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6. We would like to evaluate $\mathcal{L}(t^2\cos(t))$. For this purpose we will use the formula

$$\mathcal{L}(t^n f(t)) = (-1)^n \frac{d^n}{ds^n}F(s), \text{ with } n = 2, f(t) = \cos(t), \text{ and } F(s) = \frac{s}{s^2+1}.$$

Hence we get

$$\begin{aligned}
\mathcal{L}(t^2\cos(t)) &= (-1)^2 \frac{d^2}{ds^2}\frac{s}{s^2+1} = \frac{d}{ds}\Big(\frac{d}{ds}\frac{s}{s^2+1}\Big) \\
&= \frac{d}{ds}\Big(\frac{s^2+1-2s^2}{(s^2+1)^2}\Big) = \frac{d}{ds}\Big(\frac{1-s^2}{(s^2+1)^2}\Big) \\
&= \frac{-2s(s^2+1)^2 - 4s(s^2+1)(1-s^2)}{(s^2+1)^4} \\
&= \frac{-2s(s^2+1) - 4s(1-s^2)}{(s^2+1)^3} \\
&= \frac{2s(s^2-3)}{(s^2+1)^3}.
\end{aligned}$$

□

19. We have

$$\mathcal{L}(1 * t^3) = \mathcal{L}(1).\mathcal{L}(t^3) = \frac{1}{s}.\frac{3!}{s^4} = \frac{6}{s^5}.$$

□

23. We have

$$\mathcal{L}\Big(\int_0^t e^{\tau}d\tau\Big) = \frac{\mathcal{L}(e^t)}{s} = \frac{\frac{1}{s-1}}{s} = \frac{1}{s(s-1)}.$$

□

28. We have

$$\begin{aligned}\mathcal{L}\Big(\int_0^t \sin(\tau)\cos(t-\tau)d\tau\Big) &= \mathcal{L}(\sin * \cos) \\ &= \mathcal{L}(\sin(t)).\mathcal{L}(\cos(t)) \\ &= \frac{1}{s^2+1}.\frac{s}{s^2+1} \\ &= \frac{s}{(s^2+1)^2}.\end{aligned}$$

□

34. We would like to evaluate $\mathcal{L}^{-1}\Big(\frac{1}{s(s-a)^2}\Big)$. Note that

$$\frac{1}{s(s-a)^2} = \frac{G(s)}{s}, \text{ where } G(s) = F(s-a) \text{ and } F(s) = \frac{1}{s^2} = \mathcal{L}(t).$$

Moreover we have by the formula $\mathcal{L}^{-1}(F(s-a)) = e^{at}f(t)$,

$$g(t) = \mathcal{L}^{-1}(G(s)) = \mathcal{L}^{-1}(F(s-a)) = te^{at}.$$

Using the formula

$$\mathcal{L}^{-1}\Big(\frac{G(s)}{s}\Big) = \int_0^t g(\tau)d\tau,$$

we get

$$\begin{aligned}
\mathcal{L}^{-1}\Big(\frac{1}{s(s-a)^2}\Big) &= \int_0^t \tau e^{a\tau} d\tau \\
&= \Big[\tau \frac{e^{a\tau}}{a}\Big]_0^t - \int_0^t \frac{e^{a\tau}}{a} d\tau \\
&= t\frac{e^{at}}{a} - \Big[\frac{e^{a\tau}}{a^2}\Big]_0^t \\
&= t\frac{e^{at}}{a} - \frac{e^{at}}{a^2} + \frac{1}{a^2} \\
&= \frac{(at-1)e^{at}+1}{a^2}.
\end{aligned}$$

□

45. We consider the integral equation

$$\begin{cases} y'(t) = 1 - \sin(t) - \displaystyle\int_0^t y(\tau) d\tau, \\ y(0) = 0. \end{cases}$$

Let $Y(s) = \mathcal{L}(y(t))$. Applying the Laplace transform to the integral equation and taking into account the initial condition and the fact that $\mathcal{L}\Big(\int_0^t y(\tau)d\tau\Big) = \frac{Y(s)}{s}$, we get

$$\begin{aligned}
sY(s) - y(0) &= \frac{1}{s} - \frac{1}{s^2+1} - \frac{Y(s)}{s} \\
\Leftrightarrow \Big(s + \frac{1}{s}\Big)Y(s) &= \frac{1}{s} - \frac{1}{s^2+1} \\
\Leftrightarrow \frac{s^2+1}{s}Y(s) &= \frac{1}{s} - \frac{1}{s^2+1} \\
\Leftrightarrow Y(s) &= \frac{1}{s^2+1} - \frac{s}{(s^2+1)^2}. \qquad (1)
\end{aligned}$$

Note that

$$\frac{s}{(s^2+1)^2} = \frac{1}{2}\frac{2s}{(s^2+1)^2} = \frac{1}{2}(-1)^1\frac{d}{ds}\Big(\frac{1}{s^2+1}\Big) = \frac{1}{2}\mathcal{L}(t\sin(t)). \qquad (2)$$

We deduce from (1) and (2) that

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \sin(t) - \frac{1}{2}t\sin(t) = (1 - \frac{t}{2})\sin(t).$$

□

52. We would like to evaluate $\mathcal{L}(f(t))$, where $f(t)$ is the periodic function of period 2 defined on $[0,2]$ by

$$f(t) = \begin{cases} t, & \text{if } 0 \le t \le 1 \\ 2-t, & \text{if } 1 \le t \le 2. \end{cases}$$

The Laplace transform of f is given by

$$\begin{aligned} \mathcal{L}(f(t)) &= \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}f(t)dt \\ &= \frac{1}{1-e^{-2s}}\int_0^1 te^{-st}dt + \frac{1}{1-e^{-2s}}\int_1^2 (2-t)e^{-st}dt. \end{aligned} \tag{1}$$

Using the change of variables $\tau = 2 - t$, we get

$$\int_1^2 (2-t)e^{-st}dt = \int_0^1 \tau e^{-s(2-\tau)}d\tau = e^{-2s}\int_0^1 \tau e^{s\tau}d\tau. \tag{2}$$

Integrating by parts, we get

$$\begin{aligned} \int_0^1 te^{-st}dt &= \Big[t\frac{e^{-st}}{-s}\Big]_0^1 - \int_0^1 \frac{e^{-st}}{-s}dt \\ &= -\frac{e^{-s}}{s} + \frac{1}{s}\int_0^1 e^{-st}dt \\ &= -\frac{e^{-s}}{s} + \frac{1}{s}\Big[\frac{e^{-st}}{-s}\Big]_0^1 \\ &= -\frac{e^{-s}}{s} + \frac{1}{s}\frac{1-e^{-s}}{s} \\ &= \frac{1-(1+s)e^{-s}}{s^2}. \end{aligned} \tag{3}$$

Changing s to $-s$ in (3), we obtain

$$\int_0^1 \tau e^{s\tau} d\tau = \frac{1-(1-s)e^s}{s^2}. \tag{4}$$

Finally we get by taking into account (1), (2), (3) and (4)

$$\begin{aligned}
\mathcal{L}(f(t)) &= \frac{1}{1-e^{-2s}} \frac{1-(1+s)e^{-s}}{s^2} + \frac{1}{1-e^{-2s}} e^{-2s} \frac{1-(1-s)e^s}{s^2} \\
&= \frac{1-e^{-s}-se^{-s}+e^{-2s}-e^{-s}+se^{-s}}{s^2(1-e^{-2s})} \\
&= \frac{1-2e^{-s}+e^{-2s}}{s^2(1-e^{-2s})} \\
&= \frac{(1-e^{-s})^2}{s^2(1-e^{-2s})}.
\end{aligned}$$

□