6 Linear Equation

6.1 Equation with constant coefficients

Consider the equation

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n.$$

This equating has n independent solutions.

If the eigenvalues are distinct then the solutions are

$$c_k e^{\lambda_k t}, \quad k = 1, \ldots, n,$$

where c_k are the corresponding eigenvectors.

If an eigenvalue λ has multiplicity $m > 1$ then the it has m dependent solutions of the form

$$P_0 e^{\lambda t}, P_1 e^{\lambda t}, \ldots, P_{m-1} e^{\lambda t},$$

where $P_k(t)$, $k = 0, \ldots, m-1$ are polynomial vectors of degree k or smaller.

Let $x_1(t), \ldots, x_n(t)$ are n independent solutions then

$$\Phi(t) = (x_1(t)\ x_2(t) \ldots x_n(t))$$

is called a fundamental matrix. The solutions are given by

$$x(t) = \Phi(t)c,$$

with c a constant vector. Given initial condition $x(t_0) = x_0$ the solution of the initial value problem is

$$x(t) = \Phi(t)\Phi^{-1}(t_0)x_0$$

Often one chooses Φ such that $\Phi(t_0) = I$, the $n \times n$ identity matrix.

Theorem. *Consider the equation $\dot{x} = Ax$, with A a constant matrix , eigenvalues $\lambda_1, \ldots, \lambda_n$.*

a. *If $Re\,\lambda_k < 0$, $k = 1, \ldots, n$, then for each $x(t_0) = x_0 \in \mathbb{R}^n$ and suitable chosen positive constants C and μ we have*

$$\|x(t)\| \leq C\|x_0\|\, e^{-\mu t}, \qquad \textit{and} \qquad \lim_{t\to\infty} x(t) = 0.$$

b. *If $Re\,\lambda_k \leq 0$, $k = 1, \ldots, n$, where the eigenvalues with $Re\,\lambda_k = 0$ are distinct, then $x(t)$ is bounded for $t \geq t_0$. Explicitly*

$$\|x(t)\| \leq C\|x_0\|.$$

c. *If there exists an eigenvalue λ_k with $Re\,\lambda_k > 0$, then in each neighborhood of $x = 0$ there are initial values such that for the corresponding solutions we have*

$$\lim_{t\to\infty} \|x(t)\| = \infty.$$

The solution $x = 0$ is

- asymptotically stable in a.
- Lyapunov stable in b
- unstable in c

Example. *See example 6.1*

6.2 Equations with coefficients which have a limit

Consider the equation

$$\dot{x} = Ax + B(t)x \qquad x \in \mathbb{R}^n.$$

with A a non-singular constant $n \times n$ matrix, $B(t)$ is continuous $n \times n$ matrix with

$$\lim_{t\to\infty} \|B(t)\| = 0.$$

It is not always true that the solution of this equation tend to the solution of $\dot{x} = Ax$.

Example. *Consider the equation*

$$\ddot{x} - \frac{2}{t}\dot{x} + x = 0, \qquad t \geq 1.$$

The equation

$$\ddot{x} + x = 0, \qquad t \geq 1.$$

has bounded solutions only. However, the non-autonomous equation has the two unbounded independent solutions

$$\sin t - t\cos t, \qquad \text{and} \qquad \cos t + t\sin t.$$

So more conditions on $B(t)$ are required

Theorem. *Consider the equation*

$$\dot{x} = Ax + B(t)\,x,$$

with $B(t)$ continuous for $t \geq t_0$ with the properties that

a. the eigenvalues λ_k of A, $k = 1, \ldots, n$ have $Re\,\lambda_k \leq 0$, the eigenvalues corresponding with $Re\,\lambda_k = 0$ are distinct;

b. $\int_{t_0}^{\infty} \|B\|\,dt$ is bounded,

then the solutions are bounded and $x = 0$ is stable in the sense of Lyapunov.

Theorem. *Consider the equation*

$$\dot{x} = Ax + B(t)\,x,$$

with $B(t)$ continuous for $t \geq t_0$ such that

a. all eigenvalues of A have negative real parts;

b. $\lim_{t\to\infty} \|B\| = 0$,

then for all solutions we have

$$\lim_{t\to\infty} x(t) = 0$$

and $x = 0$ is asymptotically stable.

Example. *Consider the equation*

$$\dot{x} = -x + b(t)\,x, \qquad t > 0.$$

- $b(t) = (1+t^2)^{-1}$, $t \geq 0$, then by the first theorem $x = 0$ is stable and by the second theorem $x = 0$ is actually asymptotically stable.
- $b(t) = (1+t)^{-1}$, $t \geq 0$, then the first theorem does not apply but by the second theorem $x = 0$ is asymptotically stable.
- $b(t) = at(1+t)^{-1}$, $t \geq 0$, $a > 0$, then both theorems do not apply. On the other hand, we can write the equation in the form

  $$\dot{x} = -x + \frac{at}{1+t}\,x = \left(a - 1 - \frac{a}{1+t}\right) x, \qquad t > 0.$$

  and by integration we obtain

  $$x(t) = c(1+t)^a\, e^{(a-1)t}.$$

  Thus $x = 0$ is asymptotically stable if $0 < a < 1$ and unstable if $a > 1$.

Theorem. *Consider the equation $\dot{x} = Ax + B(t)\,x$ with $B(t)$ continuous for $t \geq t_0$ and $\lim_{t\to\infty} \|B(t)\| = 0$. If at least one eigenvalue of the matrix A has a positive real part, there exists in each neighborhood of $x = 0$ solutions $x(t)$ such that*

$$\lim_{t\to\infty} \|x(t)\| = \infty.$$

The solution $x = 0$ is unstable.

6.3 Equations with periodic coefficients (Linear Periodic equation)

Consider the homogenous linear periodic equation

$$\dot{x} = A(t)\,x, \qquad A(t+T) = A(t), \qquad t \in \mathbb{R}, \qquad A(t) \text{ continuous}, \qquad T > 0, \tag{3}$$

where $A(t)$ is an $n \times n$ matrix. This equation can have both periodic and non-periodic solutions.

For example consider the equation $\dot{x} = a(t)\,x$ with $a(t) = 1$ and $a(t) = \sin^2 t$. In both cases we have non-periodic solutions , even unbounded solutions if $x(t_0) \neq 0$.

Lemma. *If C is a nonsingular $n \times n$ matrix, then there is a matrix B such that $C = e^B$.*

Lemma. *If $\Phi(t)$ is a fundamental matrix of*

$$\dot{x} = A(t)\,x, \qquad A(t+T) = A(t), \qquad t \in \mathbb{R}, \qquad A(t) \text{ continuous}, \qquad T > 0.$$

then $\Phi(t+T)$ is also a fundamental matrix and thus

$$\Phi(t+T) = \Phi(t)\,C,$$

for some constant matrix C.

Proof. Let $\tau = t + T$, then

$$\frac{dx(\tau)}{d\tau} = A(\tau - T)\,x(\tau) = A(\tau)\,x(\tau).$$

So $\Phi(\tau) = \Phi(t+T)$ is also a fundamental matrix. The fundamental matrices $\Phi(t)$ and $\Phi(t+T)$ are linearly dependent and thus there exists a nonsingular $n \times n$ matrix C such that $\Phi(t+T) = \Phi(t)\,C$.

This follows from the fact each column in one matrix is a linear combination of the columns of the other matrix.

Theorem. *Suppose $A(t+T) = A(t)$. The system $\dot{x} = A(t)x$ has at least one nontrivial solution $\phi(t)$ such that*

$$\phi(t+T) = \mu\,\phi(t), \qquad \mu \text{ constant.}$$

Proof. Let $\Phi(t)$ be a fundamental matrix. Then $\Phi(t+T) = \Phi(t)C$, where C is a nonsingular constant matrix. Let μ be an eigenvalue of C with eigenvector v. Let $\phi(t) = \Phi(t)v$. Then

$$\phi(t+T) = \Phi(t+T)v = \Phi(t)\,C\,v = \mu\Phi(t)v = \mu\phi(t). \quad \blacksquare$$

Remark. *If $\mu = 1$ then ϕ is periodic.*

Theorem. *(Floquet)*
Consider the equation

$$\dot{x} = A(t)\,x, \qquad A(t+T) = A(t), \qquad t \in \mathbb{R}, \qquad A(t) \text{ continuous}, \qquad T > 0.$$

Each fundamental matrix $\Phi(t)$ has the form

$$\Phi(t) = P(t)\,e^{Bt},$$

where $P(t)$ is T-periodic and B is constant $n \times n$ matrix.

Proof. From the above lemmas, there exists a constant matrix B such that

$$\Phi(t+T) = \Phi(t)\,C = \Phi(t)\,e^{BT}.$$

Let

$$P(t) = \Phi(t)\,e^{-Bt}.$$

Then

$$P(t+T) = \Phi(t+T)\,e^{-B(t+T)} = \Phi(t)\,C\,e^{-BT}e^{Bt} = \Phi(t)\,e^{Bt} = P(t). \quad \blacksquare$$

Corollary. *There exists a nonsingular periodic transformation of variables which transforms*

$$\dot{x} = A(t)\,x, \qquad A(t+T) = A(t), \qquad t \in \mathbb{R}, \qquad A(t) \text{ continuous}, \qquad T > 0.$$

into an equation with constant coefficients.

Proof. Let $P(t)$ and B be as in Floquet theorem. Let

$$x = P(t)\,y.$$

Then

$$\dot{P}(t)\,y + P(t)\,\dot{y} = A(t)\,P(t)\,y.$$

Or,

$$\dot{y} = P^{-1}(AP - \dot{P})\,y.$$

On the other hand, differentiation of $P(t) = \Phi(t)e^{-Bt}$ yields

$$\dot{P} = \dot{\Phi}\,e^{-Bt} + \Phi\,e^{-Bt}(-B) = AP - PB.$$

So we find

$$\dot{y} = By. \quad \blacksquare$$

Remark.

- *The solutions of* $\dot{y} = By$, *are vector polynomials in* t *multiplied by* $e^{\lambda t}$.
- *Thus the Floquet theorem implies that any solution of the periodic equation (3) is a linear combination of a product of polynomials in* t, $e^{\lambda t}$ *and* T*-periodic terms, where the exponents* λ *are the eigenvalues of* B.
- *The matrix* C *is called the monodromy matrix of (3). It is a nonsingular matrix associated with a fundamental matrix* $\Phi(t)$ *of (3) through the relation* $\Phi(t+T) = \Phi(t)\, C$.
- *A monodromy matrix is the inverse of the fundamental matrix of a system of ODEs evaluated at zero times the fundamental matrix evaluated at the period of the coefficients of the system.*

$$\Phi(t+T) = \Phi(t)\, C \quad \Longrightarrow \quad \Phi(T) = \Phi(0)\, C \quad \Longrightarrow \quad C = [\Phi(0)]^{-1}\, \Phi(T).$$

- *The eigenvalues* ρ *of a monodromy matrix are called the* *characteristic multipliers* *of (3).*
- *Any* λ *such that* $\rho = e^{\lambda T}$ *is called a* *characteristic exponent* *of (3).*
- *Notice that the characteristic exponents are not uniquely defined, but the multipliers are. (since* $e^{2\pi i} = 1$*).*
- *The real parts of the characteristic exponents* λ *are uniquely defined and we can always choose the exponents* λ *as the eigenvalues of* B, *where* B *is any matrix so that* $C = e^{BT}$.
- *The characteristic multipliers do not depend upon the particular monodromy matrix chosen; that is, the particular fundamental solution used to define the monodromy matrix. In fact if* $\Phi(t)$ *is a fundamental matrix solution,* $\Phi(t+T) = \Phi(t)\, C$, *and* $\Psi(t)$ *is another fundamental matrix solution, then there is a nonsingular matrix* D *such that* $\Psi(t) = \Phi(t)\, D$. *Therefore,*

$$\Psi(t+T) = \Phi(t+T)\, D = \Phi(t)\, C\, D = \Psi(t)\, D^{-1} C D$$

  and the monodromy matrix for $\Psi(t)$ *is* $D^{-1}CD$. *On the other hand matrices which are similar have the same eigenvalues.*
- *We can always select the monodromy matrix for a fundamental matrix* $\Phi(T)$ *with* $\Phi(0) = I$.

Lemma. $\lambda \in \mathbb{C}$ *is a characteristic exponent of (3) iff there is a nontrivial solution of (3) of the form* $e^{\lambda t}p(t)$ *where* $p(t+T) = p(t)$. *In particular,*

there is a periodic solution of (3) of period T *(or* $2T$ *but not* T*) iff there is a multiplier = 1 (or -1).*

Proof. Suppose $e^{\lambda t}p(t)$, $p(t+T) = p(t) \neq 0$ is a solution of (3). Then Floquet theorem implies there is an $x_0 \neq 0$ such that

$$e^{\lambda t}p(t) = \underbrace{P(t)e^{Bt}}_{\Phi(t)} x_0, \qquad P(t+T) = P(t).$$

We have

$$\begin{aligned} P(t)e^{Bt}e^{BT}x_0 &= P(t)e^{B(t+T)}x_0 = P(t+T)e^{B(t+T)}x_0 = e^{\lambda(t+T)}p(t+T) = e^{\lambda(t+T)}p(t) \\ &= e^{\lambda T}e^{\lambda t}p(t) = e^{\lambda T}P(t)e^{Bt}x_0 = P(t)e^{Bt}e^{\lambda T}x_0. \end{aligned}$$

Thus

$$P(t)e^{Bt}\left[e^{BT} - e^{\lambda T}\right]x_0 = 0.$$

Therefore $\det\left(e^{BT} - e^{\lambda T}I\right) = 0$ and thus $e^{\lambda T}$ is an eigenvalue of e^{BT} which implies that λ is an eigenvalue of B and thus a characteristic exponent.

Conversely, suppose λ is a characteristic exponent, i.e. λ satisfies $\det\left(e^{BT} - e^{\lambda T}I\right) = 0$. Then there exists $x_0 \neq 0$ such that $\left(e^{BT} - e^{\lambda T}I\right)x_0 = 0$. One can choose the representation by Floquet theorem $\Phi(t) = P(t)e^{Bt}$ so that λ is an eigenvalue of B. Then $e^{Bt}x_0 = e^{\lambda t}x_0$ for all t and

$$\overset{\textit{solution}}{\underbrace{P(t)e^{Bt}}_{\Phi(t)}}\, x_0 = \underbrace{P(t)x_0}_{p(t)}\, e^{\lambda t}$$

is the desired solution. ∎

Remark.

- *The existence of periodic solution of the equation* $\dot{x} = A(t)\,x$ *and the stability of the trivial solution are both determined by the eigenvalues of the matrix* B.
- *A necessary condition for the existence of* T*-periodic solutions is that one or more of the characteristic exponents,* λ*, are purely imaginary (multiplier has modulus 1).*
- *A necessary and sufficient condition for asymptotic stability of the* <u>*trivial solution*</u> *of* $\dot{x} = A(t)\,x$, $A(t)$ *is* T*-periodic, is that all characteristic exponents have a negative real part (multipliers have moduli < 1).*
- *A necessary and sufficient condition for stability of the* <u>*trivial solution*</u> *is that all characteristic exponents have real part* ≤ 0 *while the exponents with real part zero have multiplicity one.*
- *A serious difficulty with equations with periodic coefficients is that there are no general methods available to calculate the matrix* $P(t)$ *or the characteristic exponents.*

- *Each equation requires a special study and whole books have been devoted to some of them.*
- *However the following general theorem can be useful.*

Lemma. *If $X(t)$ is a fundamental matrix of the equation $\dot{x} = A(t)x$, then $z(t) = \det X(t)$ satisfies the scalar equation $\dot{z} = trA(t)z$. Thus*

$$\det X(t) = [detX(t_0)] \exp\left(\int_{t_0}^{t} trA(s)\, ds\right).$$

Theorem. *If $\rho_j = e^{\lambda_j T}$, $j = 1, 2, \ldots, n$ are the characteristic multipliers of (3), then*

$$\Pi_{j=1}^{n} \rho_j = \exp\left(\int_0^T trA(s)\, ds\right),$$

$$\sum_{i=1}^{n} \lambda_i = \frac{1}{T}\int_0^T trA(t)\, dt \ \left(mod\ \frac{2\pi i}{T}\right).$$

Proof. Suppose C is a monodromy matrix for the matrix solution $X(t)$, $X(0) = I$ of (3). Then $C = X(T)$ and thus from the above lemma

$$\det C = \det X(T) = \exp\left(\int_0^T \mathrm{tr}A(s)\, ds\right)$$

The statements of the theorem now follow immediately from the definitions of characteristic multipliers and exponents. ■

Remark.

- *If the sum is positive then the trivial solution is unstable.*
- *If the sum is negative or zero then we <u>do not</u> have enough information to draw conclusions about the stability of the trivial solution.*
- *We have seen in section 5.4, when linearizing in a neighborhood of a periodic solution $\phi(t)$ of an autonomous equation, that one of the solutions of the linear system is $\dot{\phi}(t)$ and thus has multiplier $= 1$ or one of the exponents is 0.*
- *This implies that if the equation has order 2 then we can construct the other independent solution.*

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application

(generalized Liénard equation) Consider the equation

$$\ddot{x} + f(x)\,\dot{x} + g(x) = 0.$$

Suppose the equation has a T-period solution $x = \phi(t)$. Let

$$y = x - \phi$$

Then,

$$\ddot{\phi} + \ddot{y} + f(\phi + y)(\dot{\phi} + \dot{y}) + g(\phi + y) = 0$$

Using the expansions of f and g

$$f(\phi + y) = f(\phi) + \frac{df}{dx}(\phi)\,y + \ldots \qquad g(\phi + y) = g(\phi) + \frac{dg}{dx}(\phi)\,y + \ldots.$$

we have

$$\ddot{\phi} + \ddot{y} + \left[f(\phi) + \frac{df}{dx}(\phi)\,y\right](\dot{\phi} + \dot{y}) + g(\phi) + \frac{dg}{dx}(\phi)\,y + \cdots = 0$$

By grouping we have

$$\ddot{\phi} + f(\phi)\dot{\phi} + g(\phi) + f(\phi)\dot{y} + \ddot{y} + \frac{df}{dx}(\phi)\dot{\phi}\,y + \frac{dg}{dx}(\phi)\,y + \cdots = 0$$

Since ϕ is a solution,

$$\ddot{y} + f(\phi)\,\dot{y} + \left[\frac{df}{dx}(\phi)\,\dot{\phi} + \frac{dg}{dx}(\phi)\right] y = \ldots$$

The equivalent vector form is

$$\begin{aligned}
\dot{y_1} &= y_2 \\
\dot{y_2} &= -\left[\frac{df}{dx}(\phi)\,\dot{\phi} + \frac{dg}{dx}(\phi)\right] y_1 - f(\phi)\,y_2 + \ldots \\
&= \begin{pmatrix} 0 & 1 \\ -\left[\frac{df}{dx}(\phi)\,\dot{\phi} + \frac{dg}{dx}(\phi)\right] & -f(\phi) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \ldots
\end{aligned}$$

The trace of the linearized equation is

$$\mathrm{Tr}A(t) = -f(\phi(t)).$$

From section 5.4, $\dot{\phi}(t)$ is a solution of the linearized equation.

So we can put $\lambda_1 = 0$. From the above theorem,

$$\lambda_2 = \frac{1}{T}\int_0^T f(\phi(t))\,dt \left(\mathrm{mod}\frac{2\pi i}{T}\right).$$

The periodic solution in the linear approximation is stable if $\lambda_2 \leq 0$ and instable if $\lambda_2 > 0$.

Example. *Consider the equation* $\dot{x} = A(t)\, x$ *with*

$$A(t) = \begin{pmatrix} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\sin t \cos t \\ -1\frac{3}{2}\sin t \cos t & -1 + \frac{3}{2}\sin^2 t \end{pmatrix}$$

The above theorem yields

$$\lambda_1 + \lambda_2 = \frac{1}{2\pi}\int_0^{2\pi} \left(-2 + \frac{3}{2}\cos^2 t + \frac{3}{2}\sin^2\right)\, dt = -\frac{1}{2}$$

With this we have no immediate conclusion of the stability of th trivial solution.

The instantaneous eigenvalues of the matrix $A(t)$, $\lambda(t)$, are $(-1 \pm i\sqrt{7})/4$, which surprisingly are time independent. This suggest theat the equation has characteristic exponent with negative real part and stability of the trivial solution.

However the solution exists of the form

$$\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix} e^{t/2}.$$

The characteristic exponents are $\lambda_1 = \frac{1}{2}$, and $\lambda_2 = -1$, the trivial solution is unstable.

One of the characteristic multipliers is e^{π} and the other one is $e^{-2\pi}$.
