2.4 Zero Derivative, $Df = 0$

Theorem 115

$$\textit{jump function} \quad \underset{p332}{\overset{Kolm,\, prob\, 8}{\Longrightarrow}} \quad \textit{has a zero derivative a.e.}$$

Theorem 116

$$\boxed{\left.\begin{array}{l} f \in AC[a,b] \\ \\ Df \overset{a.e.}{=} 0 \quad in\ [a,b] \end{array}\right\} \underset{Royden\ p110}{\overset{Kolm,\, p339}{\Longrightarrow}} \quad f \text{ is constant} \quad \forall\, t \in [a,b].}$$

Proof.

$$f \in AC[a,b] \implies f(t) = I_a Df(t) + c = c, \quad \forall\, t \in [a,b]. \quad \blacksquare$$

Theorem 117

$$\boxed{\left.\begin{array}{l} f \in C[a,b] \\ \\ Df \overset{n.e.}{=} 0 \ in\ [a,b] \end{array}\right\} \implies \quad f \text{ is constant on } [a,b].}$$

Proof.

$$f \in C[a,b] \;\&\; Df \in L^1(a,b) \text{ exists n.e. in } [a,b] \underset{thm\, 90}{\Longrightarrow} f \in AC[a,b]. \quad \blacksquare$$

Theorem 118

$$\boxed{Df(t) = 0 \quad \forall t \in (a,b) \implies \left\{\begin{array}{l} f(t) = c, \quad \forall t \in (a,b). \\ \\ f(a^+) := \lim_{t \to a^+} f(t) = c \quad exists \end{array}\right.}$$

Proof. *The result follows from MVT, Theorem 73 or from Lemma 86 since $Df \in CL^1(a,b)$.*

Remark 14

$$\boxed{Df \overset{a.e.}{=} 0 \ in\ [a,b] \quad \not\Longrightarrow \quad f \text{ is constant on } [a,b],}$$

since Cantor function is monotone on $[0,1]$ with $Df = 0$ a.e. on $[0,1]$.

$$\boxed{\left.\begin{array}{l} f \in C[a,b] \\ \\ Df \overset{a.e.}{=} 0 \ in\ [a,b] \end{array}\right\} \not\Longrightarrow \quad f \text{ is constant on } [a,b],}$$

since Cantor function is continuous and non constant on $[0,1]$ with $Df \overset{a.e.}{=} 0$ but not n.e.

2.5 Zero nth Derivative, $D^n f = 0$

Lemma 119

$$\boxed{\left.\begin{array}{llll} f \in AC^n[a,b] & & & \\ & & & \\ D^n f & \overset{a.e.}{=} & 0 & in\ [a,b] \end{array}\right\} \Longrightarrow f(t) = \sum_{k=0}^{n-1} c_k\ (t-a)^k, \quad \forall\, t \in [a,b].}$$

where c_k are arbitraries.

Proof.

$$f \in AC^n[a,b] \quad \Longrightarrow \quad f(t) = I_a^n D^n f(t) + T_a^{n-1}(t) = T_a^{n-1}(t), \quad \forall\, t \in [a,b]. \quad \blacksquare$$

Lemma 120

$$\boxed{\left.\begin{array}{llll} f \in C^{n-1}[a,b] & & & \\ & & & \\ D^n f & \overset{n.e.}{=} & 0 & in\ [a,b] \end{array}\right\} \quad \Longrightarrow \quad f(t) = \sum_{k=0}^{n-1} c_k\ (t-a)^k, \quad \forall t \in [a,b].}$$

Proof.

$$D^{n-1} f \in C[a,b] \quad \& \quad D^n f = D D^{n-1} f \in L^1(a,b) \text{ exists n.e. in } [a,b] \quad \underset{thm\,90}{\Longrightarrow}$$

$$D^{n-1} f \in AC[a,b] \quad \Longrightarrow \quad f \in AC^n[a,b] \quad \underset{lem\,119}{\Longrightarrow} \quad \text{result.} \quad \blacksquare$$

Lemma 121

$$\boxed{D^n f = 0 \quad \forall t \in (a,b) \quad \Longrightarrow \quad \begin{cases} f(t) = \sum_{k=0}^{n-1} c_k\ (t-a)^k, \quad \forall t \in (a,b), \\ \\ f(a^+) := \lim_{t \to a^+} f(t) = c_0. \end{cases}}$$

Consequently,

$$\boxed{D^n f = 0 \quad \forall t \in [a,b] \quad \Longrightarrow \quad f(t) = \sum_{k=0}^{n-1} c_k\ (t-a)^k, \quad \forall t \in [a,b].}$$

Proof.

By Lemma 86, for $t \in (a,b)$,

$$D^{n-1} f(t) = I_a D^n f(t) + c_{n-1}.$$

$$D^{n-2} f(t) = I_a D^{n-1} f(t) + c_{n-2} = c_{n-1}(t-a) + c_{n-2}.$$

The result follows by induction. $\blacksquare$

12.3 Zero Fractional Derivative

12.3.1 Introduction

Warning 295

The solution of $D_a^\alpha f = 0$ depends on two conditions:

where this holds and for what class of functions.

Notation 296

$$\boxed{D_a^{n-\alpha} T_a^{n-1} I_a^{n-\alpha} f(t) = \sum_{k=0}^{n-1} \frac{D^k I_a^{n-\alpha} f(a)}{\Gamma(\alpha - n + 1 + k)} (t-a)^{\alpha-n+k}}$$

Or with change of indices

$$\boxed{D_a^{n-\alpha} T_a^{n-1} I_a^{n-\alpha} f(t) = \sum_{j=1}^{n} \frac{D_a^{\alpha-j} f(a)}{\Gamma(\alpha - j + 1)} (t-a)^{\alpha-j}}$$

Corollary 297 (Kilbas [10], Cor 2.1, p. 72)

$$\boxed{D_a^\alpha f(t) = 0 \quad \Longleftrightarrow \quad f(t) = D_a^{n-\alpha} T_a^{n-1}(t) = \sum_{j=1}^{n} c_j (t-a)^{\alpha-j},}$$

where $c_j \in \mathbb{R}$ are arbitrary constants.

Proof. In Kilbas [10] it is mentioned that this lemma follows from the properties of the power functions. This is clear for the $\Longleftarrow$ statement.

However for the $\Longrightarrow$ statement, we can use Lemma 121. ■

Remark 36 *In the above lemma we need to assume that $f \in C(a,b)$. Otherwise, we can take $f(t) \overset{a.e.}{=} t^{\alpha-1}$, $0 < \alpha < 1$. For this function $I_a^{1-\alpha} f = const.$ on $(0,1)$ and thus $D^\alpha f = 0$ for all $t \in (a,b)$.*

12.3.2 Sufficient Conditions

Lemma 298

$$\boxed{D_a^\alpha (t-a)^{\alpha-k} = 0, \qquad k = 1, 2, , n = [\alpha] + 1.}$$

Proof. *Follows from the compositon on power functions or alternatively,*

$$D_a^\alpha (t-a)^{\alpha-k} = c_k D_a^\alpha D^k (t-a)^\alpha = c_k D^k D_a^\alpha (t-a)^\alpha = D^k (const.) = 0, \qquad k = 1, 2, , n.$$

Lemma 299

$$\boxed{f(t) = D_a^{n-\alpha} T_a^{n-1}(t) = \sum_{k=1}^{n} c_k (t-a)^{\alpha-k} \quad \Longrightarrow \quad D_a^\alpha f(t) = 0, \quad \forall t.}$$

Proof. Follows form Lemma 298. Alternatively it follows from compositions on polynomials,

$$D_a^\alpha D_a^{n-\alpha} T_a^{n-1}(t) = D^n T_a^{n-1}(t) = 0. \quad \blacksquare$$

Corollary 300

$$\boxed{D_a^\alpha D_a^{n-\alpha} T_a^{n-1} g(t) = 0, \quad \forall t.}$$

12.3.3 Necessary Condition: $D_a^\alpha f = 0$ a.e

Lemma 301

$$\boxed{\left.\begin{array}{l} f \in L^1(a,b) \\ \\ I_a^{n-\alpha} f \in AC^n[a,b] \\ \\ D_a^\alpha f(t) \overset{a.e.}{=} 0 \quad \textit{in } [a,b] \end{array}\right\} \implies f(t) \overset{a.e.}{=} D_a^{n-\alpha} T_a^{n-1}(t), \quad \textit{on } [a,b].}$$

Equality holds if in addition $f \in CL^1(a,b)$.

Proof.

$$D_a^\alpha f := D^n I_a^{n-\alpha} f \overset{a.e.}{=} 0 \underset{lem\ 119}{\implies} I_a^{n-\alpha} f = T_a^{n-1}(t) \text{ on } [a,b].$$

$$\overset{lem\ 132}{\implies} f \overset{a.e.}{=} D_a^{n-\alpha} I_a^{n-\alpha} f = \text{result}.$$

For $f \in CL^1(a,b)$, $f = D_a^\alpha I_a^\alpha f$, $t \in (a,b)$. ■

Remark 37 *This Lemma is not true if* $I_a^{n-\alpha} f \notin AC^n[a,b]$ *since there is a monotone continuous function with a zero derivative a.e.*

This can not happen if the derivative is zero everywhere on (a,b).

Next we consider this case.

12.3.4 Necessary Condition: $D_a^\alpha f = 0$ on (a, b)

Lemma 302

$$\left.\begin{array}{l} f \in L^1(a,b) \\[1em] D_a^\alpha f(t) = 0 \quad on\ (a,b) \\[1em] \alpha > 0 \end{array}\right\} \quad \begin{array}{c}\Longrightarrow \\ \Longleftarrow\end{array} \quad \left\{\begin{array}{ll} f(t) \overset{a.e.}{=} D_a^{n-\alpha} T_a^{n-1}(t), & on\ (a,b) \\[1em] T_a^{n-1} = T_{a^+}^{n-1} f \quad if\ exists & \end{array}\right.$$

Equality holds if in addition $f \in CL^1(a,b)$.

Proof.

$$D^\alpha f(t) := D^n I^{n-\alpha} f(t) = 0 \underset{lem\ 121}{\Longrightarrow} I^{n-\alpha} f(t) = T_a^{n-1}(t), \quad t \in (a,b).$$

Now

$$f(t) \overset{a.e.}{=} D_a^{n-\alpha} I_a^{n-\alpha} f(t) = \text{ result} \qquad \text{on } (a,b).$$

Equality holds since

$$f(t) = D_a^{n-\alpha} I_a^{n-\alpha} f(t), \quad \forall\, t \in (a,b).$$

Now if $T_{a^+}^{n-1} f$ exists then,

$$I^{n-\alpha} f(t) = T_a^{n-1}(t) = c_0 + c_1(t-a) + \cdots + c_{n-1}(t-a)^{n-1} \quad \Longrightarrow$$

$$D_a^{k-n+\alpha} f(a^+) = D^k I_a^{n-\alpha} f(a^+) = c_k. \quad \blacksquare$$

The $\Longleftarrow$ is clear.

Remark 38 *$f \in CL^1(a,b]$, then the fractional differential equation $D_a^\alpha f = 0$, $0 < \alpha < 1$, has $f = c(t-a)^{\alpha-1}$, $c \in \mathbb{R}$ as unique solutions.*

However, if $f \in L^1(a,b)$, then $f(t) \overset{a.e.}{=} c\,(t-a)^{\alpha-1}$.

Lemma 303

$$\left.\begin{array}{l} f \in C[a,b] \\[1em] D_a^\alpha f(t) = 0 \quad on\ (a,b) \\[1em] \alpha > 0 \end{array}\right\} \quad \Longrightarrow \quad \left\{\begin{array}{c} I^{n-\alpha} f(a) = 0 \ and\ thus \\[1em] f(t) = \sum_{k=1}^{n-1} c_k\ (t-a)^{\alpha-n+k} \quad on\ [a,b] \\[1em] \Downarrow \\[1em] 0 < \alpha < 1 \quad \Longrightarrow \quad f \equiv 0\ on\ [a,b]. \end{array}\right.$$

Proof. From Lemma 302 we have

$$f(t) = D_a^{n-\alpha} T_a^{n-1}(t) = \sum_{k=1}^{n} c_k (t-a)^{\alpha-k}, \quad t \in [a,b] \quad \Longrightarrow \quad c_n = 0, \text{ since } \alpha - n < 0.$$

When $0 < \alpha < 1$, $n = 1$ and the summation vanishes. $\blacksquare$

12.4 Necessary conditions of $D_a^\alpha f$ existence

Lemma 304

$$\boxed{D_a^\alpha f \text{ defined and bounded on } [a,b] \quad \Longrightarrow \quad I_a^{n-\alpha} f \in AC^n[a,b], \quad n = [\alpha]+1.}$$

Proof.

$$D_a^\alpha f = D^n I_a^{n-\alpha} f = D D^{n-1} I_a^{n-\alpha} f \quad \overset{cor\,62}{\Longrightarrow} \quad I_a^{n-\alpha} f \in AC^n[a,b]. \quad \blacksquare$$

Lemma 305

$$\boxed{D_a^\alpha f \in C[a,b] \quad \Longrightarrow \quad I_a^{n-\alpha} f \in AC^n[a,b], \quad n = [\alpha]+1.}$$

Proof.

$$D_a^\alpha f = D^n I_a^{n-\alpha} f \in C[a,b] \quad \overset{cor\,85}{\Longrightarrow} \quad I_a^{n-\alpha} f \in C^n[a,b] \subset AC^n[a,b]. \quad \blacksquare$$

Lemma 306

$$\boxed{D_a^\alpha f := D^n I_a^{n-\alpha} f \text{ exists everywhere on } [a,b] \begin{cases} \Longrightarrow & I_a^{n-\alpha} f \in C^{n-1}[a,b], \\ \not\Longrightarrow & I_a^{n-\alpha} f \in AC^n[a,b]. \end{cases}}$$

Proof. Apply Lemma 105 to $I_a^{n-\alpha} f$. The second part is because there is an f such that Df exists everywhere but $Df \notin L^1(a,b)$.

Lemma 307

$$\boxed{\left.\begin{array}{l} D_a^\alpha f \text{ exists at every } t \in [a,b] \\ \\ D_a^\alpha f \in L^1(a,b) \end{array}\right\} \quad \Longrightarrow \quad I_a^{n-\alpha} f \in AC^n[a,b].}$$

Proof. From Corollary 106.

Corollary 308

$$\boxed{\left.\begin{array}{l} I_a^{n-\alpha} f \in C^{n-1}[a,b] \\ \\ D_a^\alpha f \text{ exists n.e. in } [a,b] \\ \\ D_a^\alpha f \in L^1(a,b) \end{array}\right\} \quad \Longrightarrow \quad I_a^{n-\alpha} f \in AC^n[a,b].}$$

Proof. From Corollary 91.

Lemma 309

Let $n = -[-\alpha]$.

$$\boxed{D_a^\alpha f \in CL^1(a,b) \quad \Longrightarrow \quad \begin{cases} D_a^{k-n+\alpha} f := D^k I_a^{n-\alpha} f \in C(a,b) \text{ and bounded in } (a,b), \\ \text{with } D_a^{k-n+\alpha} f(a^+) \text{ exists, } k = 0,\ldots,n-1. \text{ Also} \\ I_a^{n-\alpha} f(t) = I_a^n D_a^\alpha f(t) + T_{a^+}^{n-1} I_a^{n-\alpha} f(t), \qquad t \in (a,b), \\ \Downarrow \\ f(t) \overset{a.e.}{=} I_a^\alpha D_a^\alpha f(t) + D_a^{n-\alpha} T_{a^+}^{n-1} I_a^{n-\alpha} f(t),\ t \in (a,b). \\ \text{equality holds if } f \in CL^1(a,b). \end{cases}}$$

where

$$\boxed{T_{a^+}^{n-1} I^{n-\alpha} f(t) = \sum_{k=0}^{n-1} \frac{D^{k-n+\alpha} f(a^+)}{k!} (t-a)^k = \sum_{j=1}^{n} \frac{D^{\alpha-j} f(a^+)}{(n-j)!} (t-a)^{n-j}.}$$

Proof.

For $\alpha \in \mathbb{N}$ the result reduces to the result in Lemma 108.

Let $\alpha \notin \mathbb{N}$. Then the formual for $I_a^{n-\alpha} f$ follows by applying Lemma 108 to $I_a^{n-\alpha} f$. Also from that lemma $D^k I_a^{n-\alpha} f$ exists and in $C(a,b)$. Since $[k-n+\alpha]+1 = k$, we can write

$$D^k I_a^{n-\alpha} f = D_a^{[k-n+\alpha]+1} I_a^{[k-n+\alpha]+1-(k-n+\alpha)} f \overset{def}{=} D^{k-n+\alpha} f$$

(shown again in Lemma 325). By change of indices we obtain the formula for $T_a^{n-1} I_a^{n-\alpha} f$.

For the representation of f recall that $I_a^n f := I^{n-\alpha} I_a^\alpha f$ and thus we can write

$$I_a^{n-\alpha} f(t) = I^{n-\alpha} I_a^\alpha D_a^\alpha f(t) + T_{a^+}^{n-1} I_a^{n-\alpha} f(t), \qquad t \in (a,b).$$

By applying $D_a^{n-\alpha}$ to both side we obtain

$$f(t) \overset{a.e.}{=} D_a^{n-\alpha} I_a^{n-\alpha} f(t) = I_a^\alpha D_a^\alpha f(t) + D_a^{n-\alpha} T_{a^+}^{n-1} I_a^{n-\alpha} f(t), \qquad t \in (a,b).$$

Proof (Using zero derivative property).

$$D_a^\alpha f \in CL^1(a,b) \quad \begin{cases} \underset{lem\,227}{\Longrightarrow} \quad I_a^\alpha D_a^\alpha f \in CL^1(a,b) \\ \underset{lem\,132}{\Longrightarrow} \quad D_a^\alpha f = D_a^\alpha I_a^\alpha D_a^\alpha f, \quad \text{on } (a,b). \end{cases}$$

$$\Longrightarrow \quad D_a^\alpha \left[f - I_a^\alpha D_a^\alpha f \right] = 0 \text{ on } (a,b) \quad \underset{lem\,302}{\Longrightarrow} \quad f(t) \overset{a.e}{=} I_a^\alpha D_a^\alpha f(t) + D_a^{n-\alpha} T_{a^+}^{n-1} I_a^{n-\alpha} f(t).$$

$$\Longrightarrow \quad I_a^{1-\alpha} f(t) \overset{a.e}{=} I_a^{1-\alpha} \left[I_a^\alpha D_a^\alpha f + c\,(t-a)^{\alpha-1} \right] = I_a D_a^\alpha f(t) + c\,\Gamma(\alpha)$$

$$\Longrightarrow \quad I_a^{1-\alpha} f(a^+) = c\,\Gamma(\alpha). \quad \blacksquare$$

See also [5], proposition 2.4. ■

Corollary 310

$$\boxed{\left.\begin{array}{l} I^{n-\alpha} f \in C^{n-1}[a,b] \\ \\ D_a^{\alpha} f \in CL^1(a,b) \end{array}\right\} \quad \Longrightarrow \quad I^{n-\alpha} f \in AC^n[a,b].}$$

Proof. Follows from Corollary 109.

Remark 39

$$\boxed{\left.\begin{array}{l} I^{n-\alpha} f \in C^{n-1}[a,b] \\ \\ D_a^{\alpha} f \in L^1(a,b) \end{array}\right\} \quad \not\Longrightarrow \quad I^{n-\alpha} f \in AC^n[a,b].}$$

For example if $I_0^{1-\alpha} f(t) = Cn(t) \in C[0,1]$, the Cantor function, then

$$D_0^{\alpha} f = D I_0^{1-\alpha} f = DCn(t) \stackrel{a.e.}{=} 0 \in L^1(0,1).$$

However $Cn(t) \notin AC[0,1]$.

On the other hand it is not clear that

$$f(t) = D_0^{1-\alpha} Cn(t) = D I_0^{\alpha} Cn(t).$$

exists ????

Bibliography

[1] C. Aliprantis and O. Burkinshaw. *Principles of Real Analysis.* Academic Press, 1990.

[2] G. A. Anastassiou. General fractional Opial type inequalities. *Acta Applicandae Mathematicae*, 54:303–317, 1998.

[3] G. A. Anastassiou. Opial type inequalities involving fractional derivatives of functions. *Nonlinear Studies*, 6:207–230, 1999.

[4] B. Bonilla, J. J. Trujillo, and M. Rivero. Fractional order continuity and some properties about integrability and differentiability of real functions. *Journal of Mathematical Analysis and Applications*, 231:205–212, 1999.

[5] D. Delbosco and L. Rodino. Existence and uniqueness for a nonlinear fractional differential equation. *Journal of Mathematical Analysis and Applications*, 204:609–625, 1996.

[6] B. R. Gelbaum and J. M. Olmsted. *Theorems and Counterexamples in Mathematics.* Springer-Verlag, New York, 1990.

[7] R. Gorenflo and S. Vessella. *Abel Integral Equations: Analysis and Applications.* Springer-Verlag, 1991.

[8] F. Jones. *Lebesgue Integration on Euclidean Spaces.* Jones and Barlett Publishers, 1993.

[9] A. A. Kilbas, B. Bonilla, and J. J. Trujillo. Fractional integrals and derivatives, and differential equations of fractional order in weighted spaces of continuous functions (russian). *Dokl. Nats. Akad. Nauk Belarusi*, 44(6):18–22, 2000.

[10] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and Applications of Fractional Differential Equations*, volume 204 of *Mathematics Studies.* Elsevier, Amsterdam, 2006.

[11] J. J. Koliha. Fundamental theorem of calculus for Lebesgue integration. *American Mathematical Monthly*, 113:551–555, 2006.

[12] A. N. Kolmogorov and S. V. Fomin. *Introductory Real Analysis.* Dover Publications, 1975.

[13] K. S. Miller and B. Ross. *An Introduction to the Fractional Calculus and Fractional Differential Equations.* John Wiley & Sons, Inc. 1993.

[14] I. P. Natanson. *Theory of Functions of a Real Variable*, volume I. Frederick Ungar Publishing. Co., New York, english trans. 3rd printing 1964 edition, 1955.

[15] E. Pitcher and W. E. Sewell. Existence theorems for solutions of differential equations of non-integral order. *Bulletin of the American Mathematical Society*, 44(2):100–107, 1938.

[16] I. Podlubny. *Fractional Differential Equations*, volume 198 of *Mathematics in Science and Engineering*. Acad. Press, 1999.

[17] I. Podlubny. Geometric and physical interpretation of fractional integration and fractional differentiation. *Fractional Calculus & Applied Analysis*, 5(4):367–386, 2002.

[18] H. Royden. *Real Analysis*. Prentice Hall, 1988.

[19] W. Rudin. *Real and Complex Analysis*. McGraw-Hill Science, 3rd edition, 1987.

[20] S. G. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional Integrals and Derivatives, Theory and Applications*. Gordon and Breach, Amsterdam, 1993. Engl. Trans. from the Russian 1987.

[21] M. R. Spiegel. *Theory and Problems of Real Variables*. Schaum's Outline Series. McGraw-Hill Science, 1990.

[22] C. zhi Bai and J. xuan Fang. The existence of a positive solution for a singular coupled system of nonlinear fractional differential equations. *Applied Mathematics and Computation*, 150:611–621, 2004.