Studying stability of the trivial solution (x=0)

Case(1)

$$\dot{x} = Ax + B(t)x + f(t,x)$$

$$\dot{x} = Ax, \quad \lambda_k \; eigen.of \; A$$

$$Re\,\lambda_k \le 0, \quad x(t) bdd \quad \Longrightarrow x = 0 \; is \; stable$$

$$Re\,\lambda_k < 0, \quad \|x(t)\| \le C\|x_0\|e^{-\mu t}, \quad \lim_{t\to\infty} x(t) = 0 \quad x = 0 \; asymp.stable$$

$$Re\,\lambda_k > 0, \quad for \; some \; k \quad \Longrightarrow x = 0 \; is \; unstable$$

$$\dot{x} = Ax + B(t)x, \quad \lambda_k \; eigen.of \; A$$

$$Re\,\lambda_k \le 0, \quad \int_{t_0}^{\infty} \|B(t)\| \; bdd. \quad sol.x(t) is \; bdd., \quad x = 0 \; is \; stable \; in \; the$$
Lyapunov sense

$$Re\,\lambda_k < 0, \quad \lim_{t\to\infty} \|B(t)\| = 0 \quad \Longrightarrow \lim_{t\to\infty} x(t) = 0, \quad x = 0 \; asymp.stable$$

$$\dot{x} = Ax + B(t)x + f(t,x), \quad \lambda_k \; eigen.of \; A$$

$$Re\,\lambda_k < 0, \quad \lim_{t\to\infty} \|B(t)\| = 0, \quad \lim_{\|x\|\to 0} \frac{\|f(t,x)\|}{\|x\|} = 0 \quad \Longrightarrow x = 0 \; asymp.stable$$

$$Re\,\lambda_k > 0, for \; some \; k \quad \lim_{t\to\infty} \|B(t)\| = 0, \quad \lim_{\|x\|\to 0} \frac{\|f(t,x)\|}{\|x\|} = 0 \quad \Longrightarrow x = 0 \; unstable$$

Case (2)

$\dot{x} = A(t)x + f(t,x)$

$\dot{x} = A(t)x$

$A(t)\ T - periodic$
(Floquet Theorem)

$$\sum_{i=1}^{n} \lambda_i > 0$$
$\Longrightarrow x = 0\ unstable$

$$\sum_{i=1}^{n} \lambda_i \leq 0$$
No conclusion about $x = 0$

$A(t)\ non\ periodic$

Rewrite $\dot{x} = A(t)x$ in the form $\dot{x} = Ax + B(t)x$ and apply the case (1)

$\dot{x} = A(t)x + f(t,x)$,
$\#A(t)\ T - periodic$,
$$\lim_{\|x\| \to 0} \frac{\|f(t,x)\|}{\|x\|} = 0,$$
#Real part of charactr. Exponent of $\dot{y} = A(t)y$ are negative
$\Longrightarrow x = 0\ asymp.stable$

Case (3)

$\dot{x} = f(x)$

If $\dot{x} = f(x)$ has periodic soln. then rewrite it in the form $\dot{x} = A(t)x$ where $A(t)\ is\ T - periodic$. In this case one of characteristic exponent is zero say λ_1 another given by

$\lambda_2 = \frac{1}{T}\int_0^T Tr\ A(t)dt.$

If $\lambda_2 < 0$ thus the periodic soln. is stable

First integral

$F(x) = C$

$x = a\ is\ C.P.with\ \nabla F(a) = 0$

$\frac{\partial^2 F}{\partial x^2}(a) \neq 0$

Applying Morse Lemma and the characterization of the full nonlinear obtained