King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
MATH 301/Exam 1/ Term 062/ Time allowed=2 Hours

Full Name:

ID Number:

Q #	Points
1	/3
2	/8
3	/11
4	/11
5	/8
6	/9
T	/50

1. Find the length of the curve traced by $\mathbf{r}(t) = a\cos(t)\mathbf{i} + a\sin(t)\mathbf{j} + ct\mathbf{k},\ t \in [0, 2\pi]$.

Solution:
The length of the curve traced by $\mathbf{r}(t)$ is given by:

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(-a\sin(t))^2 + (a\cos(t))^2 + c^2}dt \\ &= \int_0^{2\pi} \sqrt{a^2\sin^2(t) + a^2\cos^2(t) + c^2}dt \\ &= \int_0^{2\pi} \sqrt{a^2 + c^2}dt = 2\pi\sqrt{a^2 + c^2}. \end{aligned}$$

□

2. Show that the vector $\mathbf{F}(x,y) = (x^4 + y)\mathbf{i} + (x + y^4)\mathbf{j}$ is a gradient field and find a function ϕ such that $\nabla\phi(x,y) = \mathbf{F}(x,y)$.

Solution:

Let $\mathbf{F}(x,y) = (x^4 + y)\mathbf{i} + (x + y^4)\mathbf{j} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ be a vector field. Since the functions P and Q are continuous and have partial derivatives continuous on any domain and moreover we have $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} = 1$, $\mathbf{F}$ is a gradient field i.e. there exists a function ϕ such that $\nabla\phi(x,y) = \mathbf{F}(x,y)$ i.e.

$$\frac{\partial \phi}{\partial x} = P(x,y) = x^4 + y \tag{1}$$

$$\frac{\partial \phi}{\partial y} = Q(x,y) = x + y^4. \tag{2}$$

Integrating (1), we get

$$\phi(x,y) = \int (x^4 + y)dx = \frac{1}{5}x^5 + xy + g(y). \tag{3}$$

Using (2) and (3), we get

$$x + g'(y) = x + y^4 \;\Rightarrow\; g'(y) = y^4 \;\Rightarrow\; g(y) = \frac{1}{5}y^5 + C. \tag{4}$$

Combining (3) and (4), we obtain

$$\phi(x,y) = \frac{1}{5}x^5 + \frac{1}{5}y^5 + xy + C.$$

□

3. Use Green's theorem over a region R that does not contain the origin to evaluate the line integral $\displaystyle\int_C \frac{-y^3dx + xy^2dy}{(x^2+y^2)^2}$, where C is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Solution:

Let R be the region of the $xy-$plane bounded by C and the circle C' : $x^2 + y^2 = 1$ of center $(0,0)$ and radius 1.

Let $P(x,y) = \frac{-y^3}{(x^2+y^2)^2}$ and $Q(x,y) = \frac{xy^2}{(x^2+y^2)^2}$. These functions are continuous and have partial derivatives continuous on any domain not containing the origin. Moreover we have $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}[-y^3(x^2+y^2)^{-2}] = -3y^2(x^2+y^2)^{-2}+4y^4(x^2+y^2)^{-3} = (y^4-3x^2y^2)(x^2+y^2)^{-3},$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}[xy^2(x^2+y^2)^{-2}] = y^2(x^2+y^2)^{-2}-4x^2y^2(x^2+y^2)^{-3} = (y^4-3x^2y^2)(x^2+y^2)^{-3}.$$

Using Green's theorem we have

$$\oint_{C\cup C'} Pdx + Qdy = \int\int_R \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dxdy = 0.$$

Taking into account that orientations of C and C' in the left hand-side of the previous formula are respectively counterclockwise and clockwise, we get:

$$\int_C \frac{-y^3dx + xy^2dy}{(x^2+y^2)^2} = \int_{C'} \frac{-y^3dx + xy^2dy}{(x^2+y^2)^2}, \qquad (1)$$

where now the orientations of C and C' are both counterclockwise.

It is clear that it is easier to evaluate the second integral in (1) which we will do by using the parametrization of C':

$$C : \begin{cases} x = \cos(t), \\ y = \sin(t), \quad t \in [0, 2\pi]. \end{cases}$$

Then we have

$$
\begin{aligned}
\int_{C'} \frac{-y^3dx + xy^2dy}{(x^2+y^2)^2} &= \int_{C'} \frac{-y^3dx}{(x^2+y^2)^2} + \int_{C'} \frac{xy^2dy}{(x^2+y^2)^2} \\
&= \int_0^{2\pi} \sin^4(t)dt + \int_0^{2\pi} \sin^2(t)\cos^2(t)dt \\
&= \int_0^{2\pi} \sin^4(t)dt + \int_0^{2\pi} \sin^2(t)\cos^2(t)dt = \int_0^{2\pi} \sin^2(t)(\sin^2(t)+\cos^2(t))dt \\
&= \int_0^{2\pi} \sin^2(t)dt = \int_0^{2\pi} \frac{1}{2}(1-\cos(2t))dt = \frac{1}{2}\Big[t - \frac{1}{2}\sin(2t)\Big]_0^{2\pi} = \pi. \qquad (2)
\end{aligned}
$$

Taking into account (1) and (2), we obtain

$$\int_C \frac{-y^3dx + xy^2dy}{(x^2+y^2)^2} = \pi.$$

□

4. Evaluate the line integral $\oint_C z^2dx + x^2dy + y^2dz$, where C is the trace of the cylinder $x^2 + z^2 = 1$ on the plane $y + z = 4$.

<u>Solution:</u>

Let $\mathbf{F} = z^2\mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k}$ and let S be the portion of the plane $y + z = 4$ that is located inside the cylinder $x^2 + z^2 = 1$ (draw a figure). S is defined by the equation $z = f(x, y) = 4 - y$. The components of the vector field $\mathbf{F}$ are continuous and have partial derivatives continuous on any domain. Therefore we have by Stokes' theorem

$$\oint_C z^2dx + x^2dy + y^2dz = \oint_C \mathbf{F}.d\mathbf{r} = \int\int_S curl(\mathbf{F}).\mathbf{n}dS, \tag{1}$$

where

$$\begin{aligned} curl\mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & x^2 & y^2 \end{vmatrix} \\ &= (2y - 0)\mathbf{i} - (0 - 2z)\mathbf{j} + (2x - 0)\mathbf{k} \\ &= 2y\mathbf{i} + 2z\mathbf{j} + 2x\mathbf{k}. \end{aligned} \tag{2}$$

The unit normal vector to S is given by

$$\mathbf{n} = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}}\big(- f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}\big) = \frac{1}{\sqrt{2}}\big(\mathbf{j} + \mathbf{k}\big). \tag{3}$$

Now let R be the projection of S on the $xz-$plane. Then we get by using (1), (2) and (3)

$$\begin{aligned} &\oint_C y^2dx + z^2dy + x^2dz = \int\int_S \frac{2}{\sqrt{2}}(x + z)dS \\ &= \int\int_R \sqrt{2}(x + z)\sqrt{2}dxdz \quad \text{since } S \text{ is also defined by } y = 4 - z \\ &= 2\int\int_R (x + z)dxdz. \end{aligned} \tag{4}$$

Using the polar coordinates $x = r\cos\theta$ and $z = r\sin\theta$, we obtain

$$\begin{aligned}\int\int_R (x+z)dxdz &= \int_0^{2\pi} \Big(\int_0^1 (r\cos\theta + r\sin\theta) r dr \Big) d\theta \\ &= \int_0^{2\pi} \Big(\int_0^1 \Big[\frac{1}{3} r^3 \cos\theta + \frac{1}{3} r^3 \sin\theta \Big]_0^1 \Big) d\theta \\ &= \int_0^{2\pi} \Big(\frac{1}{3}\cos\theta + \frac{1}{3}\sin\theta \Big) d\theta \\ &= \Big[\frac{1}{3}\sin\theta - \frac{1}{3}\cos\theta \Big]_0^{2\pi} = 0. \end{aligned} \tag{5}$$

Hence we deduce from (1), (4) and (5) that

$$\oint_C yz^2 dx + xz^2 dy + x^2 y dz = 0.$$

□

5. Evaluate the area of that portion of the cone $z = a - \sqrt{x^2 + y^2}$ that is within the planes $z = b$ and $z = c$, with $0 < b < c < a$.

Solution:

Let S be the portion of the cone $z = f(x, y) = a - \sqrt{x^2 + y^2}$ that is within the planes $z = b$ and $z = c$ (draw a figure). Then the area of S is given by

$$Area(S) = \int\int_R \sqrt{1 + f_x^2 + f_y^2} dxdy, \tag{1}$$

where R is the projection of S on the $xy-$plane.
Now we have

$$f_x = \frac{-x}{\sqrt{x^2 + y^2}}, \quad f_y = \frac{-y}{\sqrt{x^2 + y^2}}$$
$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} = \sqrt{2}.$$

We deduce then from (1) by using the polar coordinates and denoting by r_b and r_c the values of r for which we have respectively $z = b$ and $z = c$ i.e. $r_b = a - b$ and $r_c = a - c$

$$\begin{aligned} Area(S) &= \int\int_R \sqrt{2} dxdy \\ &= \int_0^{2\pi} \int_{r_c}^{r_b} \sqrt{2} r dr d\theta \\ &= \sqrt{2} \int_0^{2\pi} \left[\frac{r^2}{2}\right]_{r_c}^{r_b} d\theta \\ &= 2\pi\sqrt{2}\Big(\frac{(a-b)^2}{2} - \frac{(a-c)^2}{2}\Big) \\ &= \pi\sqrt{2}(c - b)(2a - b - c). \end{aligned}$$

$\square$

6. Let $\mathbf{F}(x,y,z) = xy\mathbf{i} + 2yz\mathbf{j} + x^2e^{3y}\mathbf{k}$ and let D be the region that is bounded by the three coordinate planes and the plane $x+y+z=1$. Let S be the surface representing the exterior boundary of D which we orient outward (draw a figure). Use the divergence theorem to evaluate the flux $\int\int_S \mathbf{F}.\mathbf{n}\, dS$.

<u>Solution:</u>

Note that the components of the vector field $\mathbf{F}$ are continuous and have partial derivatives continuous everywhere. So we have by the divergence formula

$$\int\int_S \mathbf{F}.\mathbf{n}\, dS = \int\int\int_D div(\mathbf{F})dV. \tag{1}$$

Then we have for the right-hand side of (1)

$$\begin{aligned}
\int\int\int_D div(\mathbf{F})dV &= \int_0^1 \Big(\int_0^{1-y}\Big(\int_0^{1-x-y}(y+2z)dz\Big)dx\Big)dy \\
&= \int_0^1 \Big(\int_0^{1-y}\big[yz+z^2\big]_0^{1-x-y}dx\Big)dy \\
&= \int_0^1 \Big(\int_0^{1-y}(y(1-x-y)+(1-x-y)^2)dx\Big)dy \\
&= \int_0^1 \Big(\int_0^{1-y}(1-2x-y+xy+x^2)dx\Big)dy \\
&= \int_0^1 \big[x-x^2-xy+\frac{1}{2}x^2y+\frac{1}{3}x^3\big]_0^{1-y}dy \\
&= \int_0^1 (1-y-(1-y)^2-y(1-y)+\frac{1}{2}y(1-y)^2+\frac{1}{3}(1-y)^3)dy \\
&= \int_0^1 (\frac{1}{2}y(1-y)^2+\frac{1}{3}(1-y)^3)dy \\
&= \int_0^1 (\frac{1}{2}(1-y)^2-\frac{1}{6}(1-y)^3)dy \\
&= \int_0^1 (\frac{1}{2}t^2-\frac{1}{6}t^3)dt = \big[\frac{1}{6}t^3-\frac{1}{24}t^4\big]_0^1 = \frac{1}{6}-\frac{1}{24} = \frac{1}{8} \qquad t = 1-y.
\end{aligned}$$

□