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MATH 531/Term 062/Hw#2(Chap. 2)/

8. Let (x_n) be a real sequence. We would like to show that l is a cluster point of (x_n) if and only if there is a subsequence (x_{n_k}) of (x_n) that converges to l. We shall distinguish three cases: $l = \infty$, $l \in \mathbb{R}$ and $l = -\infty$.

i) ($\Rightarrow$) Assume that $l = \infty$ is a cluster point of (x_n). By definition we have

$$\forall A > 0 \;\; \forall N \in \mathbb{N} \;\; \exists n \geq N \;\text{ such that }\; x_n > A. \tag{1}$$

We shall construct by induction a subsequence (x_{n_k}) of (x_n) such that

$$\forall k \in \mathbb{N} \;\; x_{n_k} > k. \tag{2}$$

It will be clear then from (2) that $\lim\limits_{k\to\infty} x_{n_k} = \infty$.
Writing (1) for $A = 1$ and $N = 1$, we get

$$\exists n_1 \in \mathbb{N} \;\text{ such that }\; x_{n_1} > 1.$$

Assume now that we have constructed x_{n_1},..., x_{n_k} such that $n_i < n_{i+1}$ for all $i \in \{1, ..., k-1\}$ and $x_{n_i} > i$ for all $i \in \{1, ..., k\}$.
Writing (1) for $A = k+1$ and $N = n_k + 1$, we get

$$\exists n_{k+1} \in \mathbb{N} \;\text{ such that }\; n_{k+1} \geq n_k + 1 \text{ and } x_{n_{k+1}} > k+1.$$

Then we have $n_k < n_{k+1}$ and $x_{n_{k+1}} > k+1$. Hence we have obtained a subsequence (x_{n_k}) of (x_n) that satisfies (2).

($\Leftarrow$) Assume that there exists a subsequence (x_{n_k}) of (x_n) such that

$$\lim_{k\to\infty} x_{n_k} = \infty.$$

Now let $A > 0$ and $N \in \mathbb{N}$. By definition of the limit we have

$$\exists K \in \mathbb{N} \;\text{ such that }\; \forall k \geq K \;\; x_{n_k} > A.$$

Let $k = \max(K, N)$. Then we have $n_k \geq n_N \geq N$ and $x_{n_k} > A$.
Hence $l = \infty$ is a cluster point of (x_n).

ii) $(\Rightarrow)$ Assume that $l = -\infty$ is a cluster point of (x_n). By definition we have

$$\forall A < 0 \;\; \forall N \in \mathbb{N} \;\; \exists n \geq N \;\text{ such that }\; x_n < A. \tag{3}$$

We shall construct by induction a subsequence (x_{n_k}) of (x_n) such that

$$\forall k \in \mathbb{N} \;\; x_{n_k} < -k. \tag{4}$$

It will be clear then from (4) that $\lim\limits_{k\to\infty} x_{n_k} = -\infty$.
Writing (3) for $A = -1$ and $N = 1$, we get

$$\exists n_1 \in \mathbb{N} \;\text{ such that }\; x_{n_1} < -1.$$

Assume now that we have constructed x_{n_1},..., x_{n_k} such that $n_i < n_{i+1}$ for all $i \in \{1, ..., k-1\}$ and $x_{n_i} < -i$ for all $i \in \{1, ..., k\}$.
Writing (3) for $A = -(k+1)$ and $N = n_k + 1$, we get

$$\exists n_{k+1} \in \mathbb{N} \;\text{ such that }\; n_{k+1} \geq n_k + 1 \text{ and } x_{n_{k+1}} < -(k+1).$$

Then we have $n_k < n_{k+1}$ and $x_{n_{k+1}} < -(k+1)$. Hence we have obtained a subsequence (x_{n_k}) of (x_n) that satisfies (4).

$(\Leftarrow)$ Assume that there exists a subsequence (x_{n_k}) of (x_n) such that

$$\lim_{k\to\infty} x_{n_k} = -\infty.$$

Now let $A < 0$ and $N \in \mathbb{N}$. By definition of the limit we have

$$\exists K \in \mathbb{N} \;\text{ such that }\; \forall k \geq K \;\; x_{n_k} < A.$$

Let $k = \max(K, N)$. Then we have $n_k \geq n_N \geq N$ and $x_{n_k} < A$.
Hence $l = -\infty$ is a cluster point of (x_n).

iii) $(\Rightarrow)$ Assume that $l \in \mathbb{R}$ is a cluster point of (x_n). By definition we have

$$\forall \epsilon > 0 \;\; \forall N \in \mathbb{N} \;\; \exists n \geq N \;\text{ such that }\; |x_n - l| < \epsilon. \tag{5}$$

We shall construct by induction a subsequence (x_{n_k}) of (x_n) such that

$$\forall k \in \mathbb{N} \;\; |x_{n_k} - l| < \frac{1}{k}. \tag{6}$$

It will be clear then from (6) that $\lim_{k\to\infty} x_{n_k} = l$.
Writing (5) for $\epsilon = 1$ and $N = 1$, we get

$$\exists n_1 \in \mathbb{N} \;\text{ such that }\; |x_n - l| < 1.$$

Assume now that we have constructed x_{n_1},..., x_{n_k} such that $n_i < n_{i+1}$ for all $i \in \{1, ..., k-1\}$ and $|x_{n_i} - l| < \frac{1}{i}$ for all $i \in \{1, ..., k\}$.
Writing (5) for $A = \frac{1}{k+1}$ and $N = n_k + 1$, we get

$$\exists n_{k+1} \in \mathbb{N} \;\text{ such that }\; n_{k+1} \geq n_k + 1 \text{ and } |x_{n_{k+1}} - l| < \frac{1}{k+1}.$$

Then we have $n_k < n_{k+1}$ and $|x_{n_{k+1}} - l| < \frac{1}{k+1}$. Hence we have obtained a subsequence (x_{n_k}) of (x_n) that satisfies (6).

($\Leftarrow$) Assume that there exists a subsequence (x_{n_k}) of (x_n) such that

$$\lim_{k\to\infty} x_{n_k} = l.$$

Now let $\epsilon > 0$ and $N \in \mathbb{N}$. By definition of the limit we have

$$\exists K \in \mathbb{N} \;\text{ such that }\; \forall k \geq K \;\; |x_{n_k} - l| < \epsilon.$$

Let $k = \max(K, N)$. Then we have $n_k \geq n_N \geq N$ and $|x_{n_k} - l| < \epsilon$.
Hence l is a cluster point of (x_n). □

9. a) Let (x_n) be a real sequence.

1) We would like to show that $L = \overline{\lim}\, x_n$ is the largest cluster point of (x_n). We recall that $\overline{\lim}\, x_n = \lim_{n\to\infty} y_n$, where (y_n) is the sequence defined by $y_n = \sup_{k\geq n} x_k$.

We shall distinguish three cases: $L = -\infty$, $L \in \mathbb{R}$ and $L = \infty$.

i) Assume that $L = -\infty$. By definition of the limit we have

$$\forall A < 0 \;\; \exists N \in \mathbb{N} \;\text{ such that }\; \forall n \geq N \;\; y_n < A.$$

which leads, since $x_n \leq y_n$ for all $n \geq 1$, to

$$\forall A < 0 \;\; \exists N \in \mathbb{N} \;\text{ such that }\; \forall n \geq N \;\; x_n < A.$$

Hence we have in this case $\lim_{n\to\infty} x_n = -\infty$. In particular $L = -\infty$ is the unique cluster point of (x_n) (see Ex. 8).

ii) Assume that $L = \infty$. By definition of the limit we have

$$\forall A > 0 \;\; \exists N \in \mathbb{N} \;\text{ such that }\; \forall n \geq N \;\; y_n > A.$$

which leads to

$$\forall A > 0 \;\; \exists N \in \mathbb{N} \;\text{ such that }\; \forall k \geq N \;\; \exists n \geq k \;\; x_n > A. \tag{1}$$

We shall construct by induction a subsequence (x_{n_k}) of (x_n) such that

$$\forall k \in \mathbb{N} \;\; x_{n_k} > k. \tag{2}$$

It will be clear then from (2) that $\lim_{k\to\infty} x_{n_k} = \infty$. Hence we will have in this case (see Ex. 8) that $L = \infty$ is a cluster point of (x_n), which is obviously the largest one.

Writing (1) for $A = 1$ and $k = N$, we get

$$\exists n_1 \in \mathbb{N} \;\text{ such that }\; n_1 \geq N \text{ and } x_{n_1} > 1.$$

Assume now that we have constructed x_{n_1},..., x_{n_j} such that $n_i < n_{i+1}$ for all $i \in \{1, ..., j-1\}$ and $x_{n_i} > i$ for all $i \in \{1, ..., j\}$.

Writing (1) for $A = j + 1$ and $k = n_j + 1$, we get

$$\exists n_{j+1} \in \mathbb{N} \;\text{ such that }\; n_{j+1} \geq n_j + 1 \text{ and } x_{n_{j+1}} > j + 1.$$

Then we have $n_j < n_{j+1}$ and $x_{n_{j+1}} > j + 1$. Hence we have obtained a subsequence (x_{n_k}) of (x_n) that satisfies (2).

iii) Assume that $L \in \mathbb{R}$. Let $\epsilon > 0$ and $N \in \mathbb{N}$. By definition of the limit we have since (y_n) is a non-increasing sequence

$$\exists N_1 \in \mathbb{N} \;\text{ such that }\; \forall n \geq N_1 \;\; 0 \leq y_n - L < \epsilon.$$

which leads to

$$\exists N_1 \in \mathbb{N} \;\text{ such that }\; \forall n \geq N_1 \;\; L \leq y_n < L + \epsilon. \tag{3}$$

In particular we deduce from (3) that

$$\exists N_1 \in \mathbb{N} \text{ such that } \forall n \geq N_1 \ \ x_n < L + \epsilon. \tag{4}$$

For $N_2 = \max(N, N_1)$, we have from (3) $L \leq y_{N_2}$. We then deduce that

$$\exists n \in \mathbb{N} \text{ such that } n \geq N_2 \ \ L - \epsilon < x_n. \tag{5}$$

We infer from (4) and (5) that there exists $n \in \mathbb{N}$ such that $n \geq N$ and $L - \epsilon < x_n < L + \epsilon$ i.e. that $|x_n - L| < \epsilon$. Hence L is a cluster point of (x_n).

Let us now show that L is the largest cluster point of (x_n). Let L' be another cluster point of (x_n). From Ex. 8, we know that there exists a subsequence (x_{n_k}) of (x_n) which converges to L'. But since $x_{n_k} \leq y_{n_k}$ for all k, we obtain $L' \leq L$.

2) We would like to show that $l = \underline{\lim}\, x_n$ is the smallest cluster point of (x_n). We recall that $\underline{\lim}\, x_n = \lim_{n\to\infty} z_n$, where (z_n) is the sequence defined by $z_n = \inf_{k \geq n} x_k$.

We shall distinguish three cases: $l = -\infty$, $l \in \mathbb{R}$ and $l = \infty$.

i) Assume that $l = \infty$. By definition of the limit we have

$$\forall A > 0 \ \ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N \ \ z_n > A.$$

which leads, since $x_n \geq z_n$ for all $n \geq 1$, to

$$\forall A > 0 \ \ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N \ \ x_n > A.$$

Hence we have in this case $\lim_{n\to\infty} x_n = \infty$. In particular $l = \infty$ is the unique cluster point of (x_n).

ii) Assume that $l = -\infty$. By definition of the limit we have

$$\forall A < 0 \ \ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N \ \ y_n < A.$$

which leads to

$$\forall A < 0 \ \ \exists N \in \mathbb{N} \text{ such that } \forall k \geq N \ \ \exists n \geq k \ \ x_n < A. \tag{6}$$

We shall construct by induction a subsequence (x_{n_k}) of (x_n) such that

$$\forall k \in \mathbb{N} \ \ x_{n_k} < -k. \tag{7}$$

It will be clear then from (7) that $\lim\limits_{k\to\infty} x_{n_k} = -\infty$. Hence we will have in this case (see Ex. 8) that $l = -\infty$ is a cluster point of (x_n), which is obviously the smallest one.
Writing (1) for $A = -1$ and $k = N$, we get

$$\exists n_1 \in \mathbb{N} \ \text{ such that } \ n_1 \geq N \text{ and } x_{n_1} < -1.$$

Assume now that we have constructed x_{n_1},..., x_{n_j} such that $n_i < n_{i+1}$ for all $i \in \{1, ..., j-1\}$ and $x_{n_i} < -i$ for all $i \in \{1, ..., j\}$.
Writing (1) for $A = -(j+1)$ and $k = n_j + 1$, we get

$$\exists n_{j+1} \in \mathbb{N} \ \text{ such that } \ n_{j+1} \geq n_j + 1 \text{ and } x_{n_{j+1}} < -(j+1).$$

Then we have $n_j < n_{j+1}$ and $x_{n_{j+1}} < -(j+1)$. Hence we have obtained a subsequence (x_{n_k}) of (x_n) that satisfies (7).

iii) Assume that $l \in \mathbb{R}$. Let $\epsilon > 0$ and $N \in \mathbb{N}$. By definition of the limit we have since (z_n) is a non-decreasing sequence

$$\exists N_1 \in \mathbb{N} \ \text{ such that } \ \forall n \geq N_1 \ \ -\epsilon < z_n - l \leq 0.$$

which leads to

$$\exists N_1 \in \mathbb{N} \ \text{ such that } \ \forall n \geq N_1 \ \ l - \epsilon < z_n \leq l. \tag{8}$$

In particular we deduce from (8) that

$$\exists N_1 \in \mathbb{N} \ \text{ such that } \ \forall n \geq N_1 \ \ l - \epsilon < x_n. \tag{9}$$

For $N_2 = \max(N, N_1)$, we have from (8) $z_{N_2} \leq l$. We then deduce that

$$\exists n \in \mathbb{N} \ \text{ such that } \ n \geq N_2 \ \ x_n < l + \epsilon. \tag{10}$$

We infer from (9) and (10) that there exists $n \in \mathbb{N}$ such that $n \geq N$ and $l - \epsilon < x_n < l + \epsilon$ i.e. that $|x_n - l| < \epsilon$. Hence l is a cluster point of (x_n).

Let us now show that l is the smallest cluster point of (x_n). Let l' be another cluster point of (x_n). From Ex. 8, we know that there exists a subsequence (x_{n_k}) of (x_n) which converges to l'. But since $z_{n_k} \leq x_{n_k}$ for all k, we obtain $l \leq l'$.

b) Let (x_n) be a real bounded sequence. We would like to show that (x_n) has a subsequence (x_{n_k}) that converges to a real number. From the assumption there exists two real numbers m and M such that

$$\forall n \geq 1 \;\; m \leq x_n \leq M.$$

It follows that $l = \underline{\lim}\, x_n$ satisfies $m \leq l \leq M$ and therefore is a real number. Moreover we know from a) that there exists a subsequence (x_{n_k}) of (x_n) which converges to l.

□

11. c) Let (x_n) be a real Cauchy sequence such that there exists a subsequence (x_{n_k}) of (x_n) satisfying for some real number l

$$\lim_{k\to\infty} x_{n_k} = l. \tag{1}$$

We would like to show that (x_n) converges to l. Let $\epsilon > 0$. Since (x_n) is a Cauchy sequence, we have

$$\exists N \in \mathbb{N} \text{ such that } \forall n, m \geq N \;\; |x_n - x_m| < \epsilon/2. \tag{2}$$

From (1) we have

$$\exists K \in \mathbb{N} \text{ such that } \forall k \geq K \;\; |x_{n_k} - l| < \epsilon/2. \tag{3}$$

Let $M = \max(K, N)$. Note that $M \geq K$ and $n_M \geq M \geq N$. Using (2) and (3), we obtain for all $n \geq M$

$$|x_n - l| \leq |x_n - x_{n_M}| + |x_{n_M} - l| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence (x_n) converges to l.

□

18. Let (x_n) be a sequence of nonnegative real numbers. We would like to show that there exists always an extended real number S such that $S = \displaystyle\sum_{k=1}^{\infty} x_k$.

Let $S_n = \sum_{k=1}^{k=n} x_k$. It is enough to prove that the sequence (S_n) has a limit. Since we have $S_{n+1} - S_n = x_{n+1} \geq 0$, the sequence (S_n) is nondecreasing. Let $S = \sup_{n\geq 1} S_n$. We shall prove that $\lim_{n\to\infty} S_n = S$. Because $S_n \geq 0$ for all n, we have two cases: $S = \infty$ or $S \in [0, \infty)$.

i) $\underline{S = \infty}$

Let $A > 0$. Since $\sup_{n\geq 1} S_n = \infty$, there exists $N \in \mathbb{N}$ such that $S_N > A$. But (S_n) is nondecreasing. So we obtain $S_n > A$ for all $n \geq N$. Hence $\lim_{n\to\infty} S_n = \infty$.

ii) $\underline{S \in [0, \infty)}$

Let $\epsilon > 0$. Since $\sup_{n\geq 1} S_n = S$, there exists $N \in \mathbb{N}$ such that $S - \epsilon < S_N$. But (S_n) is nondecreasing. So we obtain $S - \epsilon < S_n \leq S$ for all $n \geq N$. Hence $\lim_{n\to\infty} S_n = S$.

$\square$

20. Let (x_n) be a sequence of real numbers. We would like to show that

$$\lim_{n\to\infty} x_n = x \quad \Leftrightarrow \quad x = x_1 + \sum_{k=1}^{\infty} (x_{k+1} - x_k).$$

Indeed

$$\begin{aligned} x = x_1 + \sum_{k=1}^{\infty} (x_{k+1} - x_k) \quad &\Leftrightarrow \quad x = x_1 + \lim_{n\to\infty} \sum_{k=1}^{k=n} (x_{k+1} - x_k) \\ &\Leftrightarrow \quad x = \lim_{n\to\infty} x_{n+1} \\ &\Leftrightarrow \quad x = \lim_{n\to\infty} x_n. \end{aligned}$$

$\square$

25. Let A be a nonempty subset of $\mathbb{R}$ which is both open and closed. We shall prove that $A = \mathbb{R}$. We argue by contradiction and assume that $A \neq \mathbb{R}$. Let then $f = \chi_A$ be the characteristic function of the set A. It is easy to verify that f is continuous from $\mathbb{R}$ to $\mathbb{R}$. Indeed let O be an open set of $\mathbb{R}$. Then we have the following cases:

If $0, 1 \in O$, then $f^{-1}(O) = \mathbb{R}$ is open.

If $0 \in O$ and $1 \notin O$, then $f^{-1}(O) = A^c$ is open because A is closed.

If $1 \in O$ and $0 \notin O$, then $f^{-1}(O) = A$ is open.

If $0 \notin O$ and $1 \notin O$, then $f^{-1}(O) = \emptyset$ is open.

Now obviously the function f does not satisfy the intermediate-value theorem.

□

36. Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of nonempty closed sets of real numbers such that $F_{n+1} \subset F_n$ for all $n \geq 1$.
Assume that there exists $n_0 \in \mathbb{N}$ such that F_{n_0} is bounded. We would like to show that $\bigcap_{k=1}^{\infty} F_k \neq \emptyset$.
Let (x_n) be a sequence defined by choosing an element x_n in each F_n. Then (x_n) is contained in $F_{n_0} \cup \{x_1, ..., x_{n_0}\}$ which is bounded since F_{n_0} is bounded and the set $\{x_1, ..., x_{n_0}\}$ is finite. We deduce that (see Ex. 9 b))(x_n) has a subsequence (x_{n_k}) that converges to a real number x.
Since $n_k \geq k$ and the sequence (F_n) is non-increasing, we have $x_{n_k} \in F_k$ for all k. In particular we have $x_{n_k} \in F_l$ for all l and $k \geq l$. By holding l fixed and letting $k \to \infty$, we get $x \in F_l$ for all l because F_l is closed. Hence $x \in \bigcap_{l=1}^{\infty} F_l$ and $\bigcap_{k=1}^{\infty} F_k \neq \emptyset$.

The above conclusion may not be true if all the sets F_n are unbounded. Indeed let $F_n = [n, \infty)$. Then F_n is nonempty and closed, $F_{n+1} \subset F_n$ for all $n \geq 1$. However $\bigcap_{k=1}^{\infty} F_k = \emptyset$ because if $x \in \bigcap_{k=1}^{\infty} F_k$, then we will have $x \geq k$ for all $k \geq 1$, which is impossible.

□