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4. Let f be the function defined by $f(x) = x^3 - 4x$.
f is defined everywhere. Moreover we have for all x

$$f(-x) = (-x)^3 - 4(-x) = -x^3 + 4x = -f(x).$$

Hence f is an odd function.

$\square$

6. Let f be the function defined by $f(x) = e^x - e^{-x}$.
f is defined everywhere. Moreover we have for all x

$$f(-x) = e^{-x} - e^{-(-x)} = e^{-x} - e^x = -(e^x - e^{-x}) = -f(x).$$

Hence f is an odd function.

$\square$

14. Let f be the function defined by $f(x) = x$ for $-\pi \leq x \leq \pi$.
Clearly f is an odd function. Therefore the Fourier series of f on the interval $[-\pi, \pi]$ is the sine series given by

$$\sum_{n=1}^{\infty} b_n \sin\big(\frac{n\pi}{\pi} x\big)$$

or

$$\sum_{n=1}^{\infty} b_n \sin(nx), \tag{1}$$

where

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(\frac{n\pi}{\pi}x)dx = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)dx \\
&= \frac{2}{\pi}\Big[-x\frac{1}{n}\cos(nx)\Big]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi} -\frac{1}{n}\cos(nx)dx \\
&= \frac{2}{\pi}\Big[-\pi\frac{1}{n}\cos(n\pi)+0\Big] + \frac{2}{n\pi}\int_0^{\pi}\cos(nx)dx \\
&= 2\frac{(-1)^{n+1}}{n} + \frac{2}{n\pi}\Big[\frac{1}{n}\sin(nx)\Big]_0^{\pi} \\
&= 2\frac{(-1)^{n+1}}{n} + \frac{2}{n^2\pi}(\sin(n\pi)-0) \\
&= 2\frac{(-1)^{n+1}}{n}. \qquad (2)
\end{aligned}$$

Taking into account (1) and (2), it follows that the Fourier series of f on the interval $[-\pi, \pi]$ is given by

$$\sum_{n=1}^{\infty} 2\frac{(-1)^{n+1}}{n}\sin(nx).$$

□

16. Let f be the function defined by $f(x) = x|x|$ for $-1 \le x \le 1$.
Clearly f is an odd function. Therefore the Fourier series of f on the interval $[-1, 1]$ is the sine series given by

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{1}x\right)$$

or

$$\sum_{n=1}^{\infty} b_n \sin(n\pi x), \qquad (1)$$

where

$$b_n = \frac{2}{1}\int_0^1 f(x)\sin(\frac{n\pi}{1}x)dx = 2\int_0^1 x^2\sin(n\pi x)dx. \qquad (2)$$

Integration by parts twice, we get

$$\begin{aligned}
\int x^2 \sin(n\pi x)dx &= -\frac{x^2}{n\pi}\cos(n\pi x) - \int -\frac{2x}{n\pi}\cos(n\pi x)dx \\
&= -\frac{x^2}{n\pi}\cos(n\pi x) + \frac{2}{n\pi}\int x\cos(n\pi x)dx \\
&= -\frac{x^2}{n\pi}\cos(n\pi x) + \frac{2}{n\pi}\Big(\frac{x}{n\pi}\sin(n\pi x) - \int \frac{1}{n\pi}\sin(n\pi x)dx\Big) \\
&= -\frac{x^2}{n\pi}\cos(n\pi x) + \frac{2x}{n^2\pi^2}\sin(n\pi x) - \frac{2}{n^2\pi^2}\int \sin(n\pi x)dx \\
&= -\frac{x^2}{n\pi}\cos(n\pi x) + \frac{2x}{n^2\pi^2}\sin(n\pi x) + \frac{2}{n^3\pi^3}\cos(n\pi x). \qquad (3)
\end{aligned}$$

We deduce from (2) and (3) that

$$\begin{aligned}
b_n = 2\int_0^1 x^2 \sin(n\pi x)dx &= -\frac{2}{n\pi}\cos(n\pi) + \frac{4}{n^3\pi^3}(\cos(n\pi) - 1) \\
&= \frac{2(-1)^{n+1}}{n\pi} + \frac{4}{\pi^3}\frac{(-1)^n - 1}{n^3}. \qquad (4)
\end{aligned}$$

Taking into account (1) and (4), it follows that the Fourier sine series of f on the interval $[-1, 1]$ is given by

$$\sum_{n=1}^{\infty}\Big(\frac{2(-1)^{n+1}}{n\pi} + \frac{4}{\pi^3}\frac{(-1)^n - 1}{n^3}\Big)\sin(n\pi x).$$

□

26. Let f be the function defined by

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

We would like to find the half-range cosine and sine expansions of f on the interval $[0, 1]$.

i. The half-range cosine expansion of f on the interval $[0, 1]$ is the Fourier series of the even extension $\overline{f}$ of f to the interval $[-1, 1]$ defined by

$$\overline{f}(x) = \begin{cases} 1, & \text{if } -1 \le x \le -\frac{1}{2} \\ 0, & \text{if } -\frac{1}{2} < x < \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

So the half-range cosine expansion of f on the interval $[0,1]$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\big(\frac{n\pi}{1}x\big)$$

or

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x), \tag{1}$$

where

$$\begin{aligned} a_0 &= \frac{2}{1}\int_0^1 f(x)dx = 2\int_{\frac{1}{2}}^1 dx \\ &= 2[x]_{\frac{1}{2}}^1 = 1. \end{aligned} \tag{2}$$

For a_n, we have

$$\begin{aligned} a_n &= \frac{2}{1}\int_0^1 f(x)\cos(n\pi x)dx \\ &= 2\int_{\frac{1}{2}}^1 \cos(n\pi x)dx \\ &= 2\Big[\frac{1}{n\pi}\sin(n\pi x)\Big]_{\frac{1}{2}}^1 \\ &= \frac{2}{n\pi}(\sin(n\pi) - \sin\big(\frac{n\pi}{2}\big) \\ &= -\frac{2}{n\pi}\sin\big(\frac{n\pi}{2}\big). \end{aligned} \tag{3}$$

Taking into account (1), (2) and (3), it follows that the half-range cosine expansion of f on the interval $[0,1]$ is given by

$$\frac{1}{2} - \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\big(\frac{n\pi}{2}\big)\cos(n\pi x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{\pi}\frac{(-1)^{k+1}}{(2k+1)}\cos((2k+1)\pi x).$$

ii. The half-range sine expansion of f on the interval $[0,1]$ is the Fourier series of the odd extension $\widehat{f}$ of f to the interval $[-1,1]$ defined by

$$\widehat{f}(x) = \begin{cases} -1, & \text{if} \quad -1 \le x \le -\frac{1}{2} \\ 0, & \text{if} \quad -\frac{1}{2} < x < \frac{1}{2} \\ 1, & \text{if} \quad \frac{1}{2} \le x \le 1. \end{cases}$$

So the half-range sine expansion of f on the interval $[0, 1]$ is given by

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{1}x\right)$$

or

$$\sum_{n=1}^{\infty} b_n \sin(n\pi x), \tag{4}$$

where

$$\begin{aligned} b_n &= \frac{2}{1}\int_0^1 f(x)\sin(n\pi x)dx \\ &= 2\int_{\frac{1}{2}}^{1} \sin(n\pi x)dx \\ &= 2\Big[-\frac{1}{n\pi}\cos(n\pi x)\Big]_{\frac{1}{2}}^{1} \\ &= -\frac{2}{n\pi}(\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right) \\ &= \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{3n\pi}{4}\right), \end{aligned} \tag{5}$$

where we have used the formula $\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$.
Taking into account (4) and (5), it follows that the half-range sine expansion of f on

the interval $[0,1]$ is given by

$$\sum_{n=1}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{3n\pi}{4}\right) \sin(n\pi x)$$

$$= \sum_{k=1}^{\infty} \frac{4}{\pi} \frac{1}{2k+1} \sin\left(\frac{(2k+1)\pi}{2}\right) \sin\left(\frac{3(2k+1)\pi}{4}\right) \sin((2k+1)\pi x)$$

$$= \sum_{k=1}^{\infty} \frac{4}{\pi} \frac{(-1)^k}{2k+1} \sin\left(\frac{3(2k+1)\pi}{4}\right) \sin((2k+1)\pi x)$$

$$= \sum_{l=1}^{\infty} \frac{4}{\pi} \frac{(-1)^{2l}}{4l+1} \sin\left(\frac{3(4l+1)\pi}{4}\right) \sin((4l+1)\pi x)$$

$$+ \sum_{l=1}^{\infty} \frac{4}{\pi} \frac{(-1)^{2l+1}}{4l+3} \sin\left(\frac{3(4l+3)\pi}{4}\right) \sin((4l+3)\pi x)$$

$$= \sum_{l=1}^{\infty} \frac{4}{\pi} \frac{1}{4l+1} \sin\left(3l\pi + \frac{3\pi}{4}\right) \sin((4l+1)\pi x)$$

$$+ \sum_{l=1}^{\infty} \frac{4}{\pi} \frac{-1}{4l+3} \sin\left(3l\pi + \frac{9\pi}{4}\right) \sin((4l+3)\pi x)$$

$$= \sum_{l=1}^{\infty} \frac{4}{\pi} \frac{\sqrt{2}}{2} \frac{(-1)^l}{4l+1} \sin((4l+1)\pi x) - \sum_{l=1}^{\infty} \frac{4}{\pi} \frac{\sqrt{2}}{2} \frac{(-1)^l}{4l+3} \sin((4l+3)\pi x)$$

$$= \sum_{l=1}^{\infty} \frac{2\sqrt{2}}{\pi} \frac{(-1)^l}{4l+1} \sin((4l+1)\pi x) - \sum_{l=1}^{\infty} \frac{2\sqrt{2}}{\pi} \frac{(-1)^l}{4l+3} \sin((4l+3)\pi x).$$

$\square$

38. Let f be the function defined by $f(x) = 2 - x$ for $0 \le x \le 2$.
We would like to expand f in a Fourier series on the interval $[0,2]$.
The Fourier series expansion of f on the interval $[0,2]$ is obtained from the Fourier series of the function f^* defined on the interval $[-2,2]$ by

$$f^*(x) = \begin{cases} f(2+x) = -x, & \text{if} \quad -2 \le x < 0 \\ f(x) = 2 - x, & \text{if} \quad 0 \le x \le 2. \end{cases}$$

The Fourier series of f^* on the interval $[-1,1]$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x), \tag{1}$$

where

$$\begin{aligned} a_0 &= \frac{1}{1}\int_{-1}^{1} f^*(x)dx = \int_{-1}^{0} -xdx + \int_{0}^{1}(2-x)dx \\ &= \Big[-\frac{x^2}{2}\Big]_{-1}^{0} + \Big[2x - \frac{x^2}{2}\Big]_{0}^{1} \\ &= \frac{1}{2} + 2 - \frac{1}{2} = 2. \end{aligned} \tag{2}$$

$$\begin{aligned} a_n &= \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)dx = \int_{-1}^{0} -x\cos(n\pi x)dx + \int_{0}^{1}(2-x)\cos(n\pi x)dx \\ &= \int_{-1}^{1} -x\cos(n\pi x)dx + \int_{0}^{1} 2\cos(n\pi x)dx \\ &= \int_{0}^{1} 2\cos(n\pi x)dx \quad \text{since } x\cos(n\pi x) \text{ is an odd function} \\ &= \Big[2\frac{1}{n\pi}\sin(n\pi x)\Big]_{0}^{1} \\ &= \frac{2}{n\pi}(\sin(n\pi) - 0) \\ &= 0. \end{aligned} \tag{3}$$

$$
\begin{aligned}
b_n &= \frac{1}{1}\int_{-1}^{1} f(x)\sin(n\pi x)dx = \int_{-1}^{0} -x\sin(n\pi x)dx + \int_{0}^{1}(2-x)\sin(n\pi x)dx \\
&= \int_{-1}^{1} -x\sin(n\pi x)dx + \int_{0}^{1} 2\sin(n\pi x)dx \\
&= -2\int_{0}^{1} x\sin(n\pi x)dx + 2\int_{0}^{1}\sin(n\pi x)dx \quad \text{since } x\cos(n\pi x) \text{ is an even function} \\
&= -2\int_{0}^{1} x\sin(n\pi x)dx + 2\Big[-\frac{1}{n\pi}\cos(n\pi x)\Big]_0^1 \\
&= -2\int_{0}^{1} x\sin(n\pi x)dx + \frac{2}{n\pi}(1-\cos(n\pi)) \\
&= -2\int_{0}^{1} x\sin(n\pi x)dx + 2\frac{1-(-1)^n}{n\pi}. \qquad (4)
\end{aligned}
$$

Integrating by parts, we obtain

$$
\begin{aligned}
\int_{0}^{1} x\sin(n\pi x)dx &= \Big[-x\frac{1}{n\pi}\cos(n\pi x)\Big]_0^1 - \int_0^1 -\frac{1}{n\pi}\cos(n\pi x)dx \\
&= -\frac{1}{n\pi}\cos(n\pi) + 0 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)dx \\
&= \frac{(-1)^{n+1}}{n\pi} + \frac{1}{n\pi}\Big[\frac{1}{n\pi}\sin(n\pi x)\Big]_0^1 \\
&= \frac{(-1)^{n+1}}{n\pi} + \frac{1}{n^2\pi^2}(\sin(n\pi) - 0) \\
&= \frac{(-1)^{n+1}}{n\pi}. \qquad (5)
\end{aligned}
$$

Using (3) and (4), we get

$$
b_n = -2\frac{(-1)^{n+1}}{n\pi} + 2\frac{1-(-1)^n}{n\pi} = \frac{2}{n\pi}. \qquad (6)
$$

Taking into account (1), (2), (3) and (6), it follows that the Fourier series of f^* on the interval $[-1, 1]$ is given by

$$
1 + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(n\pi x).
$$

The Fourier series of f on the interval $[0, 2]$ is given by

$$
\begin{aligned}
&1 + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(n\pi x) \quad \text{if} \quad 0 \leq x \leq 1 \\
&1 + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(n\pi(x-2)) = 1 + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(n\pi x) \quad \text{if} \quad 1 < x \leq 2.
\end{aligned}
$$

Hence the Fourier series of f on the interval $[0, 2]$ is given by

$$1 + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(n\pi x).$$

□