King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

MATH 301/Exam 2/ Term 062/ Time allowed=2 Hours

Full Name:

ID Number:

Q #	**Points**
1	/8
2	/8
3	/16
4	/5
5	/7
6	/6
T	/50

1. Evaluate $\mathcal{L}^{-1}\Big(\dfrac{e^{-s}}{s(s+1)}\Big)$.

Solution:

Note that

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{(A+B)s + A}{s(s+1)}. \tag{1}$$

We deduce from (1) that

$$\begin{cases} A + B = 0, \\ A = 1 \end{cases} \quad \Leftrightarrow \quad \begin{cases} A = 1, \\ B = -1 \end{cases}$$

It follows from (1) that

$$\frac{e^{-s}}{s(s+1)} = e^{-s}\frac{1}{s} - e^{-s}\frac{1}{s+1}$$

which leads to

$$\begin{aligned} \mathcal{L}^{-1}\Big(\frac{e^{-s}}{s(s+1)}\Big) &= \mathcal{L}^{-1}\Big(e^{-s}\frac{1}{s}\Big) - \mathcal{L}^{-1}\Big(e^{-s}\frac{1}{s+1}\Big) \\ &= \mathcal{L}^{-1}(e^{-s}F_1(s)) - \mathcal{L}^{-1}(e^{-s}F_2(s)) \end{aligned}$$

where

$$\begin{aligned} F_1(s) &= \frac{1}{s} = \mathcal{L}(1) \\ F_2(s) &= \frac{1}{s+1} = \mathcal{L}(e^{-t}) \end{aligned}$$

Using the formula

$$\mathcal{L}^{-1}(e^{-as}F(s)) = f(t-a)\mathcal{U}_a(t), \text{ with } a = 1,$$

we get

$$\mathcal{L}^{-1}\Big(\frac{e^{-s}}{s(s+1)}\Big) = \mathcal{U}_1(t) - e^{-(t-1)}\mathcal{U}_1(t) = (1 - e^{1-t})\mathcal{U}_1(t).$$

$\square$

2. Solve the integral equation

$$\begin{cases} y'(t) = 1 - \sin(t) - \displaystyle\int_0^t y(\tau)d\tau, \\ y(0) = 0. \end{cases}$$

<u>**Solution:**</u>

Let $Y(s) = \mathcal{L}(y(t))$. Applying the Laplace transform to the integral equation and taking into account the initial condition and the fact that $\mathcal{L}\Big(\displaystyle\int_0^t y(\tau)d\tau\Big) = \frac{Y(s)}{s}$, we get

$$\begin{aligned} sY(s) - y(0) &= \frac{1}{s} - \frac{1}{s^2+1} - \frac{Y(s)}{s} \\ \Leftrightarrow\ \Big(s + \frac{1}{s}\Big)Y(s) &= \frac{1}{s} - \frac{1}{s^2+1} \\ \Leftrightarrow\ \frac{s^2+1}{s}Y(s) &= \frac{1}{s} - \frac{1}{s^2+1} \\ \Leftrightarrow\ Y(s) &= \frac{1}{s^2+1} - \frac{s}{(s^2+1)^2}. \end{aligned} \tag{1}$$

Note that

$$\frac{s}{(s^2+1)^2} = \frac{1}{2}\frac{2s}{(s^2+1)^2} = \frac{1}{2}(-1)^1\frac{d}{ds}\Big(\frac{1}{s^2+1}\Big) = \frac{1}{2}\mathcal{L}(t\sin(t)). \tag{2}$$

We deduce from (1) and (2) that

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \sin(t) - \frac{1}{2}t\sin(t) = (1 - \frac{t}{2})\sin(t).$$

$\square$

3. Solve the initial-value problem

$$\begin{cases} y'' - 7y' + 6y = e^t + \delta_3(t), \\ y(0) = 0, \ y'(0) = 0. \end{cases}$$

<u>**Solution:**</u>

Let $Y(s) = \mathcal{L}(y(t))$. Applying the Laplace transform to the ode and taking into account the initial conditions, we get

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) - 7(sY(s) - y(0)) + 6Y(s) &= \mathcal{L}(e^t) + \mathcal{L}(\delta_3) \\ \Leftrightarrow \ (s^2 - 7s + 6)Y(s) &= \frac{1}{s-1} + e^{-3s} \\ \Leftrightarrow \ Y(s) &= \frac{1}{(s-1)(s^2-7s+6)} + e^{-3s}\frac{1}{(s^2-7s+6)} \\ \Leftrightarrow \ Y(s) &= \frac{1}{(s-1)^2(s-6)} + e^{-3s}\frac{1}{(s-1)(s-6)}. \end{aligned} \tag{1}$$

Note that

$$\begin{aligned} \frac{1}{(s-1)^2(s-6)} &= \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s-6} \\ &= \frac{(A+C)s^2 + (-7A+B-2C)s + 6A - 6B + C}{(s-1)^2(s-6)}. \end{aligned} \tag{2}$$

We deduce from (2) that

$$\begin{cases} A + C = 0, \\ -7A + B - 2C = 0, \\ 6A - 6B + C = 1 \end{cases} \quad \Leftrightarrow \quad \begin{cases} C = -A, \\ -5A + B = 0, \\ 5A - 6B = 1 \end{cases} \quad \Leftrightarrow \quad \begin{cases} A = -1/25, \\ B = -1/5, \\ C = 1/25 \end{cases}$$

We also have

$$\frac{1}{(s-1)(s-6)} = \frac{A}{s-1} + \frac{B}{s-6} = \frac{(A+B)s - 6A - B}{(s-1)(s-6)}. \tag{3}$$

It follows from (3) that

$$\begin{cases} A + B = 0, \\ -6A - B = 1, \end{cases} \quad \Leftrightarrow \quad \begin{cases} A = -1/5, \\ B = 1/5, \end{cases}$$

We deduce from (1), (2) and (3) that

$$\begin{aligned} Y(s) = &-\frac{1}{25}\frac{1}{s-1} - \frac{1}{5}\frac{1}{(s-1)^2} + \frac{1}{25}\frac{1}{s-6} \\ &-\frac{1}{5}e^{-3s}\frac{1}{s-1} + \frac{1}{5}e^{-3s}\frac{1}{s-6}. \end{aligned} \tag{4}$$

Using the formulas

$$\begin{aligned}
\mathcal{L}^{-1}(F(s-a)) &= e^{at}f(t), \text{ with } a=1,\ f(t)=t \text{ and } F(s)=\frac{1}{s^2} \\
\mathcal{L}^{-1}(e^{-as}F(s)) &= f(t-a)\mathcal{U}_a(t), \text{ with } a=3 \text{ and } f(t)=e^t, e^{6t}
\end{aligned}$$

we get from (4)

$$y(t)=\mathcal{L}^{-1}(Y(s)) = -\frac{1}{25}e^t - \frac{1}{5}te^t + \frac{1}{25}e^{6t} - \frac{1}{5}e^{t-3}\mathcal{U}_3(t) + \frac{1}{5}e^{6(t-3)}\mathcal{U}_3(t).$$

$\square$

4. Given that the Fourier series of the function f defined by

$$f(x) = \begin{cases} 0, & \text{if } \quad -\pi \le x < 0 \\ \sin(x), & \text{if } \quad 0 \le x \le \pi. \end{cases}$$

is given by $\dfrac{1}{\pi} + \dfrac{1}{2}\sin(x) + \dfrac{1}{\pi}\displaystyle\sum_{n=2}^{\infty} \dfrac{1+(-1)^n}{1-n^2}\cos(nx)$, find the sum of the series $\displaystyle\sum_{k=1}^{\infty} \dfrac{(-1)^k}{(2k-1).(2k+1)}$. Justify your answer.

<u>Solution:</u>

The function f is continuous on the interval $[-\pi, \pi]$ and differentiable except at $x = 0$ with

$$f'(x) = \begin{cases} 0, & \text{if } \quad -\pi \le x < 0 \\ \cos(x), & \text{if } \quad 0 < x \le \pi. \end{cases}$$

In particular f is continuous at $\frac{\pi}{2}$. Therefore we get from (1)

$$f\left(\frac{\pi}{2}\right) = \frac{1}{\pi} + \frac{1}{2}\sin\left(\frac{\pi}{2}\right) + \frac{1}{\pi}\sum_{n=2}^{\infty} \frac{1+(-1)^n}{1-n^2}\cos\left(\frac{n\pi}{2}\right). \tag{1}$$

Since we have for each integer, $1 + (-1)^{(2k+1)} = 0$ and $\cos\left(\frac{2k\pi}{2}\right) = \cos(k\pi) = (-1)^k$, we deduce from (2) that

$$1 = \frac{1}{\pi} + \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{1+(-1)^{2k}}{1-(2k)^2}(-1)^k.$$

which can be written as

$$2\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)^2-1} = \pi\left(\frac{1}{\pi} + \frac{1}{2} - 1\right) = 1 - \frac{\pi}{2}$$

or

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)^2-1} = \frac{1}{2} - \frac{\pi}{4}.$$

Hence

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1).(2k+1)} = \frac{1}{2} - \frac{\pi}{4}.$$

$\square$

5. Find the Fourier series of the function f defined by $f(x) = x|x|$ on the interval $[-1, 1]$.

Solution:

Clearly f is an odd function. Therefore the Fourier series of f on the interval $[-1, 1]$ is the sine series given by

$$\sum_{n=1}^{\infty} b_n \sin\big(\frac{n\pi}{1}x\big)$$

or

$$\sum_{n=1}^{\infty} b_n \sin(n\pi x), \tag{1}$$

where

$$b_n = \frac{2}{1}\int_0^1 f(x)\sin(\frac{n\pi}{1}x)dx = 2\int_0^1 x^2 \sin(n\pi x)dx. \tag{2}$$

Integration by parts twice, we get

$$\begin{aligned}
\int x^2 \sin(n\pi x)dx &= -\frac{x^2}{n\pi}\cos(n\pi x) - \int -\frac{2x}{n\pi}\cos(n\pi x)dx \\
&= -\frac{x^2}{n\pi}\cos(n\pi x) + \frac{2}{n\pi}\int x\cos(n\pi x)dx \\
&= -\frac{x^2}{n\pi}\cos(n\pi x) + \frac{2}{n\pi}\Big(\frac{x}{n\pi}\sin(n\pi x) - \int \frac{1}{n\pi}\sin(n\pi x)dx\Big) \\
&= -\frac{x^2}{n\pi}\cos(n\pi x) + \frac{2x}{n^2\pi^2}\sin(n\pi x) - \frac{2}{n^2\pi^2}\int \sin(n\pi x)dx \\
&= -\frac{x^2}{n\pi}\cos(n\pi x) + \frac{2x}{n^2\pi^2}\sin(n\pi x) + \frac{2}{n^3\pi^3}\cos(n\pi x). \qquad (3)
\end{aligned}$$

We deduce from (2) and (3) that

$$\begin{aligned}
b_n = 2\int_0^1 x^2 \sin(n\pi x)dx &= -\frac{2}{n\pi}\cos(n\pi) + \frac{4}{n^3\pi^3}(\cos(n\pi) - 1) \\
&= \frac{2(-1)^{n+1}}{n\pi} + \frac{4}{\pi^3}\frac{(-1)^n - 1}{n^3}. \qquad (4)
\end{aligned}$$

Taking into account (1) and (4), it follows that the Fourier sine series of f on the interval $[-1, 1]$ is given by

$$\sum_{n=1}^{\infty}\Big(\frac{2(-1)^{n+1}}{n\pi} + \frac{4}{\pi^3}\frac{(-1)^n - 1}{n^3}\Big)\sin(n\pi x).$$

□

6. Find the Complex Fourier series of the function f defined by $f(x) = e^{-|x|}$ on the interval $[-1, 1]$.

Solution:

The Complex Fourier series of f on the interval $[-1, 1]$ is given by

$$\sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi}{1}x},$$

or

$$\sum_{n=-\infty}^{\infty} c_n e^{in\pi x}, \tag{1}$$

where

$$\begin{aligned}
c_n &= \frac{1}{2}\int_{-1}^{1} f(x)e^{\frac{-in\pi}{1}x}dx = \frac{1}{2}\int_{-1}^{0} e^{x}e^{-in\pi x}dx + \frac{1}{2}\int_{0}^{1} e^{-x}e^{-in\pi x}dx \\
&= \frac{1}{2}\int_{-1}^{0} e^{(1-in\pi)x}dx + \frac{1}{2}\int_{0}^{1} e^{-(1+in\pi)x}dx \\
&= \frac{1}{2}\left[\frac{1}{1-in\pi}e^{(1-in\pi)x}\right]_{-1}^{0} + \frac{1}{2}\left[\frac{1}{-1-in\pi}e^{-(1+in\pi)x}\right]_{0}^{1} \\
&= \frac{1}{2}\left(\frac{1}{1-in\pi} - \frac{e^{(-1+in\pi)}}{1-in\pi} + \frac{e^{-(1+in\pi)}}{-1-in\pi} + \frac{1}{1+in\pi}\right) \\
&= \frac{1}{2}\left(\frac{1}{1-in\pi} - \frac{(-1)^n e^{-1}}{1-in\pi} - \frac{(-1)^n e^{-1}}{1+in\pi} + \frac{1}{1+in\pi}\right) \\
&= \frac{1}{2}(1-(-1)^n e^{-1})\left(\frac{1}{1-in\pi} + \frac{1}{1+in\pi}\right) \\
&= \frac{1-(-1)^n e^{-1}}{1+n^2\pi^2}.
\end{aligned} \tag{2}$$

Using (1) and (4), we get the Complex Fourier series of f on the interval $[-1, 1]$

$$\sum_{n=-\infty}^{\infty} \frac{1-(-1)^n e^{-1}}{1+n^2\pi^2} e^{in\pi x}.$$

□