Department of Mathematical Sciences
KFUPM
Term 031

MATH 301-01/ Exam#2/ Duration=2 Hours

Solution

1. Note that $\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau$, where $F(s)=\mathcal{L}\{f(t)\}$

a) Let $F(s)=\frac{1}{s^2-3}$. Then $f(t)=\frac{1}{\sqrt{3}}\sinh(\sqrt{3}t)$ and

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2-3)}\right\}=\frac{1}{\sqrt{3}}\int_0^t \sinh(\sqrt{3}\tau)\,d\tau=\frac{1}{\sqrt{3}}\left[\frac{\cosh(\sqrt{3}\tau)}{\sqrt{3}}\right]_0^t=\frac{1}{3}\left(\cosh(\sqrt{3}t)-1\right)$$

b) Similarly, for $F(s)=\frac{1}{s^2+3}$ and $f(t)=\frac{1}{\sqrt{3}}\sin(\sqrt{3}t)$, we obtain:

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+3)}\right\}=\frac{1}{\sqrt{3}}\int_0^t \sin(\sqrt{3}\tau)\,d\tau=\frac{1}{\sqrt{3}}\left[\frac{-\cos(\sqrt{3}\tau)}{\sqrt{3}}\right]_0^t=\frac{1}{3}\left(1-\cos(\sqrt{3}t)\right).$$

2. By the convolution theorem, we have:

$$\mathcal{L}^{-1}\{F(s)G(s)\}=(f*g)(t)=\int_0^t f(\tau)g(t-\tau)\,d\tau$$

where $F(s)=\mathcal{L}\{f(t)\}$ and $G(s)=\mathcal{L}\{g(t)\}$.

Then for $F(s)=\frac{1}{s^2+4}$, $G(s)=\frac{s}{s^2+9}$, $f(t)=\frac{1}{2}\sin(2t)$, $g(t)=\cos(3t)$

We obtain:

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)(s^2+9)}\right\}=\int_0^t \sin(2\tau)\cos(3(t-\tau))\,d\tau=\frac{1}{4}\int_0^t\left(\sin(2\tau+3t-3\tau)+\sin(2\tau-3t+3\tau)\right)d\tau$$

$$=\frac{1}{4}\int_0^t\left(\sin(3t-\tau)+\sin(5\tau-3t)\right)d\tau=\frac{1}{4}\left[\cos(3t-\tau)-\frac{1}{5}\cos(5\tau-3t)\right]_0^t$$

$$=\frac{1}{4}\left(-\cos(2t)-\frac{1}{5}\cos(2t)+\cos(3t)+\frac{1}{5}\cos(3t)\right)=\frac{1}{5}\left(\cos(2t)-\cos(3t)\right).$$

3. If $F(s)=\mathcal{L}\{f(t)\}$, then $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$.

a For $f(t)=\sin(3t)$, $n=1$, we obtain:

$$\mathcal{L}\{t\sin(3t)\} = (-1)^1 \frac{d}{ds}\left(\frac{3}{s^2+9}\right) = -\frac{3(-2s)}{(s^2+9)^2} = \frac{6s}{(s^2+9)^2}.$$

b) For $f(t)=\sin(3t)$, $n=2$, we have:

$$\mathcal{L}\{t^2\sin(3t)\} = (-1)^2\frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right) = 3\frac{d}{ds}\left(\frac{d}{ds}\left(\frac{1}{s^2+9}\right)\right)$$

$$= 3\frac{d}{ds}\left(\frac{-2s}{(s^2+9)^2}\right) = -6\frac{d}{ds}\left(\frac{s}{(s^2+9)^2}\right) = -6\frac{(s^2+9)^2 - s(4s)(s^2+9)}{(s^2+9)^4}$$

$$= -6\frac{s^2+9-4s^2}{(s^2+9)^3} = \frac{6(3s^2-9)}{(s^2+9)^3} = \frac{18(s^2-3)}{(s^2+9)^3}.$$

4. Consider the IVP: $y''+k^2y(t)=\delta_a$, $y(0)=1$, $y'(0)=k$.

a) Let $Y(s)=\mathcal{L}\{y(t)\}$. Applying the Laplace equation to the differential equation, we obtain:

$$\mathcal{L}\{y''(t)\} + k^2\mathcal{L}\{y(t)\} = \mathcal{L}\{\delta_a(t)\}$$

$$\Leftrightarrow s^2Y(s) - sy(0) - y'(0) + k^2 Y(s) = e^{-as}$$

$$\Leftrightarrow (s^2+k^2)Y(s) = s+k+e^{-as} \Leftrightarrow Y(s) = \frac{s+k+e^{-as}}{(s^2+k^2)}.$$

b) $$y(t) = \mathcal{L}^{-1}\left\{\frac{s+k+e^{-as}}{s^2+k^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} + \mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-as}}{s^2+k^2}\right\}$$

$$= \cos(kt) + \sin(kt) + \frac{1}{k}\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}e^{-as}\right\}$$

$$= \cos(kt) + \sin(kt) + \frac{1}{k}\sin(k(t-a))\,\mathcal{U}_a(t).$$

5\. a) We have:

$$\int_{-\pi}^{\pi} \sin(2x)\sin(3x)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\left(\cos(2x-3x) - \cos(2x+3x)\right)dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}\left(\cos(x) - \cos(5x)\right)dx = \frac{1}{2}\left[\sin x - \frac{\sin(5x)}{5}\right]_{-\pi}^{\pi} = 0.$$

b) $$\int_{-\pi}^{\pi} \cos^2(5x)\,dx = \int_{-\pi}^{\pi} \frac{1}{2}\left(1+\cos(10x)\right)dx = \frac{1}{2}\left(x + \frac{1}{10}\sin(10x)\right)_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left(\pi + \frac{1}{10}\sin(10\pi) - \left(-\pi + \frac{1}{10}\sin(-10\pi)\right)\right) = \frac{2\pi}{2} = \pi.$$

So the norm of $\cos(5x)$ is $\left(\int_{-\pi}^{\pi}\cos^2(5x)\,dx\right)^{1/2} = \sqrt{\pi}.$

6\. a) The half-range sine series of $f(x) = \pi - x$ in $[0,\pi]$ is given by

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{\pi}x\right) \quad \text{or} \quad \sum_{n=1}^{\infty} b_n \sin(nx)$$

with $$b_n = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\sin\left(\frac{n\pi x}{\pi}\right)dx = \frac{2}{\pi}\int_0^{\pi}(\pi-x)\sin(nx)\,dx$$

$$b_n = \frac{2}{\pi}\left\{\left[(\pi - x)\frac{-\cos(nx)}{n}\right]_0^{\pi} - \int_0^{\pi}(-1)\frac{-\cos(nx)}{n}\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{\pi}{n} - \frac{1}{n}\int_0^{\pi}\cos(nx)\,dx\right\} = \frac{2}{n\pi}\left\{\pi - \left[\frac{\sin(nx)}{n}\right]_0^{\pi}\right\} = \frac{2}{n}.$$

Hence the series is given by: $\displaystyle 2\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}.$

b) $f(x) = \pi - x$ and $f'(x) = -1$ are continuous functions on $[0,\pi]$.
So the series converges to $\frac{f(x+) + f(x-)}{2}$. In particular for $x = \frac{\pi}{2}$, we obtain:

$$\pi - \frac{\pi}{2} = 2\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right).$$

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} \sin(k\pi) = 0 & \text{if } n = 2k+1 \\ \sin\left(k\pi + \frac{\pi}{2}\right) = (-1)^k & \text{if } n = 2k. \end{cases}$$

So $\quad \dfrac{\pi}{2} = 2\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \iff \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$

7. a) The Complex Fourier series of $\boxed{f(x) = e^x}$ in $[-\pi, \pi]$ is given by

$$\sum_{-\infty}^{\infty} c_n e^{i\frac{n\pi}{\pi}x} \quad \text{or} \quad \sum_{-\infty}^{\infty} c_n e^{inx},$$

with $c_n = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} e^x e^{-i\frac{n\pi}{\pi}x}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)x}\,dx = \frac{1}{2\pi}\left[\frac{e^{(1-in)x}}{1-in}\right]_{-\pi}^{\pi}$

$$= \frac{1}{2\pi}\cdot\frac{1}{1-in}\left[e^{(1-in)\pi} - e^{-(1-in)\pi}\right] = \frac{1}{2\pi(1-in)}\left[e^{\pi}e^{-in\pi} - e^{-\pi}e^{in\pi}\right]$$

$$= \frac{e^{\pi}(-1)^n - e^{-\pi}(-1)^n}{2\pi(1-in)} = \frac{(-1)^n\left(e^{\pi} - e^{-\pi}\right)}{2\pi(1-in)} = \frac{(-1)^n \sinh(\pi)}{\pi(1+n^2)}(1+in).$$

So the complex Fourier series is given by:

$$\frac{\sinh(\pi)}{\pi}\sum_{-\infty}^{\infty} \frac{(-1)^n(1+in)}{(1+n^2)} e^{inx}.$$

b) $f(x) = e^x = f'(x)$ are continuous. So the series converges to $f(x)$ for any $x \in (-\pi, \pi)$. In particular for $x = 0$, we get:

$$f(0) = \frac{\sinh(\pi)}{\pi}\sum_{-\infty}^{\infty} \frac{(-1)^n(1+in)}{1+n^2} e^0$$

$$\iff \sum_{-\infty}^{\infty} \frac{(-1)^n}{1+n^2} + i\sum_{-\infty}^{\infty} \frac{n(-1)^n}{1+n^2} = \frac{\pi}{\sinh(\pi)}.$$

Hence

$$\sum_{-\infty}^{\infty} \frac{(-1)^n}{1+n^2} = \frac{\pi}{\sinh(\pi)}.$$

8. Eigenvalues and eigenfunctions of BVP:

$$y'' + 5y' + 3y = 0, \quad y(0) = 0, \quad y(4) = 0.$$

We first solve the quadratic equation $r^2 + 5r + \lambda = 0$

$$\Delta = 25 - 4\lambda$$

We have 3 possibilities:

* $\Delta = 0$ i.e. $\lambda = \frac{25}{4}$

In this case $r_1 = r_2 = -\frac{5}{2}$ and $\boxed{y(x) = (c_1 + c_2 x)\, e^{-\frac{5x}{2}}}$

$$\left.\begin{array}{l} y(0) = 0 \Rightarrow c_1 = 0. \\ y(4) = 0 \Rightarrow 4c_2 e^{-10} = 0 \Rightarrow c_2 = 0. \end{array}\right\} \Rightarrow y(x) \equiv 0.$$

* $\Delta > 0$: $\lambda < \frac{25}{4}$

$$r_1 = \frac{-5 - \sqrt{25 - 4\lambda}}{2}, \quad r_2 = \frac{-5 + \sqrt{25 - 4\lambda}}{2}$$

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

$$y(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

$$y(4) = 0 \Rightarrow c_1 e^{4r_1} + c_2 e^{4r_2} = 0 \Rightarrow c_1 (e^{4r_1} - e^{4r_2}) = 0$$

$\Rightarrow c_1 = 0$ since $r_1 \neq r_2$.

So $c_1 = c_2 = 0$ and $y(x) \equiv 0$.

* $\Delta < 0$: $\lambda > \frac{25}{4}$

$$r_1 = \frac{-5 - i\sqrt{4\lambda - 25}}{2}, \quad r_2 = \frac{-5 + i\sqrt{4\lambda - 25}}{2},$$

$$y(x) = e^{-\frac{5}{2}x}\left(c_1 \cos(\alpha x) + c_2 \sin(\alpha x)\right), \text{ with } \alpha = \frac{\sqrt{4\lambda - 25}}{2}.$$

$$y(0) = 0 \Rightarrow c_1 = 0 \Rightarrow y(x) = c_2 e^{-\frac{5}{2}x} \sin(\alpha x).$$

$$y(4) = 0 \Rightarrow c_2 e^{-10} \sin(2\sqrt{4\lambda - 25}) = 0 \Rightarrow c_2 \sin(2\sqrt{4\lambda - 25}) = 0$$

If $c_2 \doteq 0$, then $y(x) \equiv 0$.

If $c_2 \neq 0$, then $\sin\left(2\sqrt{4\lambda-25}\right) = 0$

$$\Longleftrightarrow \quad 2\sqrt{4\lambda-25} = n\pi, \quad n = 1, 2, 3, \ldots$$

$$\Longleftrightarrow \quad 4\lambda - 25 = \frac{n^2\pi^2}{4}$$

$$\Longleftrightarrow \quad \lambda = \frac{n^2\pi^2}{16} + \frac{25}{4} = \frac{100 + n^2\pi^2}{16}, \quad n = 1, 2, 3, \ldots$$

are the eigenvalues.

The corresponding eigenfunctions are:

$$e^{-\frac{5}{2}x} \sin(\alpha x) = e^{-\frac{5}{2}x} \sin\left(\frac{n\pi}{4} x\right), \quad n = 1, 2, 3, \ldots$$