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2. Let $\mathbf{F}(x,y,z) = 6xy\mathbf{i} + 4yz\mathbf{j} + xe^{-y}\mathbf{k}$ and let D be the region that is bounded by the three coordinate planes and the plane $x+y+z=1$. Let S be the surface representing the exterior boundary of D which we orient outward (draw a figure). We shall denote by R the projection of S on the $xy-$ plane.

We would like to verify the divergence formula i.e.

$$\int\int_S \mathbf{F}.\mathbf{n}\; dS = \int\int\int_D div(\mathbf{F})dV. \tag{1}$$

Note that the components of the vector field $\mathbf{F}$ are continuous and have partial derivatives continuous everywhere. First we have for the right-hand side of (1)

$$\begin{aligned}
\int\int\int_D div(\mathbf{F})dV &= \int_0^1 \Big(\int_0^{1-y}\Big(\int_0^{1-x-y}(6y+4z)dz\Big)dx\Big)dy \\
&= \int_0^1 \Big(\int_0^{1-y}\big[6yz+2z^2\big]_0^{1-x-y}dx\Big)dy \\
&= \int_0^1 \Big(\int_0^{1-y}(6y(1-x-y)+2(1-x-y)^2)dx\Big)dy \\
&= \int_0^1 \Big(\int_0^{1-y}(2-4x+2y-2xy-4y^2+2x^2)dx\Big)dy
\end{aligned}$$

$$\begin{aligned}
\int\int\int_D div(\mathbf{F})dV &= \int_0^1 \Big(\int_0^{1-y} (2-4x+2y-2xy-4y^2+2x^2)dx\Big)dy \\
&= \int_0^1 \big[2x-2x^2+2xy-x^2y-4xy^2+\frac{2}{3}x^3\big]_0^{1-y}dy \\
&= \int_0^1 (2(1-y)-2(1-y)^2+2y(1-y)-y(1-y)^2-4y^2(1-y)+\frac{2}{3}(1-y)^3)dy \\
&= \int_0^1 (2t-2t^2+2t(1-t)-(1-t)t^2-4t(1-t)^2+\frac{2}{3}t^3)dt \qquad t=1-y \\
&= \int_0^1 (3t^2-\frac{7}{3}t^3)dt = \big[t^3-\frac{7}{12}t^4\big]_0^1 = 1-\frac{7}{12} = \frac{5}{12}. \qquad (2)
\end{aligned}$$

Next we have for the left-hand side of (1)

$$\begin{aligned}
\int\int_S \mathbf{F}.\mathbf{n}\ dS &= \int\int_{S_1} \mathbf{F}.\mathbf{n}\ dS + \int\int_{S_2} \mathbf{F}.\mathbf{n}\ dS \\
&\quad + \int\int_{S_3} \mathbf{F}.\mathbf{n}\ dS + \int\int_{S_4} \mathbf{F}.\mathbf{n}\ dS \qquad (3)
\end{aligned}$$

S_1 is defined by $z=0$ and the unit normal vector to S_1 is given by $\mathbf{n}=-\mathbf{k}$. So we have

$$\begin{aligned}
\int\int_{S_1} \mathbf{F}.\mathbf{n}\ dS &= \int\int_{S_1} -xe^{-y}\ dS = \int\int_R -xe^{-y}dxdy \\
&= \int_0^1 x\Big(\int_0^{1-x} -e^{-y}dy\Big)dx \\
&= \int_0^1 x\big[e^{-y}\big]_0^{1-x}dx = \int_0^1 (xe^{x-1}-x)dx \\
&= \big[xe^{x-1}-e^{x-1}-x^2/2\big]_0^1 \\
&= 1-1-1/2+e^{-1} = e^{-1}-1/2. \qquad (4)
\end{aligned}$$

S_2 is defined by $x=0$ and the unit normal vector to S_2 is given by $\mathbf{n}=-\mathbf{i}$. So we have

$$\int\int_{S_2} \mathbf{F}.\mathbf{n}\ dS = \int\int_{S_2} -6xy\ dS = 0. \qquad (5)$$

S_3 is defined by $y = 0$ and the unit normal vector to S_3 is given by $\mathbf{n} = -\mathbf{j}$. So we have

$$\int\int_{S_3} \mathbf{F}.\mathbf{n}\ dS = \int\int_{S_3} -4yz\ dS = 0. \tag{6}$$

S_4 is defined by $g(x, y, z) = 0$, where $g(x, y, z) = x + y + z - 1$. So the unit normal vector to S_4 is given by

$$\mathbf{n} = \frac{1}{||\nabla g||}\nabla g = \frac{1}{\sqrt{1+1+1}}(\mathbf{i}+\mathbf{j}+\mathbf{k}) = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k}).$$

Then we have

$$\begin{aligned}
\int\int_{S_4} \mathbf{F}.\mathbf{n}\ dS &= \int\int_{S_4} \frac{1}{\sqrt{3}}(6xy + 4yz + xe^{-y})\ dS \\
&= \int\int_{S_4} \frac{1}{\sqrt{3}}(6xy + 4y(1-x-y) + xe^{-y})\ dS \\
&= \int\int_{S_4} \frac{1}{\sqrt{3}}(2xy + 4y - 4y^2 + xe^{-y})\ dS \\
&= \int\int_{R} \frac{1}{\sqrt{3}}(2xy + 4y - 4y^2 + xe^{-y})\sqrt{3}dxdy \\
&= \int\int_{R} (2xy + 4y - 4y^2 + xe^{-y})dxdy \\
&= \int_0^1 \Big(\int_0^{1-x} (2xy + 4y - 4y^2 + xe^{-y})dy\Big)dx \\
&= \int_0^1 \big[xy^2 + 2y^2 - \frac{4}{3}y^3 - xe^{-y}\big]_0^{1-x}dx \\
&= \int_0^1 (x(1-x)^2 + 2(1-x)^2 - \frac{4}{3}(1-x)^3 - xe^{x-1} + x)dx \\
&= \int_0^1 ((1-t)t^2 + 2t^2 - \frac{4}{3}t^3 - (1-t)e^{-t} + 1 - t)dt \quad t = 1-x \\
&= \int_0^1 (1 - t + 3t^2 - \frac{7}{3}t^3 - e^{-t} + te^{-t})dt \\
&= \big[t - \frac{1}{2}t^2 + t^3 - \frac{7}{12}t^4 + e^{-t} - te^{-t} - e^{-t}\big]_0^1 \\
&= 1 - 1/2 + 1 - \frac{7}{12} - e^{-1} = \frac{11}{12} - e^{-1}.
\end{aligned} \tag{7}$$

Taking into account (3)-(7), we get

$$\int\int_S \mathbf{F}.\mathbf{n}\ dS = e^{-1} - 1/2 + 0 + 0 + \frac{11}{12} - e^{-1} = \frac{5}{12}. \tag{8}$$

Comparing (2) and (8), we conclude that (1) is satisfied.

□

4. Let $\mathbf{F} = 4x\mathbf{i} + y\mathbf{j} + 4z\mathbf{k}$ and let S be the sphere $x^2 + y^2 + z^2 = 4$ oriented outwardly. Let D be the domain bounded by S. The components of the vector field $\mathbf{F}$ are continuous and have partial derivatives continuous on any domain. Therefore we have by the divergence theorem

$$\begin{aligned}\int\int_S \mathbf{F}.\mathbf{n}\ dS &= \int\int\int_D div(\mathbf{F})dV = \int\int\int_D (4+1+4)dV \\ &= \int\int\int_D 9dV = 9Vol(D) = 9\frac{4\pi}{3}2^3 = 96\pi.\end{aligned}$$

□

11. Let $\mathbf{F} = 2xz\mathbf{i} + 5y^2\mathbf{j} - z^2\mathbf{k}$ and let D be the domain bounded by $z = y$, $z = 4 - y$, $z = 2 - \frac{1}{2}x^2$, $x = 0$ and $z = 0$. We denote by S the exterior boundary of D. We would like to evaluate the flux $\int\int_S \mathbf{F}.\mathbf{n}\ dS$. Since the components of the vector field $\mathbf{F}$ are continuous and have partial derivatives continuous everywhere, we have by the divergence theorem

$$\int\int_S \mathbf{F}.\mathbf{n}dS = \int\int\int_D div\mathbf{F}dxdydz, \tag{1}$$

Let D_1 be the part of D located to the left of the plane $y = 2$ and let D_2 be the part of D located to its right. Then we have

$$\begin{aligned}\int\int\int_D div\mathbf{F}dxdydz &= \int\int\int_D 10ydxdydz \\ &= \int\int\int_{D_1} 10ydxdydz + \int\int\int_{D_2} 10ydxdydz.(2)\end{aligned}$$

First we remark the symmetry of D with respect to the plane $y = 2$. Using the change of variables $y' = 4 - y$, we get

$$\begin{aligned}\int\int\int_{D_2} 10y dxdydz &= \int\int\int_{D_1} 10(4-y')dxdy'dz \\ &= 40\int\int\int_{D_1} dxdy'dz - \int\int\int_{D_1} 10y'dxdy'dz \\ &= 40Vol(D_1) - \int\int\int_{D_1} 10ydxdydz. \qquad (3)\end{aligned}$$

We deduce from (2) and (3) that have

$$\int\int\int_D div\mathbf{F}dxdydz = 40Vol(D_1). \qquad (4)$$

Now we have

$$Vol(D_1) = \int\int\int_{D_1} dxdydz = \int_0^2\int_0^2\left(\int_0^{\min\left(y,2-\frac{1}{2}x^2\right)} dz\right)dxdy$$

$$\begin{aligned}&= \int_0^2\int_0^{2-\frac{1}{2}x^2}\left(\int_0^y dz\right)dxdy + \int_0^2\int_{2-\frac{1}{2}x^2}^2\left(\int_0^{2-\frac{1}{2}x^2} dz\right)dxdy \\ &= \int_0^2\left(\int_0^{2-\frac{1}{2}x^2} ydy\right)dx + \int_0^2\int_{2-\frac{1}{2}x^2}^2 (2-\frac{1}{2}x^2)dy\Bigg)dx \\ &= \int_0^2\left[\frac{1}{2}y^2\right]_0^{2-\frac{1}{2}x^2}dx + \int_0^2(2-\frac{1}{2}x^2)\left[y\right]_{2-\frac{1}{2}x^2}^2 dx \\ &= \int_0^2\frac{1}{2}(2-\frac{1}{2}x^2)^2dx + \int_0^2(2-\frac{1}{2}x^2)(\frac{1}{2}x^2)dx \\ &= \int_0^2\frac{1}{2}(2-\frac{1}{2}x^2)(2-\frac{1}{2}x^2+x^2)dx \\ &= \int_0^2\frac{1}{2}(2-\frac{1}{2}x^2)(2+\frac{1}{2}x^2)dx = \int_0^2(2-\frac{1}{8}x^4)dx = \left[2x-\frac{1}{40}x^5\right]_0^2 \\ &= 4-\frac{2^5}{40} = \frac{160-32}{40} = \frac{128}{40}. \qquad (5)\end{aligned}$$

Finally we deduce from (1), (4) and (5) that

$$\int\int_S curl(\mathbf{F}).\mathbf{n}dS = 40.\frac{128}{40} = 128.$$

□