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## MATH 301/Term 062/Hw#14(4.5)/

**4.** We consider the initial-value problem

$$\begin{cases} y'' + 16y = \delta_{2\pi}(t), \\ y(0) = 0, \ y'(0) = 0. \end{cases}$$

Let $Y(s) = \mathcal{L}(y(t))$. Applying the Laplace transform to the ode and taking into account the initial conditions, we get

$$s^2 Y(s) - sy(0) - y'(0) + 16Y(s) = \mathcal{L}(\delta_{2\pi})$$
$$\Leftrightarrow \ (s^2 + 16)Y(s) = e^{-2\pi s}$$
$$\Leftrightarrow \ Y(s) = e^{-2\pi s} \frac{1}{s^2 + 16} = \frac{1}{4} e^{-2\pi s} \frac{4}{s^2 + 16} = \frac{1}{4} e^{-2\pi s} F(s), \qquad (1)$$

where $F(s) = \mathcal{L}(\sin(4t))$. Using the formula

$$\mathcal{L}^{-1}(e^{-as} F(s)) = f(t-a)\mathcal{U}_a(t), \ \text{with} \ a = 2\pi, \ f(t) = \sin(4t)$$

we get from $(1)$

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \frac{1}{4} \sin(4(t-2\pi))\mathcal{U}_{2\pi}(t) = \frac{1}{4} \sin(4t)\mathcal{U}_{2\pi}(t).$$

□

**8.** We consider the initial-value problem

$$\begin{cases} y'' - 2y' = 1 + \delta_2(t), \\ y(0) = 0, \ y'(0) = 1. \end{cases}$$

Let $Y(s) = \mathcal{L}(y(t))$. Applying the Laplace transform to the ode and taking into account the initial conditions, we get

$$s^2 Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) = \mathcal{L}(1) + \mathcal{L}(\delta_2)$$

$$\Leftrightarrow (s^2 - 2s)Y(s) = 1 + \frac{1}{s} + e^{-2s}$$

$$\Leftrightarrow Y(s) = \frac{1}{s(s-2)} + \frac{1}{s^2(s-2)} + e^{-2s}\frac{1}{s(s-2)}$$

$$\Leftrightarrow Y(s) = \frac{s+1}{s^2(s-2)} + e^{-2s}\frac{1}{s(s-2)}. \tag{1}$$

Note that

$$\frac{s+1}{s^2(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} = \frac{(A+C)s^2 + (B-2A)s - 2B}{s^2(s-2)}. \tag{2}$$

We deduce from (2) that

$$\begin{cases} A + C = 0, \\ B - 2A = 1, \\ -2B = 1 \end{cases} \Leftrightarrow \begin{cases} A = -3/4, \\ B = -1/2, \\ C = 3/4 \end{cases}$$

We also have

$$\frac{1}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2} = \frac{(A+B)s - 2A}{s(s-2)}. \tag{3}$$

It follows from (3) that

$$\begin{cases} A + B = 0, \\ -2A = 1, \end{cases} \Leftrightarrow \begin{cases} A = -1/2, \\ B = 1/2, \end{cases}$$

We deduce from (1), (2) and (3) that

$$Y(s) = -\frac{3}{4}\frac{1}{s} - \frac{1}{2}\frac{1}{s^2} + \frac{3}{4}\frac{1}{s-2} - \frac{1}{2}e^{-2s}\frac{1}{s} + \frac{1}{2}e^{-2s}\frac{1}{s-2}. \tag{4}$$

Using the formula

$$\mathcal{L}^{-1}(e^{-as}F(s)) = f(t-a)\mathcal{U}_a(t), \text{ with } a = 2, \ f(t) = 1 \text{ and } f(t) = e^{2t},$$

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we get from (4)

$$y(t) = \mathcal{L}^{-1}(Y(s)) = -\frac{3}{4} - \frac{1}{2}t + \frac{3}{4}e^{2t} - \frac{1}{2}\mathcal{U}_2(t) + \frac{1}{2}e^{2(t-2)}\mathcal{U}_2(t).$$

□

**12.** We consider the initial-value problem

$$\begin{cases} y'' - 7y' + 6y = e^t + \delta_2(t) + \delta_4(t), \\ y(0) = 0, \ y'(0) = 0. \end{cases}$$

Let $Y(s) = \mathcal{L}(y(t))$. Applying the Laplace transform to the ode and taking into account the initial conditions, we get

$$s^2 Y(s) - sy(0) - y'(0) - 7(sY(s) - y(0)) + 6Y(s) = \mathcal{L}(e^t) + \mathcal{L}(\delta_2) + \mathcal{L}(\delta_4)$$

$$\Leftrightarrow \ (s^2 - 7s + 6)Y(s) = \frac{1}{s-1} + e^{-2s} + e^{-4s}$$

$$\Leftrightarrow \ Y(s) = \frac{1}{(s-1)(s^2 - 7s + 6)} + e^{-2s}\frac{1}{(s^2 - 7s + 6)} + e^{-4s}\frac{1}{(s^2 - 7s + 6)}$$

$$\Leftrightarrow \ Y(s) = \frac{1}{(s-1)^2(s-6)} + e^{-2s}\frac{1}{(s-1)(s-6)} + e^{-4s}\frac{1}{(s-1)(s-6)}. \quad (1)$$

Note that

$$\frac{1}{(s-1)^2(s-6)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s-6}$$

$$= \frac{(A+C)s^2 + (-7A + B - 2C)s + 6A - 6B + C}{(s-1)^2(s-6)}. \quad (2)$$

We deduce from (2) that

$$\begin{cases} A+C = 0, \\ -7A + B - 2C = 0, \\ 6A - 6B + C = 1 \end{cases} \Leftrightarrow \begin{cases} C = -A, \\ -5A + B = 0, \\ 5A - 6B = 1 \end{cases} \Leftrightarrow \begin{cases} A = -1/25, \\ B = -1/5, \\ C = 1/25 \end{cases}$$

We also have

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$$\frac{1}{(s-1)(s-6)} = \frac{A}{s-1} + \frac{B}{s-6} = \frac{(A+B)s - 6A - B}{(s-1)(s-6)}. \tag{3}$$

It follows from (3) that

$$\begin{cases} A + B = 0, \\ -6A - B = 1, \end{cases} \quad \Leftrightarrow \quad \begin{cases} A = -1/5, \\ B = 1/5, \end{cases}$$

We deduce from (1), (2) and (3) that

$$Y(s) = -\frac{1}{25}\frac{1}{s-1} - \frac{1}{5}\frac{1}{(s-1)^2} + \frac{1}{25}\frac{1}{s-6}$$
$$- \frac{1}{5}e^{-2s}\frac{1}{s-1} + \frac{1}{5}e^{-2s}\frac{1}{s-6} - \frac{1}{5}e^{-4s}\frac{1}{s-1} + \frac{1}{5}e^{-4s}\frac{1}{s-6}. \tag{4}$$

Using the formulas

$$\mathcal{L}^{-1}(F(s-a)) = e^{at}f(t), \text{ with } a = 1, \ f(t) = t \text{ and } F(s) = \frac{1}{s^2}$$
$$\mathcal{L}^{-1}(e^{-as}F(s)) = f(t-a)\mathcal{U}_a(t), \text{ with } a = 2, 4 \text{ and } f(t) = e^t, e^{6t}$$

we get from (4)

$$y(t) = \mathcal{L}^{-1}(Y(s)) = -\frac{1}{25}e^t - \frac{1}{5}te^t + \frac{1}{25}e^{6t} - \frac{1}{5}e^{t-2}\mathcal{U}_2(t)$$
$$+ \frac{1}{5}e^{6(t-2)}\mathcal{U}_2(t) - \frac{1}{5}e^{t-4}\mathcal{U}_4(t) + \frac{1}{5}e^{6(t-4)}\mathcal{U}_4(t).$$

$\square$