5 Introduction to the theory of stability

5.1 Simple examples

- In chapters three and four we have seen equilibrium solutions and periodic solutions. These are solutions which exist for all time.
- Now we are interested in the stability of solutions, that is whether solutions which at $t = t_0$ are starting in a neighborhood of such a special solution, will stay in this neighborhood for $t > t_0$.

Example. *Consider the harmonic oscillator with damping (linear)*

$$\ddot{x} + \mu\dot{x} + x = 0, \qquad \mu > 0.$$

The critical point $x = \dot{x} = 0$ corresponds with an equilibrium solution. The equivalent system $\dot{z} = Az$ with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -\mu \end{pmatrix}$$

The eigenvalues are

$$\lambda_{1,2} = \frac{-\mu \pm \sqrt{\mu^2 - 4}}{2}$$

and thus $Re\,\lambda_{1,2} < 0$. This implies that $(0,0)$ is a positive attractor. It is natural to call $(0,0)$ <u>stable</u>. See figure 5.1.

Example. *Consider again the mathematical pendulum described by the equation*

$$\ddot{x} + \sin x = 0, \qquad \mu > 0.$$

Using the linear analysis and the first integral we obtained the phase space shown in figure 5.2.

Clearly the equilibrium points $(\pm\pi, 0)$ are unstable as solutions starting in a neighborhood of these points will generally leave this neighborhood.

Also in this example it seems natural to call the equilibrium point $(0,0)$ stable.

On the other hand do we consider periodic solutions starting in $x(0) = a$, with $0 < a < \pi$, to be stable.

Note that solution starting in a neighborhood of $(a, 0)$ have various periods. This means that phase <u>points</u> starting near each other are not necessarily staying close. We consider this in more detail.

From Example 2.11, line integral of the equation is

$$F(x, \dot{x}) = \frac{1}{2}\dot{x}^2 - \cos x$$

and thus the has integral through $(a, 0)$ is

$$\frac{1}{2}\dot{x}^2 - \cos x = -\cos a.$$

It follows that

$$\frac{dx}{dt} = \pm\sqrt{2\cos x - 2\cos a}.$$

Because of the symmetry of the periodic solutions we have the period

$$T = 4\int_0^a \frac{dx}{\sqrt{2\cos x - 2\cos a}}$$

T can be written in terms of the Jacobian elliptic functions. It is clear that the period T depends non-trivially on $x(0) = a$; see figure 5.3.

5.2 Stability of equilibrium solutions

Consider the equation

$$\dot{x} = f(t, x), \qquad x \in \mathbb{R}^n, \quad t \in \mathbb{R}$$

with $f(t, x)$ continuous in t and x, Lipschitz-continuous in x. Assume that $x = 0$ is a critical point of the vector function $f(t, x)$, so

$$f(t, 0) = 0, \qquad t \in \mathbb{R}.$$

Critical points of non-autonomous equations are fairly rare.

The assumption $x = 0$ for the critical point is of course no restriction as we can translate any critical point to the origin of phase space.

Let $x(t; t_0, x_0)$ denote the solution starting at $t = t_0$ in $x = x_0$.

Definition. *(stability in the sense of Lyapunov)*

Consider the equation $\dot{x} = f(t, x)$ and a neighborhood $D \subset \mathbb{R}^n$ of $x = 0$; The solution $x = 0$ is called stable in the sense of Lyapunov if for each $\epsilon > 0$ and t_0 a $\delta(\epsilon, t_0)$ can be found such that

$$\|x_0\| \le \delta \quad \Longrightarrow \quad \|x(t; t_0, x_0)\| \le \epsilon, \qquad \text{for } t > t_0.$$

If this solution is not stable in the sense of Lyapunov, it is called unstable. The solution $x = 0$ is called asymptotically stable if $x = 0$ is stable and if there exists a $\delta(t_0) > 0$ such that

$$\|x_0\| \le \delta(t_0) \quad \Longrightarrow \quad \lim_{t\to\infty} \|x(t; t_0, x_0)\| = 0.$$

Remark.

- *The equilibrium solution* $(0, 0)$ *of the equation for the mathematical pendulum,* $\ddot{x} + \sin x = 0$*, is stable in the since of Lyapunov.*
- *The equilibrium solution* $(0, 0)$ *of the harmonic oscillator,* $\ddot{x} + x = 0$*, and of the harmonic oscillator with damping,* $\ddot{x} + \mu\dot{x} + x = 0$*, are also stable.*
- *The solutions* $(\pm\pi, 0)$ *of the mathematical pendulum are unstable.*

Positive attraction is not sufficient for asymptotic stability.

Example. *Consider the system which in polar coordinates takes the form*

$$\dot{r} = r(1 - r), \qquad \dot{\theta} = \sin^2(\theta/2).$$

There are two critical points $(r, \theta) = (0, 0)$ and $(1, 0)$. The point (1,0) is a positive attractor (ω-limitset) for each orbit which starts outside $(0, 0)$. However, both the solutions $(0, 0)$ and $(1, 0)$ are unstable. See figure 5.4.

Since in each neighborhood of $(1, 0)$ one can find solutions which are leaving this neighborhood, although only temporarily.

In chapter 7 we shall see that for critical points which are not degenerate this behavior can not occur. That is for a critical point which, after linearization, corresponds with a positive attractor, turns out to be asymptotically stable.

5.3 Stability of periodic solutions

Definition. *(Stability in the sense of Lyapunov for periodic solutions).*

Consider the equation $\dot{x} = f(t, x)$ with periodic solution $\phi(t)$. The periodic solution in Lyapunov stable if for each t_0 and $\epsilon > 0$ we can find $\delta(\epsilon, t_0) > 0$ such that

$$\|x_0 - \phi(t_0)\| \leq \delta \quad \Longrightarrow \quad \|x(t; t_0, x_0) - \phi(t)\| \leq \epsilon, \quad \textit{for } t \geq t_0.$$

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remark.

- Lyapunov-stability of periodic solutions is an exceptional case
- The stability implies that orbits starting in a neighborhood of the periodic solution, remain in a neighborhood but in the sense that phase points which start out being close stay near each other.
- For the harmonic oscillator equation, $\ddot{x} + x = 0$, the solutions $(c\cos t, c\sin t)$ are all of the same period and thus each periodic solution is Lyapunov stable.
- For the periodic solution of the mathematical pendulum equation, $\ddot{x} + \sin x = 0$, this is not the case, since as we have seen in the example in section 5.1, the period varies with the starting point. So the solutions are not Lyapunov stable.

Remark.

- *More complications arise when considering the concept of asymptotic stability.*
- *In the theorem in section 4.2 we remarked that an orbit γ in a neighborhood of a periodic solution of an autonomous equation can never end up in a point of the periodic orbit. The orbit γ continues to move around the periodic orbit.*
- *These considerations will lead to definitions of stability, which require some geometric ideas.*

Poincaré map

Consider the autonomous equation in

$$\dot{x} = f(x), \qquad \text{in } \mathbb{R}^n,$$

with periodic solution $\phi(t)$, corresponding with a closed orbit in n-dimensional phase-space.

Let V be a transversal to the closed orbit corresponding to ϕ. i.e. an $(n-1)$-dimensional manifold that contains ordinary points only, punctured by the closed orbit and nowhere tangent to it. Let the closed orbit intersect V in the point a.

Consider an orbit $\gamma(x_0)$ starting in $x_0 \in V$. By continuity dependence on the initial value, we can choose x_0 sufficiently close to a such that $\gamma(x_0)$ returns to V. Thus we can construct a certain neighborhood of the closed orbit that will return to V.

Thus we can define a mapping $P : V \to V$. This mapping is called the return-map or Poincaré map P.

Note that $P(a) = a$, and thus a is a fixed point of P.

Now we can define stability of the periodic solution in terms of the stability of the Poincaré-map in the neighborhood of the fixed point a.

Definition. *Given the autonomous equation $\dot{x} = f(x)$ with periodic solution $\phi(t)$, transversal V and Poincaré-map P with fixed point a. The solution $\phi(t)$ is stable if for each $\epsilon > 0$ we can find $\delta(\epsilon)$ such that*

$$\|x_0 - a\| \leq \delta,\ x_0 \in V \qquad \Longrightarrow \qquad \|P^n(x_0) - a\| \leq \epsilon, \quad n = 1, 2, \ldots$$

Remark. *A periodic solution which is stable in this sense, is called orbitally stable.*

Example. *Consider the mathematical pendulum equation*

$$\ddot{x} + \sin x = 0.$$

Consider a periodic solution starting in

$$x(0) = a, \quad \dot{x}(0) = 0, \quad 0 < a < \pi.$$

Take for V a segment $I \subset (0, \pi)$ which has a as interior point. The Poincaré-map is the identity mapping. Clearly, the periodic solutions are (orbitally) stable. These periodic solutions are not Lyapunov-stable as they have different periods.

The concept of asymptotic stability of a periodic solution shall now be connected with the attraction properties of the fixed point a, when applying the Poincaré map in a neighborhood.

Definition. *Given the autonomous equation $\dot{x} = f(x)$ with periodic solution $\phi(t)$, transversal V and Poincaré-map P with fixed point a. The solution $\phi(t)$ is asymptotically stable if it is stable and if there exists a $\delta > 0$ such that*

$$\|x_0 - a\| \leq \delta,\ x_0 \in V, \qquad \Longrightarrow \qquad \lim_{n\to\infty} P^n(x_0) = a.$$

Remark.

- *The concepts of stability and asymptotic stability for the autonomous equation $\dot{x} = f(x)$ are introduced by using a mapping.*
- *This mapping maps an $(n-1)$ dimensional transversal manifold into itself.*
- *How shall we treat now periodic solutions of non-autonomous equations?*

Non-autonomous equations

Consider the equation

$$\dot{x} = f(t, x), \qquad x \in \mathbb{R}^n,\ t \in \mathbb{R},$$

with $f(t, x)$ is T-periodic in t. Let

$$\theta(t) = \mathrm{mod}(t, T) = t - qT, \qquad q = [t/T].$$

Then

$$\dot{\theta} = 1, \qquad \theta(0) = 0.$$

Thus the non-autonomous equation is equivalent with the $(n+1)$-dimensional autonomous system

$$\dot{x} = f(\theta, x), \qquad \dot{\theta} = 1, \qquad \theta(0) = 0.$$

Accordingly

- we can apply the definition of stability for the autonomous equation.
- a T-periodic solution (θ, x) corresponds with a T-periodic solution x of $\dot{x} = f(t, x)$.
- the transversal V is now n-dimensional
- a natural choice is to use the mapping $\mathbb{R}^n \to \mathbb{R}^n$ which arises by taking the value of solutions at time T, $2T$, etc.

Example. *(linear oscillations with damping and forcing)*
Consider the equation

$$\ddot{x} + 2\mu\dot{x} + \omega_0^2 x = h\cos\omega t, \qquad 0 < \mu < \omega_0, \quad h, \omega > 0. \tag{2}$$

Here μ denotes the damping rate, h is the amplitude of the periodic forcing, ω_0 is the frequency of the free ($\mu = h = 0$) oscillating system. We exclude the case of resonance $\omega_0 = \omega$.

If we take

$$\theta(t) = \mathrm{mod}(t, 2\pi/\omega) = t - q\frac{2\pi}{\omega}, \qquad q = \left[\frac{t}{2\pi/\omega}\right],$$

then (2) is equivalent to the autonomous system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega_0^2 x - 2\mu y + h\cos\omega\theta \\ \dot{\theta} &= 1, \qquad \theta(0) = 0. \end{aligned}$$

The solutions of (2) are of the form

$$x(t) = c_1 e^{-\mu t}\cos\sqrt{\omega_0^2 - \mu^2}\, t + c_2 e^{-\mu t}\sin\sqrt{\omega_0^2 - \mu^2}\, t + \alpha\cos\omega t + \beta\sin\omega t$$

where

$$\alpha = \frac{\omega_0^2 - \omega^2}{4\mu^2\omega^2 + (\omega_0^2 - \omega^2)^2}\, h, \qquad \beta = \frac{2\mu\omega}{4\mu^2\omega^2 + (\omega_0^2 - \omega^2)^2}\, h$$

The equation (2) and the equivalent system has one periodic solution corresponding to $c_1 = c_2 = 0$:

$$x(t) = \alpha \cos \omega t + \beta \sin \omega t$$

of period $2\pi/\omega$. This solution satisfies the initial conditions:

$$x(0) = \alpha, \qquad \dot{x}(0) = \omega\beta.$$

Note that

$$t = q\frac{2\pi}{\omega}, \quad q = 0, 1, 2, \ldots \qquad \Longrightarrow \qquad \theta(t) = 0.$$

The implies that at these times the orbits intersect the xy-plane.

Thus we construct a Poincaré-mapping P by considering the intersection of the orbit with the xy-plane.

Since the periodic solution starts in $(\alpha, \omega\beta)$, the point is a fixed point of P.

Substitute $t = 2\pi/\omega$ produces

$$P\begin{pmatrix} x(0) \\ \dot{x}(0) \end{pmatrix} = P\begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$$

$$= \begin{pmatrix} c_1 e^{-\mu 2\pi/\omega} \cos\gamma + c_2 e^{-\mu 2\pi/\omega} \sin\gamma + \alpha \\ e^{-\mu 2\pi/\omega} \cos\gamma \left(-c_1\mu + c_2\sqrt{\omega_0^2 - \mu^2}\right) - e^{-\mu 2\pi/\omega} \sin\gamma \left(c_1\sqrt{\omega_0^2 - \mu^2} + \mu c_2\right) + \omega\beta \end{pmatrix}.$$

Applying P again and again we find

$$\lim_{n \to \infty} P^n \begin{pmatrix} x(0) \\ \dot{x}(0) \end{pmatrix} = \begin{pmatrix} \alpha \\ \omega\beta \end{pmatrix}.$$

This implies that the periodic solution is asymptotically stable.

Figure 5.6 indicates $P^n(x(0), \dot{x}(0))$ for various values of n and several initial values. The figure shows the Poincaré-map of the system at intervals of time $2\pi/\omega$; the origin has been shifted to the fixed point $(\alpha, \omega\beta)$ which corresponds with the periodic solution.

Remark. *The above conclusion can be drawn directly from the explicit solution since it approaches the periodic solution as $t \to \infty$.*

5.4 Linearization of non-autonomous equations

Linearization in a neighborhood of a critical point

As for the autonomous equation, we can obtain linear equations by linearization in a neighborhood of a critical point.

Suppose $x = a$ is a critical point of

$$\dot{x} = f(t, x).$$

In a neighborhood of $x = a$ we consider the linear equation

$$\dot{y} = \frac{\partial f}{\partial x}(t, a)\, y.$$

But notice here that the coefficient matrix is time dependent.

Linearization in a neighborhood of a periodic solution

Suppose $\phi(t)$ is a periodic solution of $\dot{x} = f(t, x)$ and $f(t, x)$ has a Taylor expansion to degree two. We put

$$x = \phi(t) + y$$

and after substitution and expansion we obtain

$$\dot{\phi}(t) + \dot{y} = f(t, \phi(t) + y) = f(t, \phi(t)) + \frac{\partial f}{\partial x}(t, \phi(t))\, y + \ldots$$

Since $\phi(t)$ is a solution, we have

$$\dot{y} = \frac{\partial f}{\partial x}(t, \phi(t))\, y + \ldots$$

So in the neighborhood of the periodic equation we consider the linear equation

$$\dot{z} = \frac{\partial f}{\partial x}(t, \phi(t))\, z.$$

This equation is in general difficult to solve. It has n independent solutions.

If $f(t, x) = f(x)$ and $\phi(t)$ is T-periodic then the linearized equation

$$\dot{z} = \frac{\partial f}{\partial x}(\phi(t))\, z$$

has coefficients which are T-periodic. In chapter 6 we apply Floquet theorem to this problem.

For an autonomous equation $\dot{x} = f(x)$ with a T-periodic solution $\phi(t)$ we have

$$\dot{\phi}(t) = f(\phi(t))$$

and

$$\ddot{\phi}(t) = \frac{\partial f}{\partial x}(\phi(t))\, \dot{\phi}(t).$$

So if $\phi(t)$ is a periodic solution of $\dot{x} = f(x)$ then $\dot{\phi}(t)$ is a solution of the linearized equation $\dot{z} = \frac{\partial f}{\partial x}(t, \phi(t))\, z$.

One of the implications is that, in $\mathbb{R}^2$ we can solve the linearized equation by explicitly constructing the second independent solution.

Linearization of the Poincaré-mapping

One can study the linearization of the Poincaré-mapping P in the corresponding fixed point $x = a$. This means that for the mapping P of x_0 we linearized in a neighborhood of $x = a$ so that we have to calculate

$$\frac{\partial P}{\partial x_0}(a)\,(x_0 - a).$$
