(x, e) \quad \text{for all } \ x \ \text{ below } \ S_3, \quad \forall t > 0. \tag{3.2.1}$$

$$\gamma \ \text{ is continuous at each } \ x \ \text{ below } \ S_3 \text{ such that } \ \gamma(x) < 1. \tag{3.2.2}$$

$$\exists \mu_0 > 0 \ \text{ such that } k_{x_2}(x) \leq \mu_0 \quad \text{for all } \ x \ \text{ below } \ S_3. \tag{3.2.3}$$

Then we have.

Theorem 3.2.2. *Φ is continuous at each $x_{01} \in Int(\pi_{x_1}(S_3))$ such that $x_0 = (x_{01}, \Phi(x_{01})) \in \Omega$ and $\gamma(x_0) \in [0, 1)$.*

Proof. Without loss of generality, we may assume that $\mu_0 \geq 1$. Since $\gamma(x_0) < 1$, γ is continuous at x_0 and k is nondecreasing with respect to x_2, there exists $\gamma_+ \in (0, 1)$ and a ball $B(x_0, \epsilon_0) \subset \Omega$ such that

$$\gamma(x) \leq \gamma_+ < 1 \qquad \forall x \in Z = \big(\pi_{x_1}(B(x_0, \epsilon_0)) \times (\Phi(x_{01}), +\infty)\big) \cap \Omega.$$

Since $\Phi(x_{01}) < s_+(x_{01})$ and $\gamma_+ < 1$, there exists $\delta > 0$ small enough such that

$$\Phi(x_{01}) < s_+(x_{01}) - \frac{\delta}{2} \quad \text{and} \quad \gamma(x) \leq \gamma_+ < exp\big(-\frac{\delta\mu_0}{\lambda}\big) \quad \forall x \in Z. \tag{3.2.4}$$

Because $0 \leq \gamma(x_0) < 1$, we have necessarily s_+ continuous at x_{01}. So there exists $\epsilon_0' \in (0, \epsilon_0)$ small enough such that

$$s_+(x_{01}) - \frac{\delta}{2} < s_+(x_1) < s_+(x_{01}) + \frac{\delta}{2} \qquad \forall x_1 \in [x_{01} - \epsilon_0', x_{01} + \epsilon_0'].$$

Setting $h_0 = s_+(x_{01}) - \delta/2$, we obtain

$$\Phi(x_{01}) < h_0 < s_+(x_1) < h_0 + \delta \qquad \forall x_1 \in [x_{01} - \epsilon_0', x_{01} + \epsilon_0']. \tag{3.2.5}$$

Let now $\epsilon > 0$ small enough such that

$$\epsilon < \epsilon_0' \quad \text{and} \quad h_0 + 3\epsilon < s_+(x_1) \qquad \forall x_1 \in [x_{01} - \epsilon_0', x_{01} + \epsilon_0']. \tag{3.2.6}$$

Since $u(x_{01}, h_0) = h_0$ and u is continuous at (x_{01}, h_0), there exists $\epsilon_0'' \in (0, \epsilon)$ such that

$$u(x) \leq x_2 + \int_0^\epsilon (1 - exp(-s\nu_0))ds \quad \forall x \in B_{\epsilon_0''}(x_{01}, h_0) \tag{3.2.7}$$

where $\nu_0 = \dfrac{\mu_0}{\lambda(q-1)}$
Using Theorem 3.1.3, we are in one of the following situations:

$$\begin{cases} i)\ \exists \underline{x}_0 = (\underline{x}_{01}, \underline{x}_{02}) \in B_{\epsilon_0''}(x_{01}, h_0) & \text{such that} \quad \underline{x}_{01} < x_{01} \quad \text{and} \quad u(\underline{x}_0) = \underline{x}_{02} \\ ii)\ \exists \overline{x}_0 = (\overline{x}_{01}, \overline{x}_{02}) \in B_{\epsilon_0''}(x_{01}, h_0) & \text{such that} \quad \overline{x}_{01} > x_{01} \quad \text{and} \quad u(\overline{x}_0) = \overline{x}_{02}. \end{cases}$$

Assume for example that $i)$ holds and set $h_0' = \max(\underline{x}_{02}, h_0)$, $Z_0 = \big((\underline{x}_{01}, x_{01}) \times (h_0', +\infty)\big) \cap \Omega$, $g_0(t) = 1 - exp(-\nu_0(t - h_0'))$, and

$$w_0(x) = \begin{cases} x_2 + \displaystyle\int_{x_2}^{h_0'+\epsilon} g_0(t)dt & \text{if} \quad h_0' \leq x_2 \leq h_0' + \epsilon \\ x_2 & \text{if} \quad x_2 > h_0' + \epsilon. \end{cases}$$

Note that

$$w_0(x_1, h_0') = h_0' + \int_{h_0'}^{h_0'+\epsilon} (1 - exp(-\nu_0(t - h_0')))dt = h_0' + \int_0^\epsilon (1 - exp(-s\nu_0))ds.$$

This leads by (3.2.7) to

$$u(x_1, h_0') \leq w_0(x_1, h_0') \qquad \forall x_1 \in [\underline{x}_{01}, x_{01}]. \tag{3.2.8}$$

Given that $u(x_0) = x_{02}$ and $u(\underline{x}_0) = \underline{x}_{02}$, we obtain by Corollary 1.3.1 that

$$u(x_{01}, x_2) = u(\underline{x}_{01}, x_2) = x_2 \quad \forall x_2 \geq h_0'. \tag{3.2.9}$$

Using (3.2.8)-(3.2.9), we obtain $(u - w_0)^+ = 0$ on $\partial Z_0 \cap \Omega$. It follows that $\pm\chi(Z_0)(u - w_0)^+$ are test functions for (P_L) and we have

$$\int_{Z_0} \big(\mathcal{A}(x, \nabla u) - g\mathcal{A}(x, e)\big).\nabla(u - w_0)^+ dx = \int_{S_3^{Z_0}} \beta(x, \psi - u).(u - w_0)^+ d\sigma(x). \tag{3.2.10}$$

Moreover, one has for $Z_0^+ = \big((\underline{x}_{01}, x_{01}) \times (h_0', h_0' + \epsilon)\big) \cap \Omega$

$$\int_{Z_0} \big(\mathcal{A}(x, \nabla w_0) - \chi([w_0 = x_2])\mathcal{A}(x, e)\big).\nabla(u - w_0)^+ dx = \int_{Z_0^+} (1 - g_0(x_2))^{q-1} k(x).(u - w_0)^+_{x_2} dx. \tag{3.2.11}$$

Remarking that

$$\begin{aligned} \int_{Z_0} (1 - g_0(x_2))^{q-1} k(x).(u - w_0)^+_{x_2} dx &= -\int_{Z_0} \big((1 - g_0(x_2))^{q-1} k(x)\big)_{x_2}.(u - w_0)^+ dx \\ &\quad + \int_{S_3^{Z_0}} (1 - g_0(x_2))^{q-1} k(x).(u - w_0)^+ \nu_2 d\sigma(x) \end{aligned}$$

we obtain for $Z_0^0 = \big((\underline{x}_{01}, x_{01}) \times (h_0' + \epsilon, +\infty)\big) \cap \Omega$

$$\begin{aligned} &\int_{Z_0^+} (1 - g_0(x_2))^{q-1} k(x).(u - w_0)^+_{x_2} dx = -\int_{Z_0^0} (1 - g_0(x_2))^{q-1} k(x).(u - x_2)^+_{x_2} dx \\ &- \int_{Z_0} \big((1 - g_0(x_2))^{q-1} k(x)\big)_{x_2}.(u - w_0)^+ dx + \int_{S_3^{Z_0}} (1 - g_0(x_2))^{q-1} k(x).(u - w_0)^+ \nu_2 d\sigma(x). \end{aligned} \tag{3.2.12}$$

Subtracting (3.2.11) from (3.2.10) and taking into account (3.2.12), we get

$$\begin{aligned} &\int_{Z_0^+} \big(\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla w_0)\big).\nabla(u - w_0)^+ dx + \int_{Z_0^0} \big(\mathcal{A}(x, \nabla u) - g\mathcal{A}(x, e)\big).\nabla(u - x_2) dx \\ &\quad = \int_{Z_0^0} (1 - g_0(x_2))^{q-1} k(x).(u - x_2)_{x_2} dx + \int_{Z_0} \big((1 - g_0(x_2))^{q-1} k(x)\big)_{x_2}.(u - w_0)^+ dx \\ &\quad + \int_{S_3^{Z_0}} (\beta(x, \psi - u) - (1 - g_0(x_2))^{q-1} k(x)\nu_2).(u - w_0)^+ d\sigma(x). \end{aligned} \tag{3.2.13}$$

Note that by (3.2.1)

$$\int_{Z_0^0} \big(\mathcal{A}(x, \nabla u) - g\mathcal{A}(x, e)\big).\nabla(u - x_2) dx$$

$$= \int_{Z_0^0} \big(\mathcal{A}(x,\nabla u) - g\mathcal{A}(x,e)\big).(\nabla u - (1-g_0)e)dx - \int_{Z_0^0} \big(\mathcal{A}(x,\nabla u) - g\mathcal{A}(x,e)\big).g_0 e dx$$
$$= \int_{Z_0^0} \big(\mathcal{A}(x,\nabla u) - \mathcal{A}(x,(1-g_0)e)\big).(\nabla u - (1-g_0)e)dx$$
$$+ \int_{Z_0^0} ((1-g_0)^{q-1} - g)k(x)((u-x_2)_{x_2} + g_0)dx - \int_{Z_0^0} \big(\mathcal{A}(x,\nabla u) - g\mathcal{A}(x,e)\big).g_0 e dx$$
$$= \int_{Z_0^0} \big(\mathcal{A}(x,\nabla u) - \mathcal{A}(x,(1-g_0)e)\big).(\nabla u - (1-g_0)e)dx + \int_{Z_0^0} (1-g_0)^{q-1}k(x)(u-x_2)_{x_2}dx$$
$$+ \int_{Z_0^0} ((1-g_0)^{q-1} - g)k(x)g_0 dx - \int_{Z_0^0} \big(\mathcal{A}(x,\nabla u) - g\mathcal{A}(x,e)\big).g_0 e dx \tag{3.2.14}$$

Using (3.2.13)-(3.2.14), we obtain

$$\int_{Z_0^+} \big(\mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla w_0)\big).\nabla(u-w_0)^+ dx$$
$$+ \int_{Z_0^0} \big(\mathcal{A}(x,\nabla u) - \mathcal{A}(x,(1-g_0)e)\big).(\nabla u - (1-g_0)e)dx$$
$$+ \int_{Z_0^0} ((1-g_0)^{q-1} - g)k(x)g_0 dx \leq \int_{Z_0^0} \big(\mathcal{A}(x,\nabla u) - g\mathcal{A}(x,e)\big).g_0 e dx$$
$$+ \int_{Z_0} \big((1-g_0(x_2))^{q-1}k(x)\big)_{x_2}.(u-w_0)^+ dx$$
$$+ \int_{S_3^{Z_0}} k(x)\nu_2(\gamma - (1-g_0(x_2))^{q-1}).(u-x_2)d\sigma(x). \tag{3.2.15}$$

By (3.1.1), we have for $f = g_0$ and $Z_h = Z_0^0$

$$\int_{Z_0^0} \big(\mathcal{A}(x,\nabla u) - g\mathcal{A}(x,e)\big).g_0 e dx \leq \int_{Z_0^0} \gamma k g_0 dx. \tag{3.2.16}$$

Moreover by (3.2.5) and since $h_0 \leq h_0'$, one has for all $x_1 \in [x_{01} - \epsilon_0'', x_{01} + \epsilon_0'']$, $s_+(x_1) - h_0' \leq \delta$ which leads by (3.2.4) to

$$\gamma \leq \gamma_+ < exp\Big(-\frac{\mu_0}{\lambda}(s_+(x_1) - h_0')\Big) = exp\big(-\nu_0(q-1)(s_+(x_1) - h_0'\big) = (1 - g_0(s_+(x_1)))^{q-1}$$

so that

$$\int_{S_3^{Z_0}} k(x)\nu_2(\gamma - (1-g_0(x_2))^{q-1}).(u-x_2)d\sigma(x) \leq 0. \tag{3.2.17}$$

Now since $\lambda k_{x_2} \leq k\mu_0$ a.e. in Z_0, we have

$$\int_{Z_0} \big((1-g_0(x_2))^{q-1}k(x)\big)_{x_2}.(u-w_0)^+ dx$$

$$= \int_{Z_0} \big((1-g_0)^{q-1} k_{x_2} + ((1-g_0)^{q-1})_{x_2} k\big).(u-w_0)^+ dx$$
$$= \int_{Z_0} (1-g_0)^{q-1}\big(k_{x_2} - \frac{\mu_0 k}{\lambda}\big).(u-w_0)^+ dx \leq 0. \qquad (3.2.18)$$

Then (3.2.15) becomes by (3.2.16)-(3.2.18)

$$\int_{Z_0^+} \big(\mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla w_0)\big).\nabla(u-w_0)^+ dx$$
$$+ \int_{Z_0^0\cap[u>x_2]} ((1-g_0)^{q-1} - \gamma) k g_0 dx + \int_{Z_0^0\cap[u=x_2]} (1-g-\gamma) k g_0 dx$$
$$+ \int_{Z_0^0\cap[u>x_2]} \big(\mathcal{A}(x,\nabla u) - \mathcal{A}(x,(1-g_0)e)\big).(\nabla u - (1-g_0)e) dx \leq 0. \qquad (3.2.19)$$

By (3.2.4)-(3.2.5), we have

$$\gamma(x) \leq \gamma_+ < exp\big(\frac{-\delta\mu_0}{\lambda}\big) \leq exp(-\nu_0(x_2 - h_0')) = (1-g_0(x_2))^{q-1} \quad \forall x \in Z_0^0.$$

Indeed for $x \in Z_0^0$, we have $h_0 \leq h_0' < x_2 < h_0 + \delta$, so that $x_2 - h_0' \leq \delta$. Since $\mu_0 \geq 0$, we obtain

$$exp\big(-\frac{\delta\mu_0}{\lambda}\big) \leq exp\big(-\frac{\mu_0}{\lambda}(x_2 - h_0')\big) = (1-g_0(x_2))^{q-1}.$$

So

$$\int_{Z_0^0\cap[u>x_2]} ((1-g_0)^{q-1} - \gamma) k g_0 dx \geq 0. \qquad (3.2.20)$$

Using (3.2.19)-(3.2.20) and (3.1.11), we get

$$\int_{Z_0^+} \big(\mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla w_0)\big).\nabla(u-w_0)^+ dx \leq 0.$$

This leads by (2.2.1) to $\nabla(u-w_0)^+ = 0$ a.e. in Z_0^+ and then by (3.2.8) to $u \leq w_0$ in Z_0^+ which gives in particular $u(x_1, h_0'+\epsilon) = h_0'+\epsilon$, $\forall x_1 \in (\underline{x}_{01}, x_{01})$. By Corollary 1.3.1, we obtain $u(x) = x_2$ $\forall x \in Z_0^0$.

Using Theorem 3.1.3 once again, we deduce that there exists $\overline{x}_0 = (\overline{x}_{01}, \overline{x}_{02}) \in \Omega$ such that $x_{01} < \overline{x}_{01} < x_{01} + \epsilon_0''$, $h_0' + \epsilon < \overline{x}_{02} < h_0' + 2\epsilon$ and $u(\overline{x}_0) = \overline{x}_{02}$. Arguing as above, we prove that

$$u(x) = x_2 \qquad \forall x \in \big((x_{01}, \overline{x}_{01}) \times (h_0' + 2\epsilon, +\infty)\big).$$

Hence

$$u(x) = x_2 \qquad \forall x \in \big((\underline{x}_{01}, \overline{x}_{01}) \times (h_0' + 2\epsilon, +\infty)\big).$$

Now we consider a subdivision $(k_i)_{0\leq i\leq N}$ of the interval $[\Phi(x_{01})+2\epsilon, h_0'+2\epsilon]$ with $k_i - k_{i+1} = \frac{(h_0' - \Phi(x_{01}))}{N} < \epsilon/2$. Repeating the above step, we can prove successively that $u = x_2$ in

$Z_1 \cap [x_2 \geq k_1]$, $Z_2 \cap [x_2 \geq k_2]$,..., $Z_N \cap [x_2 \geq k_N]$, where Z_i are domains of type Z_0 such that $\pi_{x_1}(Z_{i+1}) \subset \pi_{x_1}(Z_i)$. Thus we obtain

$$\Phi(x_1) \leq \Phi(x_{01}) + 2\epsilon \qquad \forall x_1 \in (\underline{x}_{1N}, \overline{x}_{1N})$$

and the upper semi-continuity of Φ at x_{01} follows. Taking into account Proposition 1.3.4, the theorem is proved. □

Remark 3.2.1. *Φ is continuous at each point $x_{01} \in Int(\pi_{x_1}(S_3)) \setminus S_-$ such that $\Phi(x_{01}) = s_-(x_{01})$ and $\gamma(x_{01}, x_2) \in [0,1)$ for all $x_2 \in (s_-(x_{01}), s_+(x_{01}))$. Indeed in this case, one has for each $\epsilon > 0$ small enough, $x_0' = (x_{01}, \Phi(x_{01}) + \epsilon) \in \Omega$, $u(x_0') = \Phi(x_{01}) + \epsilon$ and γ is continuous at x_0'. Therefore one can adapt the proof of Theorem 3.2.2 to get $u = x_2$ in $\big((x_{01} - \epsilon', x_{01} + \epsilon') \times (\Phi(x_{01}) + 3\epsilon + \infty)\big) \cap \Omega$ for some $\epsilon' > 0$, which means the upper semi-continuity and thus the continuity of Φ at x_{01}.*

As a consequence of the continuity of Φ, we obtain the expression of g which is not a characteristic function of the dry region.

Corollary 3.2.1. *Under the assumptions of Theorem 3.2.2, we have*

$$g = (1-\gamma)^+ . \chi([u = x_2]). \tag{3.2.21}$$

Proof. First note that since $\gamma = 0$ below S_2, the formula (3.2.21) can be obtained below S_2 in the same way we obtained (2.2.8).
Next by (3.1.11), we have $g = 0$ a.e. in $\Omega \cap [\gamma \geq 1]$ and then

$$g = g\chi\big([\gamma < 1]\big). \tag{3.2.22}$$

By Proposition 1.3.4, we have $[u > x_2] = [x_2 < \Phi(x_1)]$. So

$$g = 0 \quad \text{ a.e. in } [x_2 < \Phi(x_1)]. \tag{3.2.23}$$

Let $x_0 = (x_{01}, x_{02}) \in [x_2 > \Phi(x_1)]$ such that x_0 is located below S_3 and $\gamma(x_0) < 1$. By continuity of γ, there exists a ball $B_r(x_0)$ contained in $[\gamma < 1]$ such that $x_{02} - r > \Phi(x_{01})$. Since γ is non-increasing with respect to x_2, we have $Z = \big((x_{01} - \frac{r}{2}, x_{01} + \frac{r}{2}) \times (x_{02} - \frac{r}{2}, +\infty)\big) \cap \Omega \subset [\gamma < 1]$. Moreover $u(x_{01}, x_2) = x_2 \ \forall x_2 \geq x_{02} - \frac{r}{2}$. Then one can adapt the proof of Theorem 3.2.2 to show that $u(x_1, x_2) = x_2 \ \forall (x_1, x_2) \in Z' = \big((x_{01} - r', x_{01} + r') \times (x_{02} - r', +\infty)\big) \cap \Omega$ for some $r' \in (0, r/2)$. By (3.1.9), we obtain $g = 1 - \gamma$ a.e. in Z'. We deduce that

$$g = 1 - \gamma \quad \text{ a.e. in } [x_2 > \Phi(x_1)] \cap [\gamma < 1].$$

Now the set $[x_2 = \Phi(x_1)] \cap [\gamma < 1]$ being of measure zero (since Φ is continuous at any point x_1 such that $(x_1, \Phi(x_1)) \in \Omega$ and $\gamma(x_1, \Phi(x_1)) < 1$), we get by (3.2.22)-(3.2.23)

$$g = (1 - \gamma)\chi([x_2 \geq \Phi(x_1)]).\chi([\gamma < 1])$$

which is (3.2.21). □

3.3 Existence and Uniqueness of Minimal and Maximal Solutions

In this section, we show the existence and uniqueness of two solutions which minimize (resp. maximize) a functional. Moreover one is minimal and the other one is maximal in the usual sense among all solutions. From now on, we assume that:

$$\beta(x,u) > 0 \quad \forall u > 0, \quad \text{a.e. } x \in S_3. \tag{3.3.1}$$

We first prove a result similar to Theorem 2.3.1.

Theorem 3.3.1. *Let (u_1, g_1) and (u_2, g_2) be two solutions of (P_L). Set $u_m = \min(u_1, u_2)$, $u_M = \max(u_1, u_2)$, $g_m = \min(g_1, g_2)$, and $g_M = \max(g_1, g_2)$. Then we have for $i = 1, 2$ and for all $\zeta \in W^{1,q}(\Omega)$*

$$i) \quad \int_\Omega \big((\mathcal{A}(x,\nabla u_i) - \mathcal{A}(x,\nabla u_m)) - (g_i - g_M)\mathcal{A}(x,e)\big).\nabla\zeta dx = 0$$

$$ii) \quad \int_\Omega \big((\mathcal{A}(x,\nabla u_i) - \mathcal{A}(x,\nabla u_M)) - (g_i - g_m)\mathcal{A}(x,e)\big).\nabla\zeta dx = 0$$

$$iii) \quad \beta(x, \psi - u_1) = \beta(x, \psi - u_2) \qquad a.e. \ x \in S_3.$$

Proof. *i)* Let $\zeta \in C^1(\bar{\Omega})$, $\zeta \geq 0$. For $\delta, \epsilon > 0$, we consider $\alpha_\delta(x) = \Big(1 - \dfrac{d(x, A_m)}{\delta}\Big)^+$ where $A_m = [u_m > x_2] \cup [\gamma \geq 1]$ and $\xi = \min\Big(\alpha_\delta\zeta, \dfrac{u_i - u_m}{\epsilon}\Big)$. We have

$$\begin{aligned}
&\int_\Omega ((\mathcal{A}(x,\nabla u_i) - \mathcal{A}(x,\nabla u_m)) - (g_i - g_M)\mathcal{A}(x,e)).\nabla\zeta dx = \\
&\quad = \int_\Omega ((\mathcal{A}(x,\nabla u_i) - \mathcal{A}(x,\nabla u_m)) - (g_i - g_M)\mathcal{A}(x,e)).\nabla(\alpha_\delta\zeta) dx \\
&\quad + \int_\Omega ((\mathcal{A}(x,\nabla u_i) - \mathcal{A}(x,\nabla u_m)) - (g_i - g_M)\mathcal{A}(x,e)).\nabla\big((1-\alpha_\delta)\zeta\big) dx.
\end{aligned} \tag{3.3.2}$$

Since $(1 - \alpha_\delta)\zeta$ is a test function for (P_L), we have

$$\int_\Omega \big(\mathcal{A}(x,\nabla u_i) - g_i\mathcal{A}(x,e)\big).\nabla\big((1-\alpha_\delta)\zeta\big) dx \leq \int_{S_3} \beta(x, \psi - u_i)((1-\alpha_\delta)\zeta) d\sigma(x). \tag{3.3.3}$$

Since $(1 - \alpha_\delta)\zeta = 0$ on A_m, we have

$$\begin{aligned}
&\int_\Omega \big(\mathcal{A}(x,\nabla u_m) - g_M\mathcal{A}(x,e)\big).\nabla\big((1-\alpha_\delta)\zeta\big) dx \\
&\qquad = \int_{[u_m = x_2]\cap[\gamma<1]} k(x)(1 - g_M)\big((1-\alpha_\delta)\zeta\big)_{x_2} dx \\
&\qquad = \int_{[u_m = x_2]\cap[\gamma<1]} \Big(\frac{\beta(x,\varphi)}{\nu_2}(1-\alpha_\delta)\zeta\Big)_{x_2} dx
\end{aligned}$$

$$= \int_\Omega \Big(\frac{\beta(x,\varphi)}{\nu_2}(1-\alpha_\delta)\zeta\Big)_{x_2} dx = \int_{\partial\Omega} \frac{\beta(x,\varphi)}{\nu_2}(1-\alpha_\delta)\zeta n_2 d\sigma(x)$$
$$= \int_{S_3} \beta(x,\varphi)(1-\alpha_\delta)\zeta d\sigma(x) + \int_{S_1'} \frac{\beta(x,\varphi)}{\nu_2}(1-\alpha_\delta)\zeta n_2 d\sigma(x) \qquad (3.3.4)$$

where $S_1' = \cup_{i=1}^{i=N} S_{1,i}$, $S_{1,i} = \{(x_1, s_-(x_1)) \ / \ x_1 \in \pi_{x_1}(S_{3,i}) \}$, and n_2 denotes the second entry of the outward unit normal vector n to S_1'.
We claim that

$$\int_{S_1'} \frac{\beta(x,\varphi)}{\nu_2}(1-\alpha_\delta)\zeta n_2 d\sigma(x) = 0.$$

Indeed first remark that

$$\int_{S_1'} \frac{\beta(x,\varphi)}{\nu_2}(1-\alpha_\delta)\zeta n_2 d\sigma(x) = \int_{S_1'\setminus\overline{A_m}} \frac{\beta(x,\varphi)}{\nu_2}(1-\alpha_\delta)\zeta n_2 d\sigma(x).$$

Next let $x_0 = (x_{01}, s_-(x_{01})) \in S_1' \setminus \overline{A_m}$ such that $\beta\big((x_{01}, s_+(x_{01})), \varphi(x_{01}, s_+(x_{01}))\big) > 0$. Without loss of generality, we assume that s_- and s_+ are continuous at x_{01}.
Since $x_0 \notin \overline{A_m}$, there exists $\epsilon_0 > 0$ such that $B_{\epsilon_0}(x_0) \cap A_m = \emptyset$ which means that $B_{\epsilon_0}(x_0) \cap \Omega \subset [u_m = x_2] \cap [\gamma < 1]$. Note that we can choose ϵ_0 small enough to ensure that s_- and s_+ are continuous in $(x_{01}-\epsilon_0, x_{01}+\epsilon_0)$. Moreover, using the continuity of γ in $[\gamma < 1]$, one can also assume that $\beta\big((x_1, s_+(x_1)), \varphi(x_1, s_+(x_1))\big) > 0$ in $(x_{01}-\epsilon_0, x_{01}+\epsilon_0)$.
By Corollary 1.3.1 and the fact that γ is non-increasing with respect to x_2, we get for some $\epsilon \in (0, \epsilon_0)$

$$u_m = x_2 \quad \text{and} \quad \gamma < 1 \quad \text{in} \quad D_\epsilon = \big((x_{01}-\epsilon, x_{01}+\epsilon) \times I\!R\big) \cap \Omega.$$

By Theorem 3.2.2, we know that Φ_1 and Φ_2 are continuous in $(x_{01}-\epsilon, x_{01}+\epsilon)$.
Now we distinguish three cases :

$*$ $\underline{\forall x_1 \in (x_{01}-\epsilon, x_{01}+\epsilon) \ \Phi_1(x_1) = s_-(x_1) :}$

In this case, we have $u_1(x) = x_2 \ \forall x \in D_\epsilon$. Let $\xi \in W^{1,q}(\Omega)$ such that $\xi = 0$ on $\partial D_\epsilon \cap \Omega$. Since $\pm\chi(D_\epsilon)\xi$ are test functions for (P_L), we have

$$\int_{D_\epsilon} (1-g_1)k\xi_{x_2} dx = \int_{\partial D_\epsilon \cap S_3} \beta(x,\varphi)\xi d\sigma(x).$$

Using Corollary 3.2.1, we get

$$\int_{D_\epsilon} (1-g_1)k\xi_{x_2} dx = \int_{D_\epsilon} \frac{\beta(x,\varphi)}{\nu_2}\xi_{x_2} dx$$
$$= \int_{\partial D_\epsilon \cap S_3} \beta(x,\varphi)\xi d\sigma(x) + \int_{\partial D_\epsilon \cap S_1'} \frac{\beta(x,\varphi)}{\nu_2} n_2 \xi d\sigma(x).$$

This leads to

$$\int_{\partial D_\epsilon \cap S_1'} \frac{\beta(x,\varphi)}{\nu_2} n_2 \xi d\sigma(x) = 0 \qquad \forall \xi \in W^{1,q}(\Omega) \quad \text{such that} \quad \xi = 0 \quad \text{on } \partial D_\epsilon \cap \Omega.$$

We obtain $\beta(x,\varphi)=0$ on $\partial D_\epsilon \cap S_3$ which contradicts (3.3.1) .

$*$ $\underline{\forall x_1 \in (x_{01}-\epsilon, x_{01}+\epsilon)\ \Phi_1(x_1)=s_+(x_1):}$

In this case, we have $u_1(x) > x_2\ \forall x \in D_\epsilon$ and then $u_2(x)=u_m(x)=x_2\ \forall x \in D_\epsilon$. This leads to $\Phi_2(x_1)=s_-(x_1)\,\forall x_1 \in (x_{01}-\epsilon, x_{01}+\epsilon)$. Then we get a contradiction as in the previous case.

$*$ $\underline{\exists x'_{01} \in (x_{01}-\epsilon, x_{01}+\epsilon)\ :\ s_-(x'_{01}) < \Phi_1(x'_{01}) < s_+(x'_{01})\ :}$

By continuity there exists a small $\delta' \in (0, \epsilon-|x_{01}-x'_{01}|)$ such that $x_{01}-\epsilon < x'_{01}-\delta' < x'_{01}+\delta' < x_{01}+\epsilon$ and $s_-(x_1) < \Phi_1(x_1) < s_+(x_1)\ \forall x_1 \in (x'_{01}-\delta', x'_{01}+\delta')$.
Since $u_m = x_2$ in D_ϵ, this leads to $\Phi_2(x_1)=s_-(x_1)\,\forall x_1 \in (x'_{01}-\delta', x'_{01}+\delta')$. Again we obtain $\beta(x,\varphi)=0$ on $\partial D_{\delta'} \cap S_3$, with $D_{\delta'} = \big((x'_{01}-\delta', x'_{01}+\delta') \times I\!R\big) \cap \Omega$ which is impossible.

We conclude that $S'_1 \setminus \overline{A_m} = \emptyset$ and therefore

$$\int_{S'_1} \frac{\beta(x,\varphi)}{\nu_2}(1-\alpha_\delta)\zeta n_2 d\sigma(x) = 0.$$

Subtracting then (3.3.4) from (3.3.3), we get by the monotonicity of β

$$\begin{aligned}\int_\Omega &((\mathcal{A}(x,\nabla u_i)-\mathcal{A}(x,\nabla u_m))-(g_i-g_M)\mathcal{A}(x,e)).\nabla\big((1-\alpha_\delta)\zeta\big)dx \\ &\le \int_{S_3}(\beta(x,\psi-u_i)-\beta(x,\psi-x_2))(1-\alpha_\delta)\zeta d\sigma(x) \le 0. \qquad (3.3.5)\end{aligned}$$

Now clearly $\pm\xi$ are test functions for (P_L). So we have for $i,\ j=1,2$ with $i \neq j$

$$\begin{aligned}\int_\Omega &((\mathcal{A}(x,\nabla u_i)-\mathcal{A}(x,\nabla u_j))-(g_i-g_j)\mathcal{A}(x,e)).\nabla\xi dx \\ &= \int_{S_3}(\beta(x,\psi-u_i)-\beta(x,\psi-u_j)).\xi d\sigma(x). \qquad (3.3.6)\end{aligned}$$

Given that we integrate only on the set $[u_i-u_m>0]$ where $u_m=u_j$, (3.3.6) becomes by the monotonicity of β

$$\int_\Omega((\mathcal{A}(x,\nabla u_i)-\mathcal{A}(x,\nabla u_m))-(g_i-g_M)\mathcal{A}(x,e)).\nabla\xi dx \le 0$$

which can be written by the monotonicity of $\mathcal{A}$

$$\begin{aligned}&\int_{[u_i-u_m\ge\epsilon\zeta]}(\mathcal{A}(x,\nabla u_i)-\mathcal{A}(x,\nabla u_m)).\nabla(\alpha_\delta\zeta)dx \\ &-\int_\Omega k(x)(g_i-g_M).(\alpha_\delta\zeta)_{x_2}dx \le \int_\Omega k(x)(g_M-g_i)\Big(\alpha_\delta\zeta-\frac{u_i-u_m}{\epsilon}\Big)^+_{x_2}dx. \qquad (3.3.7)\end{aligned}$$

Using (3.2.21), we have

$$\int_\Omega k(x)(g_M-g_i)\Big(\alpha_\delta\zeta-\frac{u_i-u_m}{\epsilon}\Big)^+_{x_2}dx$$

$$
\begin{aligned}
&= \int_{[u_i>u_m=x_2]\cap[\gamma<1]} \Big(k(x) - \frac{\beta(x,\varphi)}{\nu_2}\Big)\Big(\alpha_\delta\zeta - \frac{u_i-u_m}{\epsilon}\Big)^+_{x_2} dx \\
&= \int_{[u_i>u_m=x_2]} k(x)(1-\gamma)^+\Big(\alpha_\delta\zeta - \frac{u_i-u_m}{\epsilon}\Big)^+_{x_2} dx \\
&= \int_{D_i} dx_1 \int_{\Phi_m(x_1)}^{\Phi_i(x_1)} k(x)(1-\gamma)^+\Big(\alpha_\delta\zeta - \frac{u_i-u_m}{\epsilon}\Big)^+_{x_2} dx_2. \qquad (3.3.8)
\end{aligned}
$$

with $D_i = \{\, x_1 \in \pi_{x_1}(\Omega) \,/\, \Phi_m(x_1) < \Phi_i(x_1)\,\}$, $i = 1,2$ and $\Phi_m = \min(\Phi_1, \Phi_2)$.
Since for a.e. $x_1 \in D_i$, $k(1-\gamma)^+(x_1, .)$ is nondecreasing with respect to x_2, we deduce by the second mean-value theorem

$$
\begin{aligned}
&\int_{\Phi_m(x_1)}^{\Phi_i(x_1)} k(1-\gamma)^+\Big(\alpha_\delta\zeta - \frac{u_i-u_m}{\epsilon}\Big)^+_{x_2} dx_2 \\
&\qquad = (k(1-\gamma)^+)(x_1, \Phi_i(x_1)_-) \int_{\Phi_*(x_1)}^{\Phi_i(x_1)} \Big(\alpha_\delta\zeta - \frac{u_i-u_m}{\epsilon}\Big)^+_{x_2} dx_2 \\
&\qquad \le (k(1-\gamma)^+)(x_1, \Phi_i(x_1)_-)(\alpha_\delta\zeta)(x_1, \Phi_i(x_1)) \qquad (3.3.9)
\end{aligned}
$$

with $\Phi_*(x_1) \in [\Phi_m(x_1), \Phi_i(x_1)]$ and $(k(1-\gamma)^+)(x_1, \Phi_i(x_1)_-)$ is the left limit of $(k(1-\gamma)^+)(x_1, .)$ at $\Phi_i(x_1)$.
Taking into account (3.3.7)-(3.3.9), we get

$$
\begin{aligned}
&\int_{[u_i-u_m\ge\epsilon\zeta]} (\mathcal{A}(x,\nabla u_i) - \mathcal{A}(x,\nabla u_m)).\nabla(\alpha_\delta\zeta)dx - \int_\Omega k(x)(g_i - g_M).(\alpha_\delta\zeta)_{x_2} dx \\
&\qquad \le \int_{D_i} (k(1-\gamma)^+)(x_1, \Phi_i(x_1)_-)(\alpha_\delta\zeta)(x_1, \Phi_i(x_1))dx_1
\end{aligned}
$$

which leads by letting ϵ go to zero to

$$
\begin{aligned}
&\int_\Omega (\mathcal{A}(x,\nabla u_i) - \mathcal{A}(x,\nabla u_m) - (g_i - g_M)\mathcal{A}(x,e)).\nabla(\alpha_\delta\zeta)dx \\
&\qquad \le \int_{D_i} (k(1-\gamma)^+)(x_1, \Phi_i(x_1)_-)(\alpha_\delta\zeta)(x_1, \Phi_i(x_1))dx_1 \\
&\qquad \le \int_{D_i} (k(1-\gamma)^+\alpha_\delta\zeta)(x_1, \Phi_i(x_1))dx_1. \qquad (3.3.10)
\end{aligned}
$$

Using (3.3.2), (3.3.5) and (3.3.10), we get

$$
\begin{aligned}
&\int_\Omega (\mathcal{A}(x,\nabla u_i) - \mathcal{A}(x,\nabla u_m) - (g_i - g_M)\mathcal{A}(x,e)).\nabla\zeta dx \\
&\qquad \le \int_{D_i} \big(k(1-\gamma)^+\alpha_\delta\zeta\big)(x_1, \Phi_i(x_1))dx_1. \qquad (3.3.11)
\end{aligned}
$$

Now let $x_{01} \in D_i$ such that $\gamma(x_{01}, \Phi_i(x_{01})) < 1$. By continuity there exists a small $\eta > 0$ such that $\gamma(x) < 1\ \forall x \in B_\eta(x_{01}, \Phi_i(x_{01}))$. Moreover one can assume that $\Phi_m(x_{01}) < \Phi_i(x_{01}) - \frac{\eta}{2}$.

Since $\gamma(x_{01}, \Phi_i(x_{01}) - \frac{\eta}{2}) < 1$ and $u_m(x_{01}, \Phi_i(x_{01}) - \frac{\eta}{2}) = \Phi_i(x_{01}) - \frac{\eta}{2}$, one can argue as in the proof of Theorem 3.2.2 to get for some $\eta' \in (0, \eta)$

$$u_m(x) = x_2 \qquad \forall x \in \big((x_{01} - \eta', x_{01} + \eta') \times (\Phi_i(x_{01}) - \eta', +\infty)\big) \cap \Omega.$$

It follows that $(x_{01}, \Phi_i(x_{01})) \notin \overline{A_m}$. Thus $\alpha_\delta(x_{01}, \Phi_i(x_{01}))$ converges to 0 when δ goes to 0. Using the Lebesgue theorem, we obtain

$$\lim_{\delta \to 0} \int_{D_i} \big(k(1-\gamma)^+ \alpha_\delta \zeta\big)(x_1, \Phi_i(x_1))dx_1 = 0$$

which leads to

$$\int_\Omega (\mathcal{A}(x, \nabla u_i) - \mathcal{A}(x, \nabla u_m) - (g_i - g_M)\mathcal{A}(x, e)).\nabla \zeta dx \le 0.$$

At this step, we argue as at the end of the proof of Theorem 2.3.1 to get

$$\int_\Omega (\mathcal{A}(x, \nabla u_i) - \mathcal{A}(x, \nabla u_m) - (g_i - g_M)\mathcal{A}(x, e)).\nabla \zeta dx = 0 \qquad \forall \zeta \in W^{1,q}(\Omega).$$

ii) We argue as for the proof of *ii*) of Theorem 2.3.1.

iii) Let $\xi \in W^{1,q}(\Omega),\ \xi = 0$ on S_2. Using *i*) for $i = 1, 2$, we obtain

$$\int_\Omega (\mathcal{A}(x, \nabla u_i) - \mathcal{A}(x, \nabla u_j) - (g_i - g_j)\mathcal{A}(x, e)).\nabla \xi dx = 0.$$

But since $\pm\xi$ are test functions for (P_L), we deduce that

$$\int_{S_3} \big(\beta(x, \psi - u_2) - \beta(x, \psi - u_1)\big)\xi d\sigma(x) = 0 \qquad \forall \xi \in W^{1,q}(\Omega),\ \xi = 0 \text{ on } S_2$$

which gives $\beta(x, \psi - u_1) = \beta(x, \psi - u_2)$ a.e. on S_3. □

As a consequence of Theorem 3.3.1, we deduce by arguing as for the proof of Corollary 2.3.1:

Corollary 3.3.1. *Let* (u_1, g_1) *and* (u_2, g_2) *be two solutions of* (P_L)*. Then* $(\min(u_1, u_2), \max(g_1, g_2))$ *and* $(\max(u_1, u_2), \min(g_1, g_2))$ *are also solutions of* (P_L)*.*

We consider the set of all solutions of (P_L):

$$\mathcal{S}_L = \{\, (u, g) \in W^{1,q}(\Omega) \times L^\infty(\Omega) \,/\, (u, g) \text{ is a solution of } (P_L) \,\}.$$

Then we define the following functional $\mathcal{I}_L$ on $\mathcal{S}_L$ by:

$$\forall (u, g) \in \mathcal{S}_L, \quad \mathcal{I}_L(u, g) = \int_\Omega \mathcal{A}(x, \nabla u).\nabla u dx - \frac{1}{q}\int_\Omega g\mathcal{A}(x, e).e dx + \int_{S_3} \beta(x, \psi - u)(\psi - u) d\sigma(x). \tag{3.3.12}$$

We assume that:

$$\text{for a.e. } x \in S_3 \qquad \beta(x,.) \text{ is a continuous function.} \tag{3.3.13}$$

Here is the main result of this section.

Theorem 3.3.2. *There exist a unique minimal solution* (u_m, g_M) *and a unique maximal solution* (u_M, g_m) *in* $\mathcal{S}_L$ *in the following sense*

$$\mathcal{I}_L(u_m, g_M) = \min_{(u,g)\in\mathcal{S}_L} \mathcal{I}_L(u,g), \qquad \mathcal{I}_L(u_M, g_m) = \max_{(u,g)\in\mathcal{S}_L} \mathcal{I}_L(u,g).$$
$$u_m \le u \le u_M, \qquad g_m \le g \le g_M \qquad \forall (u,g) \in \mathcal{S}_L.$$

We first prove a monotonicity result.

Lemma 3.3.1. $\mathcal{I}_L$ *is strictly monotone i.e.* $\forall (u_1, g_1)$, $(u_2, g_2) \in \mathcal{S}_L$

i) $\quad u_1 \le u_2, \quad g_2 \le g_1 \quad$ *in* $\quad \Omega \quad \Longrightarrow \quad \mathcal{I}_L(u_1, g_1) \le \mathcal{I}_L(u_2, g_2)$

ii) $\quad u_1 \le u_2, \quad g_2 \le g_1 \quad$ *in* $\quad \Omega \quad$ *and* $\quad u_1 \neq u_2 \quad \Longrightarrow \quad \mathcal{I}_L(u_1, g_1) < \mathcal{I}_L(u_2, g_2)$.

Proof. *i)* Let (u_1, g_1), $(u_2, g_2) \in \mathcal{S}_L$ satisfying $u_1 \le u_2$ and $g_2 \le g_1$ a.e. in Ω. Then we have, since by Theorem 3.3.1 *iii)* $\beta(x, \psi - u_1) = \beta(x, \psi - u_2)$ a.e. in S_3.

$$\begin{aligned}
&\mathcal{I}_L(u_1, g_1) - \mathcal{I}_L(u_2, g_2) = \\
&\quad = \int_\Omega \big(\mathcal{A}(x,\nabla u_1).\nabla u_1 - g_1 \mathcal{A}(x,e).e\big)dx - \int_{S_3} \beta(x,\psi - u_1) u_1 d\sigma(x) \\
&\quad - \int_\Omega \big(\mathcal{A}(x,\nabla u_2).\nabla u_2 - g_2 \mathcal{A}(x,e).e\big)dx + \int_{S_3} \beta(x,\psi - u_2) u_2 d\sigma(x) \\
&\quad + \frac{1}{q'} \int_\Omega (g_1 - g_2)\mathcal{A}(x,e).edx \\
&\quad = \int_\Omega \big(\mathcal{A}(x,\nabla u_1) - g_1 \mathcal{A}(x,e)\big).\nabla(u_1 - x_2)dx - \int_{S_3} \beta(x,\psi - u_1)(u_1 - x_2) d\sigma(x) \\
&\quad - \int_\Omega \big(\mathcal{A}(x,\nabla u_2) - g_2 \mathcal{A}(x,e)\big).\nabla(u_2 - x_2)dx + \int_{S_3} \beta(x,\psi - u_2)(u_2 - x_2) d\sigma(x) \\
&\quad + \frac{1}{q'} \int_\Omega (g_1 - g_2)\mathcal{A}(x,e).edx + \int_\Omega \big((\mathcal{A}(x,\nabla u_1) - \mathcal{A}(x,\nabla u_2)) - (g_1 - g_2)\mathcal{A}(x,e)\big).\nabla x_2 dx.
\end{aligned}$$

By Theorem 3.3.1 *i)* and since $\pm(u_i - x_2)$ $(i = 1, 2)$ are test functions for (P_L), we get

$$\mathcal{I}_L(u_1, g_1) - \mathcal{I}_L(u_2, g_2) = \frac{1}{q'} \int_\Omega (g_1 - g_2) k(x) dx \le 0.$$

ii) Assume that $u_1 \le u_2$, $g_2 \le g_1$ in Ω and $\mathcal{I}_L(u_1, g_1) = \mathcal{I}_L(u_2, g_2)$. From the proof of *i)*, we get $\displaystyle\int_\Omega (g_1 - g_2) k(x) dx = 0$ which leads to $g_1 = g_2$ a.e. in Ω. Using Theorem 3.3.1 *i)* for

$\xi = u_1 - u_2$, we obtain

$$\int_\Omega \big(\mathcal{A}(x,\nabla u_1) - \mathcal{A}(x,\nabla u_2)\big).\nabla(u_1-u_2)dx = 0$$

which leads by (2.2.1) to $\nabla(u_1-u_2)=0$ a.e. in Ω. But since $u_1-u_2=0$ on S_2, we deduce that $u_1=u_2$. □

Proof of Theorem 3.3.2. First remark that for each $(u,g)\in\mathcal{S}_L$, we have by (1.1.2) and the monotonicity of $\beta(x,.)$

$$\mathcal{I}_L(u_1,g_1) \geq -\frac{1}{q}\int_\Omega k(x)gdx \geq -\frac{|\Omega|}{q}M$$

from which we deduce that there exists a minimizing sequence $(u_k,g_k)_{k\in I\!N}$ for $\mathcal{I}_L$ i.e.

$$\forall k\in I\!N \quad (u_k,g_k)\in\mathcal{S}_L \quad \text{ and } \quad \lim_{k\to+\infty}\mathcal{I}_L(u_k,g_k) = m = \inf_{(u,g)\in\mathcal{S}_L}\mathcal{I}_L(u,g). \qquad (3.3.14)$$

As in the Dirichlet case, we define the sequence $(v_k,f_k)_{k\in I\!N}$ by

$$\begin{cases}(v_0,f_0)=(u_0,g_0)\\ (v_{k+1},f_{k+1}) = \big(\min(u_{k+1},v_k),\max(g_{k+1},f_k)\big) \qquad \forall k\in I\!N.\end{cases}$$

Using Corollary 3.3.1 and Lemma 3.3.1 and arguing as in the proof of Theorem 2.2.2, one can verify that for each $k\in I\!N$, $(v_k,f_k)\in\mathcal{S}_L$ and that we have for some $(v,f)\in L^q(\Omega)\times L^{q'}(\Omega)$

$$m = \lim_{k\to+\infty}\mathcal{I}_L(v_k,f_k). \qquad (3.3.15)$$

$$v_k \longrightarrow v \quad \text{in} \quad L^q(\Omega) \quad \text{and a.e. in } \Omega. \qquad (3.3.16)$$

$$f_k \longrightarrow f \quad \text{in} \quad L^{q'}(\Omega) \quad \text{and a.e. in } \Omega. \qquad (3.3.17)$$

Now since $\pm(v_k - x_2)$ are test functions for (P_L), we have by $(P_L)ii)$, Proposition 1.3.1 and $(1.1.2)iii)$)

$$\lambda\int_\Omega|\nabla v_k|^q dx \leq \int_\Omega \mathcal{A}(x,\nabla v_k).\nabla v_k dx = \int_{S_3}\beta(x,\psi-v_k)(v_k-x_2)d\sigma(x)$$
$$+\int_\Omega\mathcal{A}(x,\nabla v_k).\nabla x_2 dx \leq \int_{S_3}(h_0-x_2)\beta(x,\psi-x_2)d\sigma(x) + M|\Omega|^{1/q}\Big(\int_\Omega|\nabla v_k|^q dx\Big)^{1/q'}.$$

We deduce that $(v_k)_k$ is bounded in $W^{1,q}(\Omega)$. So we have up to a subsequence

$$v_{k_p} \rightharpoonup v \quad \text{in} \quad W^{1,q}(\Omega) \qquad (3.3.18)$$

$$v_{k_p} \longrightarrow v \quad \text{in} \quad L^q(S_3) \quad \text{and a.e. in } S_3. \qquad (3.3.19)$$

By the continuity of the trace operator and (3.3.19), we have $v = 0$ on S_2. By (3.3.16)-(3.3.17), we deduce that

$$v \geq x_2, \qquad 0 \leq f \leq 1, \qquad f(v - x_2) = 0 \quad \text{a.e. in } \Omega.$$

Hence to prove that $(v, f) \in \mathcal{S}_L$, it remains to prove that it satisfies $(P_L)i)$.
From the fact that (v_k) is bounded in $W^{1,q}(\Omega)$, we deduce that up to a subsequence still denoted by (v_{k_p}), one has

$$\mathcal{A}(x, \nabla v_{k_p}) \rightharpoonup \mathcal{A}_0 \qquad \text{in} \quad I\!\!L^{q'}(\Omega). \tag{3.3.20}$$

Let $p, s \in I\!\!N$ such that $p \leq s$. We have by Theorem 3.3.1

$$\int_\Omega \{\big(\mathcal{A}(x, \nabla v_{k_p}) - \mathcal{A}(x, \nabla v_{k_s})\big) - (f_{k_p} - f_{k_s})\mathcal{A}(x, e)\}.\nabla v_{k_p} dx = 0$$

from which we deduce by letting first $s \to +\infty$ and then $p \to +\infty$

$$\lim_{p \to +\infty} \int_\Omega \mathcal{A}(x, \nabla v_{k_p}).\nabla v_{k_p} dx = \int_\Omega \mathcal{A}_0.\nabla v dx. \tag{3.3.21}$$

Using the monotonicity of $\mathcal{A}$, (3.3.18) and (3.3.20)-(3.3.21), we easily obtain

$$\mathcal{A}(x, \nabla v_{k_p}) \rightharpoonup \mathcal{A}(x, \nabla v) \qquad \text{in} \quad I\!\!L^{q'}(\Omega). \tag{3.3.22}$$

Finally, let $\xi \in W^{1,q}(\Omega)$ such that $\xi \geq 0$ on S_2. For each $p \in I\!\!N$, we have

$$\int_\Omega \big(\mathcal{A}(x, \nabla v_{k_p}) - f_{k_p}\mathcal{A}(x, e)\big).\nabla \xi dx \leq \int_{S_3} \beta(x, \psi - v_{k_p})\xi d\sigma(x). \tag{3.3.23}$$

Using (3.3.13), (3.3.17), (3.3.19) and (3.3.22), we get by letting $p \to +\infty$ in (3.3.23)

$$\int_\Omega \big(\mathcal{A}(x, \nabla v) - f\mathcal{A}(x, e)\big).\nabla \xi dx \leq \int_{S_3} \beta(x, \psi - v)\xi d\sigma(x).$$

Thus (v, f) is a solution of (P_L).
Now, using (3.3.15)-(3.3.17) and (3.3.19)-(3.3.22), we obtain

$$\begin{aligned} m = \lim_{p \to +\infty} \mathcal{I}_L(v_{k_p}, f_{k_p}) &= \lim_{p \to +\infty} \int_\Omega \mathcal{A}(x, \nabla v_{k_p}).\nabla v_{k_p} dx - \frac{1}{q}\int_\Omega f_{k_p}\mathcal{A}(x, e).edx \\ &+ \int_{S_3} \beta(x, \psi - v_{k_p})(\psi - v_{k_p})d\sigma(x) \\ &= \int_\Omega \mathcal{A}(x, \nabla v).\nabla v dx - \frac{1}{q}\int_\Omega f\mathcal{A}(x, e).edx + \int_{S_3} \beta(x, \psi - v)(\psi - v)d\sigma(x) = \mathcal{I}_L(v, f). \end{aligned}$$

Let $(u, g) \in \mathcal{S}_L$. Since by Corollary 3.3.1, $\big(\min(u, v), \max(g, f)\big) \in \mathcal{S}_L$, we deduce by Lemma 3.3.1 $i)$ that

$$m \leq \mathcal{I}_L\big(\min(u, v), \max(g, f)\big) \leq \mathcal{I}_L(v, f) = m$$

which leads by Lemma 3.3.1 *ii*) to $(v,f) = \big(\min(u,v), \max(g,f)\big)$ i.e. $v \le u$ and $g \le f$ a.e. in Ω. The uniqueness of (v,f) is then clear. This achieves the proof of the first part of Theorem 3.3.2.

Let us prove the second part of the theorem. First we have by (1.1.2)*iii*) and the monotonicity of $\beta(x,.)$

$$\begin{aligned}\mathcal{I}_L(u,g) &= \int_\Omega \mathcal{A}(x,\nabla u).\nabla u dx - \frac{1}{q}\int_\Omega gk(x)dx + \int_{S_3}\beta(x,\psi-u)(\psi-u)d\sigma(x)\\ &\le \int_\Omega \mathcal{A}(x,\nabla u).\nabla u dx + \int_{S_3}\beta(x,\varphi)\varphi d\sigma(x)\\ &\le M\int_\Omega |\nabla u|^q dx + \int_{S_3}\beta(x,\varphi)\varphi d\sigma(x).\end{aligned}$$

Moreover since $\pm(u-x_2)$ are test functions for (P_L), we have

$$\int_\Omega \big(\mathcal{A}(x,\nabla u) - g\mathcal{A}(x,e)\big).\nabla(u-x_2)dx = \int_{S_3}\beta(x,\psi-u)(u-x_2)d\sigma(x)$$

This leads by (1.1.2)*iii*) to

$$\begin{aligned}\lambda\int_\Omega |\nabla u|^q dx &\le \int_\Omega \mathcal{A}(x,\nabla u).\nabla u dx\\ &= \int_\Omega \mathcal{A}(x,\nabla u).e dx + \int_{S_3}\beta(x,\psi-u)(u-x_2)d\sigma(x)\\ &\le M|\Omega|^{1/q}\Big(\int_\Omega |\nabla u|^q dx\Big)^{1/q'} + \int_{S_3}\beta(x,\varphi)(h_0-x_2)d\sigma(x)\end{aligned}$$

and

$$\int_\Omega |\nabla u|^q dx \le C \qquad \text{for some positive constant } C.$$

Thus $\mathcal{I}_L(u,g)$ is bounded for all $(u,g)\in\mathcal{S}_L$. Let then $(u_k,g_k)_{k\in I\!N}$ be a sequence of solutions of (P_L) such that

$$\lim_{k\to+\infty}\mathcal{I}_L(u_k,g_k) = \sup_{(u,g)\in\mathcal{S}_L}\mathcal{I}_L(u,g).$$

We consider the following sequence $(w_k,h_k)_{k\in I\!N}$ defined by

$$\begin{cases}(w_0,h_0) = (u_0,g_0)\\ (w_{k+1},h_{k+1}) = \big(\max(u_{k+1},w_k),\min(g_{k+1},h_k)\big) \quad \forall k\in I\!N.\end{cases}$$

Then we have by Corollary 3.3.1 and Lemma 3.3.1 *i*)

$$\forall k\in I\!N \qquad (w_k,h_k)\in\mathcal{S}_L \quad \text{and} \quad \mathcal{I}_L(u_k,g_k)\le\mathcal{I}_L(w_k,h_k)\le\sup_{(u,g)\in\mathcal{S}_L}\mathcal{I}_L(u,g).$$

It follows that

$$\sup_{(u,g)\in\mathcal{S}_L} \mathcal{I}_L(u,g) = \lim_{k\to+\infty} \mathcal{I}_L(w_k,h_k). \tag{3.3.24}$$

Using the monotonicity of $(w_k,h_k)_{k\in I\!N}$, (3.3.24) and arguing as above, we prove that for a subsequence $(w_{k_p},h_{k_p})_{p\in I\!N}$, we have

$$\begin{aligned}
&w_{k_p} \rightharpoonup w \quad \text{in} \quad W^{1,q}(\Omega)\\
&w_{k_p} \longrightarrow w \quad \text{in} \quad L^q(\Omega) \quad \text{and a.e. in } \Omega\\
&w_{k_p} \longrightarrow w \quad \text{in} \quad L^q(S_3) \quad \text{and a.e. in } S_3\\
&\mathcal{A}(x,\nabla w_{k_p}) \rightharpoonup \mathcal{A}(x,\nabla w) \quad \text{in} \quad I\!L^{q'}(\Omega)\\
&h_k \longrightarrow h \quad \text{in} \quad L^{q'}(\Omega) \quad \text{and a.e. in } \Omega
\end{aligned}$$

Thus it becomes easy to verify that (w,h) is a solution of (P_L) which satisfies $\mathcal{I}_L(w,h) = \sup_{(u,g)\in\mathcal{S}_L} \mathcal{I}_L(u,g)$. We can then prove, as in the case of minimal solution, that for any $(u,g) \in \mathcal{S}_L\,:\, u \le w,\, h \le g$ a.e. in Ω. □

3.4 Reservoirs-Connected Solution

In this section, we first extend the definition of the reservoirs-connected solution. Then we prove the uniqueness of this solution in three situations.

Theorem 3.4.1. *Let (u,g) be a solution of (P_L). For each $i \in \{1,...,N\}$ and each interval $I \subset \pi_{x_1}(S_{3,i})$, one cannot have $u \equiv x_2$ in $(I\times I\!R)\cap\Omega$.*

Proof. Assume that $u \equiv x_2$ in $Z = (I\times I\!R)\cap\Omega$. By (3.1.9), one has $g = 1-\gamma$ a.e. in Z. Let now $\xi \in W^{1,q}(\Omega)$ such that $\xi = 0$ on $\partial Z\cap\Omega$. Then $\pm\xi$ are test functions for (P_L) and we have

$$\int_Z \gamma\mathcal{A}(x,e)\nabla\xi dx = \int_{S_3^Z} \beta(x,\varphi)\xi d\sigma(x)$$

or

$$\int_Z \frac{\beta(x,\varphi)}{\nu_2}\xi_{x_2} dx = \int_{S_3^Z} \beta(x,\varphi)\xi d\sigma(x)$$

which leads to

$$\int_{\partial Z\cap S_1} \frac{\beta(x,\varphi)}{\nu_2}\xi n_2 d\sigma(x) = 0$$

where n_2 is the second entry of the outward unit normal vector n to S_1. Hence we obtain $\beta(x,\varphi) = 0$ on $\partial Z\cap S_1$, which contradicts (3.3.1). □

In the following we give similar results as in 2.4.

Definition 3.4.1. *A solution* (u, g) *of* (P_L) *is called a reservoirs-connected solution if for each connected component* C *of* $[u > x_2]$*, we have* $\overline{\pi_{x_1}(C)} \cap \pi_{x_1}(S_3) \neq \emptyset$*.*

Remark 3.4.1. *Suppose that we have* $\beta(x, \varphi) \geq k(x)\nu_2$ *a.e. in some open connected subset* T *of* $S_{3,i}$ *for some* $i \in \{1, \dots, N\}$*. If* C *is a connected component of* $[u > x_2]$ *such that* $\overline{\pi_{x_1}(C)} \cap \pi_{x_1}(T) \neq \emptyset$*, then by Corollary 3.1.1,* C *contains the strip of* Ω *below* T *and* T *on its boundary.*

Theorem 3.4.2. *Let* (u, g) *be a solution of* (P_L) *and* C *a connected component of* $[u > x_2]$ *such that* $\overline{\pi_{x_1}(C)} \cap \pi_{x_1}(S_3) = \emptyset$*. If we set* $h_c = \sup\{\, x_2 \,/\, (x_1, x_2) \in C \,\}$*, then we have*

$$\begin{cases} C = \{\, (x_1, x_2) \in \Omega \,/\, x_1 \in \pi_{x_1}(C) \,,\, x_2 < h_c \,\}, \\ u = x_2 + (h_c - x_2)^+ . \chi(C), \quad g \,=\, 1 - \chi(C) \qquad \text{in} \quad Z = (\pi_{x_1}(C) \times \mathbb{R}) \cap \Omega. \end{cases}$$

Proof. See The proof of Theorem 2.4.1. □

Definition 3.4.2. *We call a pool in* Ω *a pair of functions defined in* Ω *by* $(p, \chi) = ((h - x_2)^+, 1)\chi(C)$*, where* C *is a connected component of* $\Omega \cap [x_2 < h]$*.*

Remark 3.4.2. *Thanks to this definition, Theorem 3.4.2 becomes:*
For each solution (u, g) *of* (P_L) *and each connected component* C *of* $[u > x_2]$ *such that* $\overline{\pi_{x_1}(C)} \cap \pi_{x_1}(S_3) = \emptyset$*,* $(u - x_2, 1 - g)$ *agrees with a pool in the strip* $\Omega \cap (\pi_{x_1}(C) \times I\!R)$*.*

We get by adapting the proof of Theorem 2.4.2.

Theorem 3.4.3. *Any solution* (u, g) *of* (P_L) *can be written as*

$$u = u_r + \sum_{i \in I} p_i \quad \text{and} \quad g = g_r - \sum_{i \in I} \chi_i$$

where (u_r, g_r) *is a reservoirs-connected solution and* (p_i, χ_i) *are pools.*

It follows from Theorem 3.4.3.

Corollary 3.4.1. *The minimal solution* (u_m, g_M) *is a reservoirs-connected solution.*

3.5 Uniqueness of the Reservoirs-Connected Solution

In this section, we establish the uniqueness of the reservoirs-connected solution in three situations.

3.5.1 The case of Linear Darcy's Law with a Diagonal Permeability Matrix

Here we assume that:

$$\mathcal{A}(x,\xi) = a(x).\xi \ \forall \xi \in \mathbb{R}^2, \text{ a.e. } x \in \Omega, \ a(x) = (a_{ij}(x)) \text{ is a } 2\times 2 \text{ matrix.} \tag{3.5.1}$$

$$\exists \lambda, M > 0: \quad \lambda|\xi|^2 \leq a(x)\xi.\xi \leq M|\xi|^2 \quad \forall \xi \in I\!R^2, \text{ a.e. } x \in \Omega. \tag{3.5.2}$$

$$a_{12}(x) = a_{21}(x) = 0 \quad \text{for a.e. } x \in \Omega. \tag{3.5.3}$$

$$\tfrac{\partial a_{22}}{\partial x_2} \geq 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{3.5.4}$$

$$\text{For a.e. } x \in S_3, \ \beta(x,.) \text{ is an increasing function.} \tag{3.5.5}$$

Then we have:

Theorem 3.5.1. *Assume that (3.5.1)-(3.5.5) are satisfied. Then there is one and only one reservoirs-connected solution.*

We first prove a lemma.

Lemma 3.5.1. *Let (u, g) be a solution of (P_L). Then we have*

$$\nabla(u - u_m) = (g - g_M)e \qquad a.e. \ in \ \Omega. \tag{3.5.6}$$

Proof. Set $p = u - x_2$, $\chi = 1 - g$, $p_m = u_m - x_2$ and $\chi_m = 1 - g_M$. Then from Theorem 3.3.1 *i*), we have by taking $\zeta = p - p_m$ and $\zeta = x_2$ respectively

$$\int_\Omega a(x)\big(\nabla(p - p_m) + (\chi - \chi_m)e\big).\nabla(p - p_m)dx = 0 \tag{3.5.7}$$

$$\int_\Omega a(x)\big(\nabla(p - p_m) + (\chi - \chi_m)e\big).edx = 0. \tag{3.5.8}$$

Using the fact that $0 \leq \chi - \chi_m \leq 1$ and $\chi\nabla(p - p_m) = \nabla(p - p_m)$ a.e. in Ω, (3.5.8) becomes

$$\int_\Omega a(x)\big(\nabla(p - p_m)\big).\chi edx + \int_\Omega a(x)\big((\chi - \chi_m)e\big).(\chi - \chi_m)edx \leq 0. \tag{3.5.9}$$

Since $\pm p_m$ and $\pm p$ are test functions for (P_L), we have

$$\int_\Omega a(x)(\nabla p + \chi e).\nabla p_m dx = \int_{S_3} \beta(x, \varphi - p)p_m d\sigma(x). \tag{3.5.10}$$

$$\int_\Omega a(x)(\nabla p_m + \chi_m e).\nabla p dx = \int_{S_3} \beta(x, \varphi - p_m)p d\sigma(x). \tag{3.5.11}$$

Subtracting (3.5.11) from (3.5.10) and taking into account the fact that $p = p_m$ a.e. in S_3 which is a consequence of Theorem 3.3.1 *iii*) and (3.5.5), we obtain

$$\int_\Omega a(x)(\chi e).\nabla p_m dx - \int_\Omega a(x)(\chi_m e).\nabla p dx$$

$$= \int_\Omega a(x)(\nabla p_m).\nabla p dx - \int_\Omega a(x)(\nabla p).\nabla p_m dx = 0 \tag{3.5.12}$$

since by (3.5.3) a is a symmetric matrix.
Now because $\chi \nabla p_m = \chi_m \nabla p_m$ a.e. in Ω, (3.5.12) becomes

$$\int_\Omega a(x)(-\chi_m e).\nabla(p - p_m)dx = 0$$

or

$$\int_\Omega a(x)\nabla(p - p_m).(-\chi_m e)dx = 0. \tag{3.5.13}$$

Then, if we add (3.5.9) and (3.5.13), we get

$$\int_\Omega a(x)\big(\nabla(p - p_m) + (\chi - \chi_m)e\big).(\chi - \chi_m)edx \le 0. \tag{3.5.14}$$

Finally, adding (3.5.7) and (3.5.14), we obtain

$$\int_\Omega a(x)\big(\nabla(p - p_m) + (\chi - \chi_m)e\big).\big(\nabla(p - p_m) + (\chi - \chi_m)e\big)dx \le 0$$

which leads by (3.5.2) to

$$\nabla(p - p_m) = -(\chi - \chi_m)e \quad \text{a.e. in} \quad \Omega$$

and (3.5.6) is proved. □

Proof of Theorem 3.5.1. Let (u, g) be a reservoirs-connected solution of (P_L). Let $C_{m,i}$ be a connected component of $[u_m > x_2]$ such that $\pi_{x_1}(C_{m,i}) \cap \pi_{x_1}(S_{3,i}) \neq \emptyset$ and let C_i be the connected component of $[u > x_2]$ which contains $C_{m,i}$.
First we deduce from (3.5.6) that we have for some nonnegative constant c_i

$$u - u_m = c_i \qquad \text{in} \quad C_{m,i} \tag{3.5.15}$$

We shall prove that $u = u_m$ in $C_{m,i}$. To do this, we distinguish two cases:

i) $\underline{\overline{C_{m,i}} \cap (S_2 \cup S_3) \neq \emptyset}$:

Note that $w = u - u_m$ satisfies

$$\begin{cases} div(a(x)(\nabla w)) = div\big((g - g_M)a(x)(e)\big) & \text{in} \quad \mathcal{D}'(\Omega) \\ w = 0 \qquad \text{on} \quad S_2 \cup S_3. \end{cases}$$

So $w \in C^{0,\alpha}(\Omega \cup S_2 \cup S_3)$ and from (3.5.15), we obtain $w = 0$ in $C_{m,i}$.

ii) $\underline{\overline{C_{m,i}} \cap (S_2 \cup S_3) = \emptyset}$:

Again we distinguish two cases:

. • $\underline{\partial C_i \cap \partial C_{m,i} \cap \Omega \neq \emptyset}$:

Since $u - u_m = 0$ on $\partial C_i \cap \partial C_{m,i} \cap \Omega$, we deduce that $u - u_m = c_i = 0$ in $C_{m,i}$ and therefore $u = u_m$ in $C_{m,i}$.

. • $\underline{\partial C_i \cap \partial C_{m,i} \cap \Omega = \emptyset}$:

Since $\partial C_i \cap \partial C_{m,i} \cap \Omega = \emptyset$, we have $C_i \setminus C_{m,i} \neq \emptyset$ and $\partial C_{m,i} \cap \Omega \subset C_i$. Again we deduce from (3.5.6) that for some constant c_i'

$$u = c_i' \qquad \text{in} \quad C_i \setminus C_{m,i}. \tag{3.5.16}$$

Using (3.5.15)-(3.5.16) and the continuity of u, we get

$$\Phi_m(x_1) = c_i' - c_i = k_i \qquad \forall x_1 \in \pi_{x_1}(C_{m,i}). \tag{3.5.17}$$

Since $\partial C_{m,i} \cap (S_2 \cup S_3) = \emptyset$, we deduce that $\pm\xi = \pm(u_m - k_i)\chi(C_{m,i})$ are suitable test functions for (P_L) and then we have

$$\int_\Omega a(x)(\nabla u_m - g_M e).\nabla \xi dx = 0$$

which can be written

$$\int_{C_{m,i}} a(x)\nabla u_m.\nabla u_m dx = 0.$$

Using (3.5.2), we deduce that $\nabla u_m = 0$ a.e. in $C_{m,i}$ and

$$u_m = k_i \qquad \text{in} \quad C_{m,i}. \tag{3.5.18}$$

Now let $x_0 = (x_{01}, \Phi_m(x_{01})) = (x_{01}, k_i) \in \partial C_{m,i} \cap \Omega$ and let $r > 0$ small enough. Let $\zeta \in \mathcal{D}(B_r(x_0))$. Since $\pm\zeta$ are test functions for (P_L), we deduce by taking into account (3.5.17)-(3.5.18)

$$\int_{B_r(x_0)} a(x)(\nabla u_m - g_M e).\nabla \zeta dx = 0$$

or

$$\int_{B_r(x_0)\cap[x_2=k_i]} \frac{\beta(x,\varphi)}{\nu_2}\zeta dx_1 = 0 \qquad \forall \zeta \in \mathcal{D}(B_r(x_0))$$

which leads to $\beta(.,\varphi) = 0$ a.e. in $\pi_{x_1}(B_r(x_0))$. But this contradicts (3.3.1).

Finally, we have proved that $u = u_m$ in $C_{m,i}$ for all $i \in \{1, ..., N\}$. As in 2.5, one can prove that $u = u_m$ in Ω and by Corollary 3.2.1, we get $g = g_M$ in Ω. □

Remark 3.5.1. *Theorem (3.5.1) remains valid if we assume that*

$$\beta(x,\varphi) \geq a_{22}(x)\nu_2 \quad \textit{for a.e. } x \textit{ below } S_3, \tag{3.5.19}$$

and if we replace (3.5.3) and (3.5.5) respectively by

$$a_{12}(x) = 0 \quad \textit{for a.e. } x \in \Omega, \tag{3.5.20}$$

and

$$\text{for each } i \in \{1,...,N\}, \text{ there exists a nonempty domain } S'_{3,i} \subset S_{3,i} :$$
$$\text{for a.e. } x \in S'_{3,i},\ u \to \beta(x,u) \text{ is an increasing function.} \tag{3.5.21}$$

Indeed by (3.5.19) and Corollary 3.1.1, we know that for each $i \in \{1,...,N\}$*, the strip* Z_i *below* $S_{3,i}$ *is completely saturated. Now let* (u,g) *be a reservoirs-connected solution of* (P_L) *and let* C_i *(resp.* $C_{m,i}$*) be the connected component of* $[u > x_2]$ *(resp.* $[u_m > x_2]$*) that contains* Z_i*. We have* $g = g_M = 0$ *a.e. in* Z_i*. Moreover by Theorem 3.3.1 iii) and (3.5.21), we have also* $u = u_m$ *a.e. in* $S'_{3,i}$*. Then one can adapt the proof of Theorem 2.5.1 to show that* $u = u_m$ *in* $C_i = C_{m,i}$*.*

3.5.2 The case of a Nonlinear Darcy's law

In this section, we prove the uniqueness of the reservoirs-connected solution for a Darcy's law corresponding to:

$$\mathcal{A}(x,\xi) = |a(x)\xi.\xi|^{\frac{q-2}{2}} a(x)\xi, q > 1, q \neq 2 \text{ and } a(x) = (a_{ij}) \text{ is a } 2\times 2 \text{ matrix,} \tag{3.5.22}$$

subject to the assumptions:

$$a(x) \text{ is symmetric and belongs to } C^{0,1}(\Omega). \tag{3.5.23}$$
$$\exists \lambda, M > 0 :\ \lambda|\xi|^2 \leq a(x)\xi.\xi \leq M|\xi|^2\ \forall \xi \in I\!R^2,\ \text{a.e. } x \in \Omega. \tag{3.5.24}$$
$$a_{12}(x) = 0 \quad \text{for a.e. } x \in \Omega. \tag{3.5.25}$$
$$\frac{\partial a_{22}^{q/2}}{\partial x_2} \geq 0 \quad \text{in } \mathcal{D}'(\Omega) \tag{3.5.26}$$
$$\forall i \in \{1,...,N\},\ \varphi = h_i - x_2 \text{ on } S_{3,i}. \tag{3.5.27}$$
$$\forall i \in \{1,...,N\},\ \exists r_i > 0, \exists x_i \in S_{3,i}, \exists \alpha_i \in (0,1) :$$
$$S_{3,i}^{r_i} = S_{3,i} \cap B_r(x_i) \text{ is } C^{1,\alpha_i}. \tag{3.5.28}$$
$$\forall i \in \{1,...,N\},\ \exists \kappa_i \in (0,1) : \quad \beta \in C^{0,\kappa_i}(S_{3,i}^{r_i} \times I\!R). \tag{3.5.29}$$
$$\beta(x,\varphi) \geq a_{22}^{q/2}(x)\nu_2 \quad \text{for a.e. } x \text{ below } S_3. \tag{3.5.30}$$
$$\forall x \in S_{3,i}^{r_i}, u \longmapsto \beta(x,u) \quad \text{is an increasing function.} \tag{3.5.31}$$

Then we have:

Theorem 3.5.2. *Under the assumptions (3.5.22)-(3.5.31), there is one and only one reservoirs-connected solution of* (P_L)*.*

To prove Theorem 3.5.2, we need three lemmas. We shall denote by (u, g) a reservoirs-connected solution of (P_L). By (3.5.30) and Corollary 3.1.1, we know that for all $i \in \{1, ..., N\}$, the strip Z_i below $S_{3,i}$ is completely saturated. For each $i \in \{1, ..., N\}$, we denote by C_i (resp. $C_{m,i}$) the connected component of $[u > x_2]$ (resp. $[u_m > x_2]$) which contains Z_i.

Lemma 3.5.2. *For each $i \in \{1, \ldots, N\}$, we have the following alternatives:*

i) either $\exists x_i' \in B(x_i, r_i) \cap S_{3,i}, \quad \exists r_i' \in (0, r_i) : \quad \forall x \in \overline{B(x_i', r_i') \cap \Omega} \qquad \nabla u(x) \neq 0$

ii) or $u = h_i$ *in* C_i.

Proof. First note that since $g = 0$ a.e. in Z_i, we have by (1.3.3) $div(\mathcal{A}(x, \nabla u)) = 0$ in $B(x_i, r_i) \cap \Omega$. Taking into account the assumptions (3.5.22)-(3.5.24) and (3.5.27)-(3.5.29), we have (see [34]) $u \in C^{1,\alpha_i}(\overline{B(x_i, r) \cap \Omega})$ for all $r \in (0, r_i)$. So either $i)$ is true or we must have $\nabla u(x) = 0 \ \forall x \in B(x_i, r_i) \cap S_{3,i}$ which leads to

$$\beta(x, \psi - u) = |a(x)\nabla u.\nabla u|^{\frac{q-2}{2}} a(x)\nabla u.\nu = 0 \qquad \text{in} \quad S_{3,i}^{r_i}.$$

Using (3.5.31), we obtain $u = \psi = h_i$ in $S_{3,i}^{r_i}$.
Now set

$$\begin{cases} w_0(x) = u(x) - h_i & \forall x \in B(x_i, r_i) \cap \Omega \\ w_0(x) = 0 & \forall x \in B(x_i, r_i) \setminus \Omega. \end{cases}$$

Since $w_0 = 0$ and $\nabla w_0 = 0$ on $S_{3,i}^{r_i}$, it is clear that $w_0 \in C^1(B(x_i, r_i))$ and that $div(\mathcal{A}(x, \nabla w_0)) = 0$ in $\mathcal{D}'(B(x_i, r_i))$. The rest of the proof follows the proof of Lemma 2.5.1. □

Lemma 3.5.3. *If u and u_m are not both constant in C_i and $C_{m,i}$ respectively, then there exists $x_i' \in B(x_i, r_i) \cap S_{3,i}$, $r_i' \in (0, r_i)$, $0 < \lambda_0$, $\lambda_1 < +\infty$ such that*

$$\forall x \in \overline{B(x_i', r_i') \cap \Omega}, \qquad \lambda_0 \leq \lambda(x) \leq \lambda_1,$$

where $\lambda(x) = \displaystyle\int_0^1 |\nabla w_t(x)|^{q-2} dt$ *and* $w_t = tu + (1-t)u_m$.

Proof. One can adapt the proof of Lemma 2.5.2.

Lemma 3.5.4. *For each $i \in \{1, \ldots, N\}$, there exists $x_i' \in B(x_i, r_i) \cap S_{3,i}$, $r_i' \in (0, r_i)$ such that*

$$u = u_m \qquad \text{in } B(x_i', r_i') \cap \Omega.$$

Proof. If u is constant in C_i and u_m is constant in $C_{m,i}$, we have by Lemma 3.5.2 $u = h_i$ in C_i and $u_m = h_i$ in $C_{m,i}$ and Lemma 3.5.4 follows in this case.
In the following we assume that either u is not constant in C_i or u_m is not constant in $C_{m,i}$.

By Lemma 3.5.3, we know that there exists $x_i' \in S_{3,i}^{r_i}$ and $r_i' \in (0, r_i)$, λ_0, $\lambda_1 > 0$ such that

$$\forall x \in \overline{B(x_i', r_i') \cap \Omega} \qquad \lambda_0 \leq \lambda(x) \leq \lambda_1 < +\infty.$$

Since $B(x_i', r_i') \cap \Omega \subset C_i \cap C_{m,i}$ and $g = g_M = 0$ a.e. in $C_i \cap C_{m,i}$, we deduce from Theorem 3.3.1 i) that

$$\int_{B(x_i', r_i') \cap \Omega} (\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla u_m)).\nabla \zeta dx = 0 \qquad \forall \zeta \in \mathcal{D}(B(x_i', r_i')).$$

The rest of the proof follows the proof of Lemma 2.5.3. $\square$

Proof of Theorem 3.5.2. Using Lemma 3.5.4 and arguing as in the proof of Theorem 2.5.2, we prove that for each $i \in \{1, ..., N\}$, $u = u_m$ in $C_{m,i}$ and that $C_{m,i} = C_i$. Hence $u = u_m$ in Ω and by Corollary 3.2.1, we deduce that $g = g_M$ in Ω. $\square$

Remark 3.5.2. *The case $q = 2$ was discussed in Remark 3.5.1. In particular we neither need the $C^{0,1}$ regularity of the matrix $a(x)$ nor the $C^{0,\alpha}$ regularity of any part of $S_{3,i}$.*
As a special case of Theorem 3.5.2, we obtain the uniqueness of the solution for a rectangular dam supplied by two reservoirs.
The uniqueness of the reservoirs-connected solution in general is still an open problem for $q \neq 2$ even for $\mathcal{A}(x, \xi) = |\xi|^{q-2}\xi$.

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