8 Stability analysis by the direct method

8.1 Introduction

- So far we used the method of linearization to study the stability of a solution.
- Linearization starts with small perturbations of the equilibrium or periodic solution and then study the local effects of these local perturbations.
- In this chapter we introduce direct methods for characterizing the solutions in a way with respect to stability which is not necessarily local.

Example. *Consider the system*

$$\begin{aligned}\dot{x} &= ax - y + kx(x^2+y^2) \\ \dot{y} &= x - ay + ky(x^2+y^2), \qquad a > 0, \quad k \text{ constant.}\end{aligned}$$

Where we are interested in the stability of the trivial solution.

Linearization in a neighborhood of $(0,0)$ yields

$$A = \begin{pmatrix} a & -1 \\ 1 & -a \end{pmatrix}, \qquad \lambda_{1,2} = \pm\sqrt{a^2-1}$$

So in the linear approximation we find for the critical point $(0,0)$

$a^2 > 1$	saddle
$a^2 = 1$	degenerate case
$0 < a^2 < 1$	center

For the nonlinear problem, the instability theorem (theorem 7.3) implies that for $a^2 > 1$ the trivial solution is unstable.

If $0 < a^2 \leq 1$ the method of linearization of chapter 7 is not conclusive.

Alternative approach for $0 < a^2 \leq 1$. Consider a one-parameter family of ellipses around $(0,0)$:

$$x^2 - 2axy + y^2 = c.$$

Let ψ be the intersection angle of an orbit and an ellipse defined by the angle between the tangent vector of the orbit and the outward directed normal vector in the point of intersection.

If $\pi/2 < \psi < 3\pi/2$, $\cos\psi < 0$, the orbit enters the interior of this particular ellipse. Thus, if $\cos\psi < 0$ for all solutions and all ellipses in a neighborhood of $(0,0)$ the trivial solution is asymptotically stable.

Computation of ψ. Let

$$V = x^2 - 2axy + y^2 = c.$$

Then the normal vector field is

$$\nabla V = (V_x, v_y) = 2(x - ay, -ax + y)$$

The tangent vector $\vec{\tau}$ of the orbit is given by $\vec{\tau} = (\dot{x}, \dot{y})$. Thus

$$\cos\psi = \frac{\nabla V \cdot \vec{\tau}}{\|\nabla V\| \, \|\vec{\tau}\|}$$

The sign of $\cos\psi$ is determined by the numerator

$$\nabla V \cdot \vec{\tau} = (V_x, V_y) \cdot (\dot{x}, \dot{y}) = V_x\dot{x} + V_y\dot{y} = L_t V.$$

which is the orbital derivative of the function V. From the system

$$L_t V = (2x - 2ay)\,\dot{x} + (-2ax + 2y)\,\dot{y} = 2k(x^2 + y^2)(x^2 + y^2 - 2axy).$$

So

$$\cos\psi \begin{cases} < 0, & k < 0, \\ > 0, & k > 0. \end{cases}$$

The result holds for all ellipses and all orbits in a neighborhood of $(0,0)$ so we conclude that the trivial solution is asymptotically stable if $k < 0$, unstable if $k > 0$.

Actually, since the result holds for all orbits, for $k < 0$ and $0 < a^2 < 1$ we have global stability of $(0,0)$.

8.2 Lyapunov functions and applications

Consider the equation

$$\dot{x} = f(t,x), \qquad t \geq t_0, \quad x \in D \subset \mathbb{R}^n.$$

Assume $x = 0$ is a solution, so $f(t,0) = 0$, $t \geq t_0$, $0 \in D$.

Definition. *$V(t,x)$ is a scalar function defined and continuously differentiable in $[t_0,\infty)\times D$, $0 \in D \subset \mathbb{R}^n$, and*

$$V(t,0) = 0.$$

We write $V(x)$ if V is independent of t.

Definition. *The function $V(x)$ (with $V(0) = 0$) is called positively (negatively) definite in D if $V(x) > 0$ (< 0) for $x \in D$, $x \neq 0$.*

Definition. *The function $V(x)$ (with $V(0) = 0$) is called positively (negatively) semidefinite in D if $V(x) \geq 0$ (≤ 0) for $x \in D$.*

Definition. *The function $V(t,x)$ is called positively (negatively) definite in D if there exists a function $W(x)$ with the following properties:*

- *$W(x)$ is defined and continuous in D.*
- *$W(0) = 0$,*
- *$0 < W(x) \leq V(t,x)$, ($V(t,x) \leq W(x) < 0$) for $x \neq 0$, $t \geq t_0$.*

To define semidefinite functions $V(t,x)$ we replace $<$ by $\leq$.

Example. *Quadratic functions with positive coefficient are definite functions which are used very often.*

Consider in $\mathbb{R}^3$ the subset $D = \{(x,y,z) : x^2 + y^2 + z^2 \leq 1\}$ and for $t \geq 0$ the function

$x^2 + 2y^2 + 3z^2 + z^3$	*positive definite*
$x^2 + z^2$	*positive semidefinite*
$x^2 + y^2 + \cos^3 tz^2$	*not sign definite*

Extension of the concept of orbital derivative.

Definition. *The orbital derivative* L_t *of the function* $V(t,x)$ *in the direction of the vector field* $f(t,x)$*, where* x *is a solution of* $\dot{x} = f(t,x)$ *is*

$$\begin{aligned} L_t V = V_t + V_x \dot{x} &= V_t + V_x f(t,x) \\ &= V_t + V_{x_1} f_1(t,x) + \cdots + V_{x_n} f_n(t,x). \end{aligned}$$

with $x = (x_1, \ldots, x_n)$ *and* $f = (f_1, \ldots, f_n)$*.*

Theorem. *Consider the equation* $\dot{x} = f(t,x)$ *with* $f(t,0) = 0$*,* $x \in D \subset \mathbb{R}^n$*,* $t \geq t_0$*. If a function* $V(t,x)$ *can be found, defined in a neighborhood of* $x = 0$ *such that in this neighborhood*

- $V(t,x)$ *positively definite for* $t \geq t_0$
- $L_t V$ *is negatively semidefinite.*

Then the solution $x = 0$ *is stable in the sense of Lyapunov.*

Proof. By assumption, in a neighborhood of $x = 0$ we have for some $R > 0$ and $\|x\| \leq R$,

$$V(t,x) \geq W(x) > 0, \qquad x \neq 0, \quad t > t_0,$$

and

$$L_t V \leq 0.$$

Consider the ball

$$B = \{x : 0 < r \leq \|x\| \leq R\}.$$

Let

$$m = \min_{x \in B} W(x).$$

Consider the neighborhood of $x = 0$,

$$S = \{x : V(t,x) < m\}$$

S exist since $V(t,x)$ is continuous and positively definite while $V(t,0) = 0$.

Starting a solution in S at $t = t_0$, we have

$$V(t, x(t)) - V(t_0, x(t_0)) = \int_{t_0}^{t} L_t V(\tau, x(\tau))\, d\tau \leq 0, \qquad t \geq t_0.$$

Thus the function $V(t, x(t))$ can not increase along a solution and this would be necessary to enter B as initially $V(t_0, x(t_0)) < m$. Therefor, the solution can never enter B. Since R is arbitrary ($\rightarrow 0$), the result follows. ■

Remark.

- *The scalar function* $V(t,x)$ *is called a* <u>*Lyapunov function*</u>*.*
- *For each class of problems, the construction of the function varies and there is not general method.*
- *In the previous theorem we have assumed that the orbital derivative* $L_t V$ *is semidefinite negative. This includes the case that* $L_t V = 0$*,* $t \geq t_0$*,* $x \in D$*. This means that* $V(t,x)$ *is a first integral of the equation.*

Next, the orbital derivative is used to obtain a strong form of stability.

Theorem. *Consider the equation*

$$\dot{x} = f(t,x), \qquad f(t,0) = 0, \qquad x \in D \subset \mathbb{R}^n, \quad t \geq t_0.$$

If a function $V(t,x)$ can be found, defined in a neighborhood of $x = 0$, such that in this neighborhood

- *$V(t,x)$ is positively definite $t \geq t_0$.*
- *$L_t V$ is negative definite.*

Then the solution $x = 0$ is asymptotically stable.

Proof. From the previous theorem, $x = 0$ is stable. If $x = 0$ is not asymptotically stable then there is a solution $x(t)$ and a constant a such that

$$\|x(t)\| \geq a, \qquad \text{for } t \geq t_0,$$

when starting arbitrarily close to zero. Let

$$B = \{x : 0 < a \leq \|x(t)\| \leq R\}, \qquad t \geq t_0.$$

By assumption we have $L_t V(t,x) \leq W(x) < 0$, $x \neq 0$. So we have in B

$$L_t V \leq -\mu, \qquad \mu > 0,$$

so that

$$V(t,x(t)) - V(t_0, x(t_0)) = \int_{t_0}^{t} L_t V(\tau, x(\tau))\, d\tau \leq -\mu(t - t_0).$$

or

$$V(t,x) \leq V(t_0, x(t_0)) - \mu(t - t_0)$$

Thus for sufficiently large time, $V(t,x)$ becomes negative. This contradicts the assumption that $V(t,x)$ is positively definite. ■

Example. *Consider the equation $\dot{x} = Ax + f(x)$ with*

$$\lim_{\|x\| \to 0} \frac{\|f(x)\|}{\|x\|} = 0.$$

$A = diag(\lambda_1, \ldots, \lambda_n)$, $\lambda_k < 0$. i.e. $\dot{x}_i = \lambda_i x_i + f_i(x)$.

By Poincaré-Lyapunov theorem, $x = 0$ is asymptotically stable. Alternatively, consider the Lyapunov function

$$V(x) = \sum_{i=1}^{n} x_i^2.$$

Then

$$L_t V = 2 \sum_{i=1}^{n} x_i \dot{x}_i = 2 \sum_{i=1}^{n} \lambda_i x_i^2 + 2 \sum_{i=1}^{n} x_i f_i(x).$$

As $\lambda_i < 0$, $i = 1, \ldots, n$, and from the limit of f, $L_t V$ is negatively definite in a neighborhood of $x = 0$. $V(x)$ is a Lyapunov function and by previous theorem $x = 0$ is asymptotically stable.

Definition. *Consider the equation $\dot{x} = f(x)$ and suppose that $x = 0$ is an asymptotically stable solution. A set of point x_0 with the property that for the solution of*

$$\dot{x} = f(x), \qquad x(0) = x_0$$

we have $x(t) \to 0$ for $t \to \infty$, is called a domain of attraction of $x = 0$.

The following corollary to the previous theorem shows how to use Lyapunov function to characterize the domain of attraction.

Corollary. *Consider the equation*

$$\dot{x} = f(x), \qquad f(0) = 0.$$

Suppose the following

- *The Lyapunov function $V(x)$ is positively definite for $\|x\| \le R$.*
- *S is a closed $n-1$-dimensional manifold which encloses $x = 0$ and which is contained in the ball with radius R.*

Suppose

- *$L_t V < 0$, x in the interior of S.*
- *$L_t V = 0$, $x \in \partial S$.*
- *$L_t V > 0$, x outside S.*
- *The $(n-1)$-dimensional manifold $V(x) = c$, positive constant, is entirely contained in the interior of S.*

Then the set defined by $V(x) \le c$ in the ball $\|x\| \le R$ is a domain of attraction of $x = 0$

Using a Lyapunov function one can also establish the instability of a solution.

Theorem. *Consider equation*

$$\dot{x} = f(t, x), \qquad f(0) = 0, \qquad x \in D \subset \mathbb{R}^n, \quad t \ge t_0.$$

If there exists a function $V(t, x)$ in a neighborhood of $x = 0$ such that

a. $V(t, x) \to 0$ for $\|x\| \to 0$, uniformly in t;

b. $L_t V$ is positively definite in a neighborhood of $x = 0$;

c. from a certain value $t = t_1 \ge t_0$, $V(t, x)$ takes positive values in each sufficiently small neighborhood of $x = 0$;

then the trivial solution is unstable.

Proof. By contradiction. Suppose $x = 0$ is stable and shows that V becomes arbitrarily large in the neighborhood of zero. ■

Example. *Consider the system*

$$\begin{aligned} \dot{x} &= a(t)\, y + b(t)\ x(x^2+y^2) \\ \dot{y} &= -a(t)\, x + b(t)\, y\, (x^2+y^2) \end{aligned}$$

The functions $a(t)$ and $b(t)$ are continuous for $t \geq t_0$. Consider the Lyapunov function

$$V(x,y) = x^2 + y^2.$$

Then V is positively definite and

$$L_t V = 2b(t)\,(x^2+y^2)^2.$$

Thus, from the first theorem of this section the trivial solution $(0,0)$ is stable if $b(t) \leq 0$. By the previous theorem, $(0,0)$ is unstable if $b(t) > 0$ for $t \geq t_0$.

Example. *Consider the nonlinear oscillator with linear damping equation*

$$\ddot{x} + \mu\dot{x} + x + ax^2 + bx^3 = 0, \qquad \mu > 0.$$

As in example 7.1, it follows from Poincaré-Lyapunov theorem that the trivial solution is asymptotically stable.

Alternatively, we introduce the energy of the nonlinear oscillator without damping

$$V(x,\dot{x}) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{1}{3}ax^3 + \frac{1}{4}bx^4.$$

We can find a neighborhood D of $(0,0)$, dependent in size on a and b, in which V is positive definite. Furthermore,

$$L_t V = \dot{x}\ddot{x} + x\dot{x} + ax^2\dot{x} + bx^3\dot{x} = -\mu\dot{x}^2.$$

and thus $L_t V \leq 0$ in D. So we can conclude that $(0,0)$ is stable but we can not obtain asymptotically stable.

Note that $L_t V = 0 \implies \dot{x} = 0$. However, if $x \neq 0$, $\dot{x} = 0$ is a transversal of the phase-flow so we conclude that D is a domain of attraction of the trivial solution.

8.3 Hamiltonian systems and systems with first integrals

- Hamilton's equations provide a new and equivalent way of looking at classical mechanics.
- Generally, these equations do not provide a more convenient way of solving a particular problem. Rather, they provide deeper insights into both the general structure of classical mechanics and its connection to quantum mechanics as understood through Hamiltonian mechanics, as well as its connection to other areas of science.
- The value of the Hamiltonian is the total energy of the system being described. For a closed system, it is the sum of the kinetic and potential energy in the system. There is a set of differential equations known as the Hamilton equations which give the time evolution of the system.
- Hamiltonians can be used to describe such simple systems as a bouncing ball, a pendulum or an oscillating spring in which energy changes from kinetic to potential and back again over time.
- Hamiltonians can also be employed to model the energy of other more complex dynamic systems such as planetary orbits in celestial mechanics and also in quantum mechanics.

Hamilton's equations are written as

$$\dot{p} = -\frac{\partial H}{\partial q_i}, \qquad \dot{q} = \frac{\partial H}{\partial p_i}, \qquad i = 1, \dots n,$$

where $H = H(p, q) : \mathbb{R}^{2n} \to \mathbb{R}$ is a twice continuously differentiable function. Note that

$$L_t H = 0$$

and thus H is a first integral of the equations.

Theorem. *Consider Hamilton's equations. We assume that they admit the trivial solution. If $H(p, q) - H(0, 0)$ is sign definite in a neighborhood of $(p, q) = (0, 0)$, then the trivial solution is stable in the sense of Lyapunov (since $L_t H \leq 0$).*

Proof. Let $V(p, q) = H(p, q) - H(0, 0)$. Then V is sign definite, $L_t V = L_t H \leq 0$ and $V(0, 0) = 0$. Thus $(0, 0)$ is stable.

Hamiltonian systems

Mechanical systems in which the force field can be derived from a potential $\phi(q)$ are characterized in many cases by the Hamiltonian

$$H(p,q) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \phi(q).$$

Theorem. *Consider the Hamilton function*

$$H(p,q) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \phi(q), \qquad (p,q) \in \mathbb{R}^{2n},$$

with potential $\phi(q)$ which can be expanded in a Taylor series in a neighborhood of each critical point. An isolated minimum of the potential corresponds with a stable equilibrium solution, an isolated maximum corresponds with an unstable equilibrium solution.

Proof.
The Hamilton's equations for this mechanical system

$$\dot{p}_i = -\frac{\partial \phi}{\partial q_i}, \qquad \dot{q}_i = p_i.$$

Equilibrium solutions corresponding with the critical points determined by

$$p_i = 0, \qquad \frac{\partial \phi}{\partial q_i} = 0, \qquad i = 1, \dots n.$$

If the equilibrium point corresponds with an isolated minimum of $\phi(q)$, then $H(p,q) - H(CritPoint) > 0$, and thus by theorem 8.4 this equilibrium solution is stable.
If the equilibrium point corresponds with an isolated maximum of $\phi(q)$ then we can introduce the Lyapunov function

$$V(p,q) = \sum_{i=1}^{n} p_i q_i.$$

Then we can show that in neighborhoods of this critical point V is positive and $L_t V$ is positive definite. Thus the critical point is unstable. **Proof.**

Remark. *The equation of motion of the Hamiltonian system is*

$$\ddot{q} = -\frac{\partial \phi}{\partial q_i}, \qquad i = 1, \dots n.$$

Example. *Consider the equation of motion*

$$\ddot{x} + f(x) = 0$$

Let $(p, q) = (\dot{x}, x)$ then

$$\dot{p} = \ddot{x} = -f(q), \qquad \dot{q} = p$$

and we obtain the Hamilton function

$$H(p, q) = \frac{1}{2}p^2 + \int^q f(\tau)\, d\tau.$$

In this case $\phi(q) = \int^q f(\tau)\, d\tau$

According to the previous theorem, The isolated minima correspond with stable equilibrium solutions and isolated maxima correspond with unstable equilibrium solutions. See figure 8.3.

Example. *Consider the Hamilton function*

$$H(p, q) = \frac{1}{2}(p_1^2 + p_2^2) + \underbrace{\frac{1}{2}(q_1^2 + q_2^2) + q_1^2 q_2 - \frac{1}{3}q_2^3}_{\phi(q_1, q_2)}.$$

The function $\phi(q)$ has the following critical points:

$$\phi_{q_1} = q_1 + 2q_1q_2 = q_1(1 + 2q_2), \qquad \phi_{q_2} = q_2 + q_1^2 - q_2^2$$

$(0, 0)$,	isolated minimum
$(0, 1), (\pm\sqrt{3}/2, -1/2)$,	no maximum or minimum is assumed

By previous theorem $(0, 0, 0, 0)$ is a stable equilibrium solution.

For the other critical points, $(0, 0, 0, 1)$ and $(0, 0, \pm\sqrt{3}/2, -1/2)$ we can perform linearization then apply theorem 7.3 for the system $\dot{x} = Ax + f(x)$ to conclude instability.
