7 Stability by Linearization

- One way to study the stability of the equilibrium solutions and periodic solutions is by analyzing the system, linearized in a neighborhood of these special solutions.
- Justification of linearization methods has been started by Poincaré and Lyapunov since 1900.
- A more recent result is the "stable and unstable manifold theorem" 3.3. This theorem is concerned with autonomous equations of the form

  $$\dot{x} = Ax + g(x),$$

  with A a constant $n \times n$ matrix of which all eigenvalues have nonzero real part.
- Theorem 3.3 establishes that the stable and unstable manifold E_s and E_u of the linearized equation can be continued on adding the nonlinearity $g(x)$; the manifolds of the nonlinear equation W_s and W_u emanating from the origin are tangent to E_s and E_u.
- Theorem 3.3 discusses only the existence of invariant manifolds and not very explicitly the behavior with time of individual solutions.
- Thus the theorem does not characterize the stability of the trivial solution.
- In this chapter we
  - add the quantitative element to the theory.
  - obtain more general results
  - consider the stability of the trivial solution for nonautonomous equations.

7.1 Asymptotic stability of the trivial solution

Note. (Fundamental matrix of linear equation with constant coefficient)

For the equation $\dot{x} = Ax$, *the matrix* e^{At} *is a fundamental matrix. Thus any fundamental matrix* $\Phi(t)$ *can be written as*

$$\Phi(t) = e^{At}C \quad \Longrightarrow \quad \Phi(t) = e^{A(t-t_0)}\Phi(t_0)$$

In particular, any fundamental matrix $\Phi(t)$ *with* $\Phi(t_0) = I$ *satisfies*

$$\Phi(t) = e^{A(t-t_0)},$$

and

$$\Phi(t)\,\Phi^{-1}(\tau) = e^{A(t-t_0)}e^{-A(\tau-t_0)} = e^{A(t-\tau)} = \Phi(t-\tau+t_0).$$

Lemma. *Consider the equation in* $\mathbb{R}^n$

$$\dot{x} = Ax + B(t)x + f(t,x), \qquad x(t_0) = x_0, \quad t \in \mathbb{R}.$$

with $B(t)$ *and* $f(t,x)$ *are continuous for* $t \geq t_0$. *Let* $\Phi(t)$ *be a fundamental matrix of* $\dot{y} = Ay$ *with* $\Phi(t_0) = I$. *Then the initial value problem is equivalent to the integral equation*

$$x(t) = \Phi(t)x_0 + \int_{t_0}^{t} \Phi(t-s+t_0)\,[B(s)x(s) + f(s,x(s))\,]\,ds.$$

Proof. From the note above $\Phi(t) = e^{A(t-t_0)}$. Let $x = \Phi(t)z$, then $z(t_0) = x_0$. Substitute into the equation to obtain

$$\dot{\Phi(t)}z + \Phi(t)\dot{z} = A\Phi(t)\,z + B(t)\,\Phi(t)\,z + f(t,\Phi(t)z).$$

Since $\dot{\Phi} = A\Phi$, we have

$$\Phi(t)\dot{z} = B(t)\,\Phi(t)\,z + f(t,\Phi(t)z)$$

or

$$\dot{z} = \Phi^{-1}(t)\,[B(t)\,x(t) + f(t,x(t))\,].$$

Integration produces

$$z(t) - z(t_0) = \int_{t_0}^{t} \Phi^{-1}(s)\,[B(s)\,x(s) + f(s,x(s)]\,ds$$

or

$$z(t) = x(t_0) + \int_{t_0}^{t} \Phi^{-1}(s)\,[B(s)\,x(s) + f(s,x(s)]\,ds$$

Multiplication by $\Phi(t)$ and using the expression of $\Phi(t)\Phi^{-1}(\tau)$ yields the required result.
■

Theorem. *(Poincaré-Lyapunov) Consider the equation in* $\mathbb{R}^n$

$$\dot{x} = Ax + B(t)x + f(t,x), \qquad x(t_0) = x_0, \quad t \in \mathbb{R}. \tag{4}$$

- A *is a constant* $n \times n$ *matrix with eigenvalues which all have negative real parts;*
- $B(t)$ *is a continuous* $n \times n$ *matrix with the property*
$$\lim_{t\to\infty} \|B(t)\| = 0.$$
- $f(t,x)$ *is continuous in* t *and* x *and Lipschitz continuous in* x *in a neighborhood of* $x = 0$*; moreover,*
$$\lim_{\|x\|\to 0} \frac{\|f(t,x)\|}{\|x\|} = 0 \quad \textit{uniformly in } t.$$
This condition implies that $x = 0$ *is a solution.*

Then there exist positive constants C*,* t_0*,* δ*,* μ *such that*

$$\|x_0\| \le \delta \quad \Longrightarrow \quad \|x(t)\| \le C\|x_0\|\, e^{-\mu(t-t_0)}, \qquad t \ge t_0.$$

The solution $x = 0$ *is asymptotically stable and the attraction is exponential in a* δ*-neighborhood of* $x = 0$*.*

Proof.
Estimates. From theorem 6.1, since all the eigenvalues of A have negative real part, there exist positive constants C and λ such that

$$\|\Phi(t)\| \le C\, e^{-\lambda(t-t_0)}, \qquad t \ge t_0.$$

From the assumption on f, for $\delta_0 > 0$ sufficiently small there exists a constant $b(\delta_0)$ such that

$$\|x\| \le \delta_0 \quad \Longrightarrow \quad \|f(t,x)\| \le b(\delta_0)\|x\|, \qquad t \ge t_0.$$

From the assumption on B, for t_0 sufficiently large

$$\|B(t)\| \le b(\delta_0), \qquad t \ge t_0.$$

Existence & Uniqueness. The existence and uniqueness theorem yields that in a neighborhood of $x = 0$, the solution of the initial value problem (4) exists for $t_0 \le t \le t_1$. From the above theorem, if $\Phi(t)$ is a fundamental set with $\Phi(t_0) = I$, then

$$\|x(t)\| \le \|\Phi(t)\|\|x_0\| + \int_{t_0}^{t} \|\Phi(t-s+t_0)\| \, [\|B(s)\|\|x(s)\| + \|f(s,x(s))\| \,] \, ds.$$

Let $t_0 \le t_2 \le t_1$ be determined by the condition $\|x\| \le \delta_0$. Using the estimates for Φ, B, and f we have for $t_0 \le t \le t_2$

$$\|x(t)\| \le C\, e^{-\lambda(t-t_0)}\|x_0\| + \int_{t_0}^{t} C\, e^{-\lambda(t-s)} 2b\|x(s)\|\, ds$$

so that

$$e^{\lambda(t-t_0)}\|x(t)\| \le C\,\|x_0\| + \int_{t_0}^{t} 2bC\; e^{\lambda(s-t_0)}\|x(s)\|\,ds.$$

We use Gronwal's inequality we obtain

$$e^{\lambda(t-t_0)}\|x(t)\| \le C\,\|x_0\|\; e^{2Cb(t-t_0)},$$

or

$$\|x(t)\| \le C\,\|x_0\|\; e^{(2Cb-\lambda)(t-t_0)}. \tag{5}$$

If δ and consequently b are small enough, the quantity $\mu = \lambda - 2Cb$ is positive and we have the required estimate for $t_0 \le t \le t_2$.

Now we choose $\|x_0\|$ such that $\|x_0\| \le \delta_0$, then $\|x(t)\|$ decreases and the estimate can be repeated on a longer time interval. So the estimate (5) holds for $t \ge t_0$ if $\delta = \min(\delta_0, \delta_0/C)$.
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Example. *(Oscillator with damping)*
Consider the equation

$$\ddot{x} + \mu\dot{x} + \sin x = 0, \qquad \mu > 0. \tag{6}$$

The equivalent system can be written as

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & -\mu\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ x_1 - \sin x_1\end{pmatrix} = A\begin{pmatrix}x_1\\ x_2\end{pmatrix} + f(x_1, x_2)$$

The eigenvalues of the linearized system are

$$\lambda_{1,2} = \frac{-\mu \pm \sqrt{\mu^2 - 4}}{2}.$$

so $Re\,\lambda_{1,2} < 0$. *System (6) satisfies the requirements of the Poincaré-Lyapunov theorem, so the equilibrium solution* $(0,0)$ *is asymptotically stable.*

In the case that the linear part of the equation has periodic coefficients we can apply the theory of Floquet.

Theorem. *Consider the equation in* $\mathbb{R}^n$

$$\dot{x} = A(t)x + f(t,x), \tag{7}$$

with

- $A(t)$ *a* T*-periodic continuous matrix,*
- $f(t,x)$ *is continuous in* t *and* x *and Lipschitz continuous in* x *for* $t \in \mathbb{R}$, x *in a neighborhood of* $x = 0$. *Moreover,*

$$\lim_{\|x\|\to 0} \frac{\|f(t,x)\|}{\|x\|} = 0, \quad \textit{uniformly in } t$$

If the real parts of the characteristic exponents of the linear periodic equation

$$\dot{y} = A(t)\, y$$

are negative, the solution $x = 0$ *of (7) is asymptotically stable. Also the attraction is exponential in a* δ*-neighborhood of* $x = 0$.

Proof. From Floquet theorem, any fundamental matrix $\Phi(t)$ of $\dot{y} = A(t)y$ can be written as $\Phi(t) = P(t)e^{Bt}$, P is T-periodic. Consider the transformation

$$x = P(t)z.$$

Substitution in (7) yields

$$P\dot{z} + \dot{P}z = APz + f(t, Pz).$$

Then

$$P\dot{z} = (AP - \dot{P})\, z + f(t, Pz).$$

But

$$\dot{P} = \dot{\Phi}\, e^{-Bt} + \Phi\, e^{-Bt}(-B) = AP - PB$$

and thus

$$P\dot{z} = PB\, z + f(t, Pz).$$

Thus the transformed equation is

$$\dot{z} = B\, z + P^{-1}(t)\, f(t, P(t)z). \tag{8}$$

By assumption, the constant matrix B has only eigenvalues with negative real parts. The solution $z = 0$ of (8) satisfies the requirements of the Poincaré-Lyapunov theorem from which follows the result.

Remark. *Recall Theorem 3.1:*

Consider the equation $\dot{x} = Ax + g(x)$; if $x = 0$ is a positive (negative) attractor for the linearized equation $\dot{y} = Ay$ then $x = 0$ is a positive (negative) attractor for the nonlinear equation $\dot{x} = Ax + g(x)$

In other wards,

If $x = 0$ is a positive (negative) attractor for the linear equation $\dot{x} = Ax$ then $x = 0$ is a positive (negative) attractor for the nonlinear equation $\dot{x} = Ax + g(x)$ if $\lim_{\|x\|\to 0} \frac{\|g(x)\|}{\|x\|} = 0$.

Then clearly that theorem is a special case of Poincaré-Lyapunov theorem.

Remark. *Example 5.3 show that*

$$\textit{positive attraction by a critical point fo nonlinear equation} \quad \not\Longrightarrow \quad \textit{stability}$$

Since some orbits leaves initially then go back to the point.

Poincaré-Lyapunov theorem tells that under the stated conditions

$$\begin{aligned} &\text{linear approximation has a positive attractor} \\ &\Longrightarrow \quad \text{the solution of the nonlinear problem is asymptotically stable.} \end{aligned}$$

Remark.

$$\begin{aligned} &y = 0 \textit{ is a positive attractor of the linear equation } \dot{y} = Ay + B(t)y \\ &\not\Longrightarrow \quad 0 \textit{ is a asymptotically stable if we add a smooth nonlinear term.} \end{aligned}$$

The condition

$$\lim_{t\to\infty} \|B(t)\| = 0$$

is essential.

Example. *Consider for $t \geq 1$ the system*

$$\begin{aligned} \dot{x}_1 &= -ax_1, \\ \dot{x}_2 &= [-2a + \sin(\ln t) + \cos(\ln t)]x_2 + x_1^2, \qquad a > \frac{1}{2}. \end{aligned}$$

Note that $\lim_{t\to\infty} B(t) \neq 0$ for this system and thus this conditions of Poincaré-Lyapunov theorem is not satisfied. In a neighborhood of 0 the linearized system $\dot{x} = Ax + B(t)x$ is

$$\begin{aligned} \dot{y}_1 &= -ay_1, \\ \dot{y}_1 &= [-2a + \sin(\ln t) + \cos(\ln t)]y_2. \end{aligned}$$

with independent solutions

$$y_1(t) = e^{-at}, \qquad y_2(t) = e^{t\sin(\ln t) - 2at}.$$

This solutions tend to zero as $t \to \infty$.

Substitution of the solution $x_1(t) = c_1 e^{-at}$ into the second equation yields a linear inhomogeneous equation. Using the variation of parameters method we obtain

$$x_2(t) = e^{t \sin(\ln t) - 2at} \left(c_2 + c_1^2 \int_0^t e^{-\tau \sin(\ln \tau)} \, d\tau \right).$$

The solutions are not bounded as $t \to \infty$ unless $c_1 = 0$.

7.2 Instability of the trivial solution

Recall Theorem 3.2:

Consider the equation $\dot{x} = Ax + g(x)$; if A has an eigenvalue with positive real part, then the critical point $x = 0$ is not a positive attractor for the nonlinear equation $\dot{x} = Ax + g(x)$.

In other wards,

If $x = 0$ is unstable for the linear equation $\dot{x} = Ax$ then $x = 0$ is unstable for the nonlinear equation $\dot{x} = Ax + g(x)$.

The following theorem is a more general version of this result.

Theorem. *Consider the equation in* $\mathbb{R}^n$

$$\dot{x} = Ax + B(t)\,x + f(t,x), \qquad t \geq t_0 \tag{9}$$

with

- *A is a constant $n \times n$ matrix with eigenvalues of which at least one has positive real part.*
- *$B(t)$ is a continuous $n \times n$ matrix with the property*

$$\lim_{t\to\infty} \|B(t)\| = 0.$$

- *$f(t,x)$ is continuous in t and x, Lipschitz continuous in x in a neighborhood of $x = 0$ and*

$$\lim_{\|x\|\to 0} \frac{\|f(t,x)\|}{\|x\|} = 0, \qquad \textit{uniformly in } t.$$

Then the trivial solution of (9) is unstable.

Proof. See the textbook.

Example: competing species

Read the textbook.

7.3 Stability of periodic solutions of autonomous equations

In section 5.4, linearization of the autonomous equation $\dot{x} = f(x)$ in a neighborhood of a periodic solution $\phi(t)$ yields the equation

$$\dot{y} = \frac{df}{dx}(\phi(t))\, y$$

This linear equation always has the nontrivial T-periodic solution $\dot{\phi}(t)$. This implies that at least one of the real parts of the characteristic exponents is zero, thus the above theorem does not apply. Instead we have the following result.

Definition.

- *Recall. Let $M \subset \mathbb{R}^n$. The set*

$$U_\eta(M) = \{x \in \mathbb{R}^n : dist(x, M) < \eta\}$$

  is call the η-neighborhood of M.

- *An invariant set M of $\dot{x} = f(x)$ is said to be stable if for any $\epsilon > 0$ there exist $\delta > 0$ such that*

$$x_0 \in U_\delta(M) \quad \Longrightarrow \quad x(t, x_0) \subset U_\epsilon(M),\ \forall\, t \geq 0.$$

  M is asymptotically stable if it is stable and if there $U_b(M)$ for some $b > 0$ such that $x_0 \in U_b(M)$ implies that the solution $x(t, x_0)$ approaches M as $t \to \infty$.

- *If $\phi(t)$ is a nonconstant periodic solution of $\dot{x} = f(x)$ is orbitally stable, asymptotically orbitally stable if the corresponding invariant closed curve Γ generated by $\phi(t)$ is stable, asymptotically stable, respectively.*

- *A periodic solution is said to be asymptotically orbitally stable with asymptotic phase θ_0 if it is asymptotically orbitally stable and there is a δ such that*

$$dist(x_0, \Gamma) < \delta \quad \Longrightarrow \quad \exists\, \theta_0 = \theta(x_0) \ni \lim_{t \to \infty} \|x(t, x_0) - \phi(t + \theta_0)\| = 0.$$

Theorem. *Consider the equation $\dot{x} = f(x)$ which has a T-periodic solution $\phi(t)$; $f(x)$ is continuously differentiable in a domain in $\mathbb{R}^n$, $n > 1$, containing $\phi(t)$. Suppose that linearization of $\dot{x} = f(x)$ in a neighborhood of $\phi(t)$ yields the equation*

$$\dot{y} = \frac{df}{dx}(\phi(t))\, y$$

with characteristic exponents of which one has real part zero (characteristic multiplier one is simple) and $n-1$ exponents have real parts negative (all other characteristic multipliers have modulus < 1). Then the periodic solution $\phi(t)$ is asymptotically orbitally stable with asymptotic phase.

Example. *(generalized Liénard equation)*

Consider the equation

$$\ddot{x} + f(x)\,\dot{x} + g(x) = 0.$$

Suppose this equation has a periodic solution $x = \phi(t)$.

As in chapter 6, the linearized equation is $\dot{y} = A(t)z$ with

$$A(t) = \begin{pmatrix} 0 & 1 \\ \dots & -f(\phi(t)) \end{pmatrix}.$$

The trace of the linearized equation is $\mathrm{tr}A(t) = -f(\phi(t))$ and thus the characteristic exponents

$$\lambda_1 = 0, \qquad \lambda_2 = -\frac{1}{T}\int_0^T f(\phi(s))\,ds \left(\mathrm{mod}\frac{2\pi i}{T}\right)$$

and thus the periodic solution in the linear approximation is stable if $\lambda_2 \leq 0$.

Now, using this calculation and the above theorem we conclude that the periodic solution of the nonlinear equation $\ddot{x} + f(x)\,\dot{x} + g(x) = 0$ is asymptotically orbitally stable solution if $\int_0^T f(\phi(s))\,ds > 0$.

Example. *Consider the two dimensional system*

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y)$$

with T-periodic solution $x = \phi(t)$, $y = \psi(t)$.

By linearizing in a neighborhood of this periodic solution using

$$x = \phi(t) + u, \qquad y = \psi(t) + v$$

we obtain

$$\begin{aligned} \dot{u} &= f_x(\phi(t), \psi(t))\,u + f_y(\phi(t), \psi(t))\,v + \dots \\ \dot{v} &= g_x(\phi(t), \psi(t))\,u + g_y(\phi(t), \psi(t))\,v + \dots \end{aligned}$$

Using theorem 6.6 and the above theorem, asymptotically orbitally stable solution if

$$\int_0^T [f_x(\phi(t), \psi(t)) + g_y(\phi(t), \psi(t)]\,dt < 0.$$
