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3. Let S be that portion of the cylinder $x^2 + z^2 = 16$ that is above the $xy-$plane and within the planes $x = 0$, $x = 2$, $y = 0$ and $y = 5$. We would like to evaluate the surface area of S.
Since S is defined by the equation $z = f(x, y) = \sqrt{16 - x^2}$, the area of S is given by

$$Area(S) = \int\int_R \sqrt{1 + f_x^2 + f_y^2}dxdy, \tag{1}$$

where R is the projection of S on the $xy-$plane. Moreover we have

$$f_x = \frac{-x}{\sqrt{16 - x^2}}, \quad f_y = 0$$
$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + \frac{x^2}{16 - x^2}} = \frac{4}{\sqrt{16 - x^2}}.$$

We deduce then from (1)

$$\begin{aligned} Area(S) &= \int\int_R \frac{4}{\sqrt{16 - x^2}}dxdy = 4\int_0^5 \Big(\int_0^2 \frac{dx}{\sqrt{16 - x^2}}\Big)dy \\ &= 4\int_0^5 \big[\sin^{-1}(\frac{x}{4})\big]_0^2 dy = 4\int_0^5 \frac{\pi}{6}dy = \frac{10\pi}{3}. \end{aligned}$$

□

10. Let S be the surface defined by the portions of the cone $z^2 = \frac{1}{4}(x^2 + y^2)$ that are within the cylinder $(x - 1)^2 + y^2 = 1$ (draw a figure). We have $S = S^+ \cup S^-$, where S^+ (resp. S^-) is the part of S located above (resp. below) the $xy-$plane. By symmetry we have $Area(S^+) = Area(S^-)$ and therefore $Area(S) = 2Area(S^+)$. Moreover by shifting S 1 unit in the direction of

the $x-$axis, one can assume that S is defined by the portions of the cone $z^2 = \frac{1}{4}((x+1)^2 + y^2)$ that are within the cylinder $x^2 + y^2 = 1$.
Since S^+ is defined by the equation $z = f(x,y) = \frac{1}{2}\sqrt{(x+1)^2 + y^2}$, the area of S^+ is given by

$$Area(S^+) = \int\int_R \sqrt{1 + f_x^2 + f_y^2} dxdy, \tag{1}$$

where R is the projection of S^+ on the $xy-$plane. Moreover we have

$$f_x = \frac{x+1}{2\sqrt{(x+1)^2 + y^2}}, \quad f_y = \frac{y}{2\sqrt{(x+1)^2 + y^2}}$$

$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + \frac{(x+1)^2}{4((x+1)^2 + y^2)} + \frac{y^2}{4((x+1)^2 + y^2)}} = \frac{\sqrt{5}}{2}.$$

We deduce then from (1) that

$$Area(S^+) = \int\int_R \frac{\sqrt{5}}{2} dxdy = \frac{\sqrt{5}\pi}{2}.$$

Hence $Area(S) = 2Area(S^+) = \sqrt{5}\pi$.

□

26. We would like to evaluate the surface integral $\int\int_S (3z^2 + 4yz)dS$, where S is that portion of the plane $x + 2y + 3z = 6$ located in the first octant.
Considering that S is defined by the equation $x = h(y,z) = 6 - 2y - 3z$ and denoting by R the projection of S on the $yz-$plane, we get

$$
\begin{aligned}
\int\int_S (3z^2+4yz)dS &= \int\int_R \sqrt{1+h_y^2+h_z^2}(3z^2+4yz)dydz \\
&= \int\int_R \sqrt{1+4+9}(3z^2+4yz)dydz \\
&= \sqrt{14}\int\int_R (3z^2+4yz)dydz \\
&= \sqrt{14}\int_0^3\int_0^{2-\frac{2}{3}y} (3z^2+4yz)dydz \\
&= \sqrt{14}\int_0^3 \Big[z^3+2yz^2\Big]_0^{2-\frac{2}{3}y} dy \\
&= \sqrt{14}\int_0^3 \Big(\Big(2-\frac{2}{3}y\Big)^3+2y\Big(2-\frac{2}{3}y\Big)^2\Big)dy \\
&= \sqrt{14}\int_0^2 \big(y'^3+(3(2-y'))y'^2\big)dy' \quad \text{where } y'=2-\frac{2}{3}y \\
&= \sqrt{14}\int_0^2 \big(6y'^2-2y'^3\big)dy' \\
&= \sqrt{14}\big[2y'^3-\frac{1}{2}y'^4\big]_0^2 = 16-8=8.
\end{aligned}
$$

$\square$

33. Let $\mathbf{F}(x,y,z)=\frac{1}{2}x^2\mathbf{i}+\frac{1}{2}y^2\mathbf{j}+z\mathbf{k}$ and S be the portion of the paraboloid $z=4-x^2-y^2$ for $0\le z\le 4$. We would like to evaluate the flux $\int\int_S \mathbf{F}.\mathbf{n}\,dS$ of $\mathbf{F}$ through the surface S assuming that S is oriented upward.
S is defined by $z=f(x,y)=4-x^2-y^2$. So the unit normal vector to S is given by

$$\mathbf{n}=\frac{1}{\sqrt{1+f_x^2+f_y^2}}(-f_x\mathbf{i}-f_y\mathbf{j}+\mathbf{k})=\frac{1}{\sqrt{1+f_x^2+f_y^2}}(2x\mathbf{i}+2y\mathbf{j}+\mathbf{k}).$$

Let R be the projection of S on the $xy-$plane. Then we have

$$Flux=\int\int_S \mathbf{F}.\mathbf{n}\,dS=\int\int_S \frac{1}{\sqrt{1+f_x^2+f_y^2}}(x^3+y^3+1)\,dS=\int\int_R (x^3+y^3+1)dxdy. \quad (1)$$

Since R is bounded by the circle centered at the origin and with radius 2, we can use the polar coordinates. We obtain

$$\begin{aligned}
\int\int_R (x^3+y^3+1)dxdy &= \int_0^{2\pi}\Big(\int_0^2 (r^3\cos^3\theta + r^3\sin^3\theta+1)rdr\Big)d\theta \\
&= \int_0^{2\pi}\Big(\int_0^2 (r^4\cos^3\theta + r^4\sin^3\theta + r)dr\Big)d\theta \\
&= \int_0^{2\pi}\Big[\frac{1}{5}r^5\cos^3\theta + \frac{1}{5}r^5\sin^3\theta + \frac{1}{2}r^2\Big]_0^2 d\theta \\
&= \int_0^{2\pi}\Big(\frac{32}{5}\cos^3\theta + \frac{32}{5}\sin^3\theta + \frac{1}{2}\Big)d\theta \\
&= \int_0^{2\pi}\Big(\frac{32}{5}\cos\theta(1-\sin^2\theta) + \frac{32}{5}\sin\theta(1-\cos^2\theta) + \frac{1}{2}\Big)d\theta \\
&= \int_0^{2\pi}\Big(\frac{32}{5}\cos\theta - \frac{32}{5}\cos\theta\sin^2\theta + \frac{32}{5}\sin\theta - \frac{32}{5}\sin\theta\cos^2\theta + \frac{1}{2}\Big)d\theta \\
&= \Big[\frac{32}{5}\sin\theta - \frac{32}{15}\sin^3\theta) - \frac{32}{5}\cos\theta + \frac{32}{15}\cos^3\theta + \frac{1}{2}\theta\Big]_0^{2\pi} = \pi. \qquad (2)
\end{aligned}$$

Hence we get the flux from (1) and (2): $Flux = \int\int_S \mathbf{F}.\mathbf{n}\; dS = \pi$.

$\square$