4 Periodic Solutions

We have shown that in the case of an autonomous equation the periodic solutions correspond with closed orbits in phase-space.

Autonomous two-dimensional systems with phase-space $\mathbb{R}^2$ are called plane (or planer) systems.

Closed orbits in $\mathbb{R}^2$, corresponding with periodic solutions, do not intersect because of uniqueness.

According to the Jordan separation theorem they split $\mathbb{R}^2$ into two parts, the interior and the exterior of the closed orbit.

This topological property leads to a number of special results named after Poincaré and Bendixson.

Periodic solutions of autonomous systems with dimension larger than tow or of non-autonomous systems are more difficult to analyze.

4.1 Bendixson's criterion

Theorem. (criterion of Bendixson)

Suppose that the domain $D \subset \mathbb{R}^2$ is simply connected; (f, g) is continuously differentiable in D. Then the equations

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y), \qquad (x, y) \in D,$$

can only have periodic solutions if $\nabla \cdot (f, g)$ changes sign in D or if $\nabla \cdot (f, g) = 0$ in D.

Proof. Suppose that we have a closed orbit C in D, corresponding with a solution of the above equation; the interior of C is G. By Gauss theorem

$$\int_G \nabla \cdot (f, g)\, d\sigma = \int_C (f\, dy - g\, dx) = \int_C \left(f\frac{dy}{ds} - g\frac{dx}{ds} \right) ds.$$

The integrand in the last integral is zero as the closed orbit C corresponds with a solution of the equation. So the integral vanishes, but this means that the divergence of (f, g) can not be sign definite. ■

Example.

We know that the damped linear oscillator contains no periodic solutions.

Consider now a nonlinear oscillator with nonlinear damping represented by the equation

$$\ddot{x} + p(x)\,\dot{x} + q(x) = 0.$$

We assume that $p(x)$ and $q(x)$ are smooth and that $p(x) > 0$, $x \in \mathbb{R}$ (damping). The equivalent vector equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -q(x_1) - p(x_1)x_2.$$

The divergence of the vector function is $-p(x)$ which is negative definite. It follows from Bendixson's criterion that the equation has no periodic solutions.

Example. *Consider in* $\mathbb{R}^2$ *the van der Pol equation*

$$\ddot{x} + x = \mu(1 - x^2)\dot{x}, \qquad \mu \text{ constant.}$$

The vector form is

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + \mu(1 - x_1^2)\, x_2.$$

The divergence of vector function is $\mu(1 - x_1^2)$. A periodic solution, if it exists has to intersect with $x_1 = 1$, $x_1 = -1$ or both. See figure 2.9.

Example. *Consider in* $\mathbb{R}^2$ *the system*

$$\dot{x} = -x + y^2, \qquad \dot{y} = -y^3 + x^2.$$

The critical points are: $(0, 0)$ and $(1, 1)$ and

$$\frac{\partial f}{\partial(x, y)}(x, y) = \begin{pmatrix} -1 & 2y \\ 2x & -3y^2 \end{pmatrix}.$$

$$\frac{\partial f}{\partial(x, y)}(0, 0) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \frac{\partial f}{\partial(x, y)}(1, 1) = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}.$$

Therefore the point $(1, 1)$ is a saddle while the point $(0, 0)$ is a degenerate critical point. The phase flow is shown in Figure 3.17.

The divergence of the vector function is $-1-3y^2$; as this expression is negative definite, the system has no periodic solutions.

Remark. *The above theorem can not be generalized to systems with dimension larger than 2.*

4.2 Geometric auxiliaries for the Poincaré-Bendixson theorem

This theorem holds for plane autonomous systems. However in this section we introduce concepts that are meaningful in the more general setting of $\mathbb{R}^n$.

Let $\gamma(x_0)$ denote the orbit in phase space corresponding to the solution $x(t)$ of the initial value problem

$$\dot{x} = f(x), \qquad x(0) = x_0.$$

So if $x(t_1) = x_1$ then we have $\gamma(x_1) = \gamma(x_0)$. We use the following notation

$$\gamma(x_0) = \begin{cases} \gamma^-(x_0), & t \leq 0, \\ \gamma^+(x_0), & t \geq 0. \end{cases}$$

Thus $\gamma(x_0) = \gamma^-(x_0) \cup \gamma^+(x_0)$. For periodic solutions we have $\gamma^-(x_0) = \gamma^+(x_0)$. If we do not want to distinguish a particular point on an orbit, we will write γ, γ^+, γ^- for the orbit.

Definition. *A point $p \in \mathbb{R}^n$ is called positive limitpoint of the orbit $\gamma(x_0)$ corresponding with the solution $x(t)$ if an increasing sequence $t_1, t_2, \cdots \to \infty$ exists such that the sequence $\{x(t_i)\} \subset \gamma(x_0)$ have limit point p.*

Similarly we define negative limitpoints.

Example. *Every positive attractor is a positive limitpoint for all orbits in its neighborhood.*

Definition.

- *The set of all positive limitpoints of an orbit γ is called the ω-limitset of γ.*
- *The set of all negative limitpoints of an orbit γ is called the α-limitset of γ.*
- *These sets are denoted by $\omega(\gamma)$ and $\alpha(\gamma)$.*

It is easy to prove that equivalent definitions of the ω-limitset and α-limitset of an orbit γ of $\dot{x} = f(x)$ are:

$$\omega(\gamma) = \bigcap_{x_0 \in \gamma} \overline{\gamma^+(x_0)}, \qquad \alpha(\gamma) = \bigcap_{x_0 \in \gamma} \overline{\gamma^-(x_0)}.$$

Recall that a set $M \subset \mathbb{R}^n$ is called an invariant set of $\dot{x} = f(x)$ if for any $x_0 \in M$, the solution $x(t; x_0)$ through x_0 belongs to M for $-\infty < t < \infty$. A set M is called a positive (negative) invariant if for each $x_0 \in M$, $x(t; x_0) \subset M$ for $t \geq 0$ $(t \leq 0$.

Example. *Consider the system*

$$\dot{x_1} = -x_1, \qquad \dot{x_2} = -2x_2.$$

Thus the ω-limitset of all orbits is the origin. The α-limitset is empty for all orbits except when starting in the origin.

Example. *Consider the system*

$$\dot{x_1} = x_2, \qquad \dot{x_2} = -x_1.$$

The origin is a center and all orbits are closed. For each orbit γ we have

$$\omega(\gamma) = \alpha(\gamma) = \gamma.$$

Theorem. *The sets $\alpha(\gamma)$ and $\omega(\gamma)$ of an orbit γ are closed and invariant.*

Proof. $\alpha(\gamma)$ and $\omega(\gamma)$ are closed is obvious from the definition. Now we shaw that $\omega(\gamma)$ is invariant. If $p \in \omega(\gamma)$ then by definition there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $x(t_n; x_0) \to p$ as $n \to \infty$. Consequently, for any fixed $t \in (-\infty, \infty)$ we have

$$x(t + t_n; x_0) = x(t, x(t_n; x_0)) \quad \longrightarrow \quad x(t; p).$$

by continuity of x. This shows that the orbit through p belongs to $\omega(\gamma)$. so $\omega(\gamma)$ is invariant.

Corollary. *The limit sets of an orbit must contain only complete orbits.*

Since if $p \in \omega(\gamma)$, then $x(t; p) \subset \omega(\gamma)$. In particular, they contain critical points and periodic solutions.

Theorem. *If γ^+ (γ^-) is bounded, then $\omega(\gamma)$ ($\alpha(\gamma)$) is nonempty, compact, and connected.*

Proof. If γ^+ is bounded then $\omega(\gamma)$ is nonempty since each bounded subset of $\mathbb{R}^n$ has at least one accumulation point. Also it is obvious that $\omega(\gamma)$ is bounded (otherwise γ^+ should be unbounded). Being closed and bounded, $\omega(\gamma)$ is compact
$x(t; x_0)$ corresponds with $\gamma^+(x_0)$ for $t \geq 0$, then from above we have

$$x(t + t_n; x_0) \longrightarrow x(t; p)$$

and $\gamma(p) \subset \omega((\gamma(x_0))$. It follows that

$$d\,[x(t; x_0), \omega(\gamma)] \longrightarrow 0, \qquad \text{as } t \to \infty.$$

This implies that $\omega(\gamma)$ is connected. ■

Theorem. *If K is a positively invariant set of $\dot{x} = f(x)$ and K is homeomorphic to the closed unit ball in $\mathbb{R}^n$, there is at least one equilibrium point of $\dot{x} = f(x)$ in K.*

Consequences.

Let γ be an orbit which does not correspond with a periodic solution and approach a closed orbit C. If γ ends up in one point $p \in C$ then $p \in \omega(\gamma)$. But this can not happen since also we must have $C \subset \omega(\gamma)$ being a solution through p and the limitsets contain complete solutions.

Or alternatively, γ can not end up in one point of the closed orbit as this has to be a solution, in this case a critical point. This contradicts the assumption that we have one closed orbit.

Accordingly, the orbit γ approaches the closed orbit arbitrarily close but will keep on moving around the closed orbit. See figure 4.3).

Definition. *A set $M \subset \mathbb{R}^n$ is called a minimal set of $\dot{x} = f(x)$ if M is closed, invariant, nonempty and if M has no smaller subsets with these three properties.*

Theorem. *Suppose that A is a nonempty, compact (bounded and closed), invariant set of $\dot{x} = f(x)$ then there exists a minimal set $M \subset A$.*

In the above two examples, the ω-limitsets are all minimal. In the first example $\omega(\gamma) = \{0\}$ and the second example $\omega(\gamma) = \gamma$.

Example. *(see figure 4.4)*

Consider the equation

$$\dot{x} = x - y - x(x^2 + y^2), \quad \dot{y} = y + x - y(x^2 + y^2).$$

Clearly $(0, 0)$ is a critical point and the system has the periodic solution

$$x(t) = \cos t, \qquad y(t) = \sin t,$$

corresponding to the cycle $K : x^2 + y^2 = 1$. In polar coordinates

$$r^2 = x^2 + y^2 \quad \Longrightarrow \quad \dot{r} = \frac{x\dot{x} + y\dot{y}}{r^2}$$

$$\theta = \tan^{-1} \frac{y}{x} \quad \Longrightarrow \quad \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$$

By multiplying one equation by x and the other by y and adding or subtracting we obtain

$$\dot{r} = r(1 - r^2), \qquad \dot{\theta} = 1.$$

Thus if $r > 1$, then $\dot{r} < 0$ and thus the orbits are spirals outside K and tend to K. If $0 < r < 1$, then $\dot{r} > 0$ and thus the orbits are spirals inside K and tend to K.

Explicitly, the equation in r is separable and using $\frac{1}{r(1-r^2)} = \frac{1}{r} + \frac{r}{1-r^2}$ we obtain by integration

$$\ln \frac{r^2}{|1-r^2|} = 2t + c_1 \quad \Longrightarrow \quad \frac{r^2}{1-r^2} = c_2 e^{2t}.$$

Thus the solution for the system is

$$\theta(t) = t + \alpha, \qquad r(t) = \frac{1}{\sqrt{1 + ce^{-2t}}}.$$

These represent a family of spirals that are inside the unit circle K and tend toward K as $t \to \infty$ when $c > 0$. For $c < 0$ they are outside K and tend towards K as $t \to \infty$. Therefore the cycle k is a limit cycle.

- All circular domains with center $(0,0)$ and radius larger than 1 are positive invariant. Moreover, $(0,0)$ and the circle $r = 1$ are invariant sets.
- The origin is α-limitset for each orbit which starts with $r(0) < 1$.
- The circle $r = 1$, corresponding with the periodic solution $(x_1, x_2) = (\cos t, \sin t)$ is ω-limitset for all orbits except the critical point in the origin.
- Both the α- and ω-limitset are minimal.

Example. *(see figure 4.5)*
Consider the equation

$$\dot{r} = r(1-r), \qquad \dot{\theta} = \sin^2\theta + (1-r)^3.$$

- Clearly, the origin and the circle $r = 1$ are invariant sets.
- Along any orbit that does not lie on the circle $r = 1$ we have

$$\dot{r} \text{ is } \begin{cases} < 0 \text{ if } r > 1 & \Longrightarrow \quad r \to 1^+, \\ > 0 \text{ if } 0 < r < 1 & \Longrightarrow \quad r \to 1^-. \end{cases}$$

  This implies that the circle $r = 1$ is a limit cycle.
- $r = 1 \quad \Longrightarrow \quad \dot{\theta} = \sin^2\theta = 0$ if $\theta = 0, \pi \quad \Longrightarrow \quad \{r = 1, \theta = 0, \pi\}$ are equilibrium solutions. Thus the invariant set $\{r = 1\}$ consists of 4 orbits, given by

$$\theta = 0, \quad \theta = \pi, \quad 0 < \theta < \pi, \quad \pi < \theta < 2\pi.$$

- The set $\{r = 1\}$ has two minimal sets, the points $\{r = 1, \theta = 0\}$ and $\{r = 1, \theta = \pi\}$.

Why Minimal sets?

- Classification of the possible minimal sets is essential to understand the behavior of solutions of autonomous equations $\dot{x} = f(x)$.
- It is desired to describe the manner in which the ω-limitset of any orbit can be built up from minimal sets and orbits connecting the various minimal sets.

4.3 The Poincaré-Bendixson theorem

Consider the equation

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^2,$$

$f : \mathbb{R}^2 \to \mathbb{R}^2$ has continuous first partial derivatives. We assume that the solutions we are considering, exist for $-\infty < t < \infty$.

Lemma. *If M is a bounded minimal set of $\dot{x} = f(x)$ in $\mathbb{R}^2$, then M is a critical point or a periodic orbit.*

Lemma. *If $\omega(\gamma^+)$ contains both ordinary points and a periodic orbit γ_0 then $\omega(\gamma^+) = \gamma_0$.*

Theorem. *(Poincaré-Bendixson) Consider the equation $\dot{x} = f(x)$ in $\mathbb{R}^2$.*

If γ^+ is a <u>bounded</u> positive orbit and $\omega(\gamma^+)$ contains ordinary points only then $\omega(\gamma^+)$ is a periodic orbit.

If $\omega(\gamma^+) \neq \gamma^+$ the periodic orbit is called a limit cycle.

That is, having a positive orbit γ^+ which is bounded but does not correspond with a periodic solution, the ω-limitset $\omega(\gamma^+)$ contains a critical point or it consists of a closed orbit.

An analogous result is valid for a bounded negative orbit.

Another version:

If γ^+ is a bounded positive semiorbit and $\omega(\gamma^+)$ does not contain a critical point, then either

$$\gamma^+ = \omega(\gamma^+), \qquad \text{or} \qquad \omega(\gamma^+) = \overline{\gamma^+} \setminus \gamma^+.$$

In either case, the ω-limitset is a periodic orbit.

Proof. From above theorems we have:

- $\omega(\gamma^+)$ is compact, connected and nonempty.
- There exists a bounded minimal set $M \subset \omega(\gamma^+)$;
- By assumption M contains ordinary points only. This implies that M is a periodic orbit γ_0.
- Since $\omega(\gamma^+)$ contains both ordinary points and a periodic orbit γ_0 then we have $\omega(\gamma^+) = \gamma_0$.

4.4 Applications of the Poincaré-Bendixson theorem

To apply PB theorem one has to

- find a domain $D \subset \mathbb{R}^2$ which contains ordinary points only
- find at least one orbit which for $t \geq 0$ enters the domain D without leaving it.

Then D must contain at least one periodic orbit.

Example.

$$\begin{aligned} \dot{x} &= x - y - x(x^2 + y^2), \\ \dot{y} &= y + x - y(x^2 + y^2). \end{aligned}$$

The only critical point is $(0, 0)$.

$$\frac{\partial f}{\partial(x, y)}(x, y) = \begin{pmatrix} 1 - 3x^2 - y^2 & -1 - 2xy \\ 1 - x^2 - 3y^2 & 1 - x^2 - 3y^2 \end{pmatrix}$$

$$\frac{\partial f}{\partial(x, y)}(0, 0) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

The eigenvalues are $1 \pm i$. Thus $(0, 0)$ is a spiral point with negative attraction.

In polar coordinates the system can be written as

$$\dot{r} = r(1 - r^2), \qquad \dot{\theta} = 1.$$

Consider the annular domain $r_1 < r < r_2$, $r_1 < 1 < r_2$, and center $(0, 0)$

For any orbit inside the circle $r = 1$, $\dot{r} > 0$ and thus will eventually enter the annulus. For any orbit outside the circle $r = 1$, $\dot{r} < 0$ and thus will eventually enter the annulus. Therefore, all orbits will eventually enter the annulus. Thus according to PB theorem, the annulus should contain one or more limit cycles.

Example.

$$\begin{aligned} \dot{x} &= x(x^2 + y^2 - 2x - 3) - y, \\ \dot{y} &= y(x^2 + y^2 - 2x - 3) + x. \end{aligned}$$

The only critical point is $(0, 0)$; this is a spiral point with positive attraction since

$$\frac{\partial f}{\partial(x, y)}(x, y) = \begin{pmatrix} 3x^2 + y^2 - 4x - 3 & 2xy - 1 \\ 2x - 2 & x^2 + 3y^2 - 2x - 3 \end{pmatrix},$$

$$\frac{\partial f}{\partial(x, y)}(0, 0) = \begin{pmatrix} -3 & -1 \\ -2 & -3 \end{pmatrix}.$$

The eigenvalues are -1 and -5.

We can apply Bendixson's criterion to see whether closed orbits are possible.

$$\begin{aligned} \nabla \cdot (f,g) &= 3x^2 + y^2 - 4x - 3 + x^2 + 3y^2 - 2x - 3 \\ &= 4x^2 + 4y^2 - 6x - 6 = 4\left[\left(x - \frac{3}{4}\right)^2 + y^2 - \frac{33}{16}\right] \end{aligned}$$

Inside the (Bendixson-) circle with center $(3/4, 0)$ and radius $\sqrt{33}/4$ the expression is sign definite and no closed orbit can be contained in the interior of this circle.

Closed orbits are possible which enclose or which intersect this Bendixson-circle. In polar coordinates the system takes the form

$$\dot{r} = r(r^2 - 2r\cos\theta - 3), \qquad \dot{\theta} = 1.$$

Note that

$$(r-3)(r+1) = r^2 - 2r - 3 \leq \boxed{r^2 - 2r\cos\theta - 3} \leq r^2 + 2r - 3 = (r+3)(r-1).$$

Thus

$$\dot{r} \text{ is } \begin{cases} < 0 \quad if \quad r < 1, \\ \\ > 0 \quad if \quad r > 3. \end{cases}$$

Moreover, for $r = 1$ and θ in the neighborhood of π we have $\dot{r} < 0$. Thus any orbit that passes through such a point remains

The annulus $1 < r < 3$ contains only ordinary points.

Since the annulus contains no critical points, by PB theorem the annulus must contain at least one periodic orbit.

Liénard and van der Pol Equation

Consider the equation

$$\ddot{x} + f(x)\,\dot{x} + x = 0$$

with $f(x)$ Lipschitz-continuous in $\mathbb{R}$. Let

$$F(x) = \int_0^x f(s)\,ds.$$

We assume the following

a. $F(x)$ is an odd function.

b. $F(x) \to \infty$ as $x \to \infty$ and there exists a constant $\beta > 0$ such that for $x > \beta$, $F(x) > 0$ and monotonically increasing.

c. There is an $\alpha > 0$ such that for $F(x) < 0$ for $0 < x < \alpha$ and $F(\alpha) = 0$.

Note that $F'(x) = f(x)$. Since $F(0) = 0$, it follows from assumption (c) that

$$f(0) = F'(0) \leq 0.$$

We can write the equation as

$$\frac{d}{dt}\left[\dot{x} + F(x)\right] + x = 0.$$

Using the transformation

$$x \longrightarrow x, \qquad \dot{x} \longrightarrow y = \dot{x} + F(x),$$

we obtain the equivalent system

$$\begin{aligned} \dot{x} &= y - F(x), \\ \dot{y} &= -x. \end{aligned}$$

Note that on replacing (x, y) in the system by $(-x, -y)$ the system does not change since $F(x)$ is odd. This means that if $(x(t), y(t))$ is a solution, the reflection through the origin $(-x(t), -y(t))$ is also a solution.

Linear Analysis

The system has only one critical point $(0,0)$. The linearized system in the neighborhood of $(0,0)$ has the matrix

$$\begin{pmatrix} -f(x) & 1 \\ -1 & 0 \end{pmatrix}(0,0) = \begin{pmatrix} -f(0) & 1 \\ -1 & 0 \end{pmatrix}.$$

With $F'(0) = f(0)$ the eigenvalues are

$$\lambda_{1,2} = \frac{-f(0) \pm \sqrt{f^2(0) - 4}}{2}.$$

Note that if $f(0) \neq 0$ then $f(0) < 0$ and clearly $Re\,\lambda_{1,2} > 0$, so $(0,0)$ is a negative attractor for the linearized system and thus for the nonlinear equation.

Theorem. *Under the conditions (a-c) the equation has at least one periodic solution. If $\alpha = \beta$ there exists only one periodic solution and the corresponding orbit is ω-limitset for all orbits except the critical point $(0,0)$.*

Proof. The orbits are solutions of the first order equation

$$\frac{dy}{dx} = -\frac{x}{y - F(x)}.$$

Note that the slopes of the orbits $y = y(x)$ are horizontal on the x-axis and vertical on the y-axis.

Consider the function

$$R = \frac{1}{2}(x^2 + y^2).$$

Along any orbit we have

$$\frac{dR}{dx} = x + y\frac{dy}{dx} = x - \frac{xy}{y - F(x)} = -\frac{x\,F(x)}{y - F(x)},$$

and

$$\frac{dR}{dy} = \frac{dR}{dx}\frac{dx}{dy} = \frac{x\,F(x)}{y - F(x)}\,\frac{y - F(x)}{x} = F(x).$$

Now we show that for an orbit starting in $(0, y_0)$ with y_0 sufficiently large the behavior is as shown in figure 4.12 with $|y_1| < y_0$. If this is the case, the reflection of the orbit produces an invariant set bounded by the two orbits and the segments $[-y_1, y_0]$ and $[-y_0, y_1]$. We assume that $y_1 \to -\infty$ as $y_0 \to \infty$, otherwise, we have the required behavior.

The line integral along such an orbit yields

$$\begin{aligned} R(0,y_1) - R(0,y_0) = \int_{ABECD} dR &= \int_{AB} + \int_{CD} \frac{dR}{dx}\,dx + \int_{BEC} \frac{dR}{dy}\,dy \\ &= \int_{AB} + \int_{CD} \frac{-xF(x)}{y-F(x)}\,dx + \int_{BEC} F(x)\,dy. \end{aligned}$$

Since $F(x)$ is bounded for $0 \le x \le \beta$, the first expression tends to zero as $y_0 \longrightarrow \infty$.

Along the line integral of the second expression, $x > \beta$ and thus $F(x) > 0$. Since dy is negative, the integral is negative. The integral approaches $-\infty$ as $y_0 \to \infty$ because of the unbounded increase of length of the curve BEC. We conclude that if y_0 is large enough

$$R(0,y_1) - R(0,y_0) < 0.$$

The PB theorem implies the existence of at least one periodic solution.

Now suppose that $\alpha = \beta$. As shown in figure 4.13, if we choose y_0 sufficiently small, the orbits behave like C_1 and

$$R_{D_1} - R_{A_1} = \int_{A_1E_1D_1} F(x)\,dy > 0,$$

since $F(x) < 0$ for $0 < x < \alpha$. So no periodic orbit can start in $(x_0, 0)$ with $0 < x_0 < \alpha$.

Next consider a curve C_2 intersecting the x-axis in E_2, to the right of $(\alpha, 0)$.

$$I = R_{D_2} - R_{A_2} = \int_{A_2E_2D_2} F(x)\,dy.$$

For $x \ge \alpha$, $F(x)$ is monotonically increasing from 0 to ∞. As above this integral tends to $-\infty$ as $y_0 \to \infty$. Because of the monotonicity of $F(x)$, the integral I has one zero, i.e. one y_0 such that $R_{A_2} = R_{D_2}$ and so one periodic solution. ■

Remark. *For the function R we have*

$$\dot{R} = x\dot{x} + y\dot{y} = x(y - F(x)) + y(-x) = -x\,F(x).$$

For $-\alpha < x < \alpha$ we have $\dot{R} \ge 0$; this is in agreement with the negative attraction of $(0,0)$. Orbits starting on the boundary of a circular domain with radius smaller than α, can not enter this circular domain.

Application. (van der Pol equation)

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \quad \mu > 0.$$

In this case

$$f(x) = \mu(x^2 - 1), \qquad F(x) = \mu\left(\frac{1}{3}x^3 - x\right).$$

The conditions (a-c) are all satisfied with $\alpha = \beta = \sqrt{3}$.

4.5 Periodic Solution in $\mathbb{R}^n$

- We know that because of the translation property of solutions of autonomous equations, periodic solutions of these equations correspond with closed orbits in phase space. This does not hold for non-autonomous equations.
- Consider the equation
$$\dot{x} = f(t,x), \qquad x \in \mathbb{R}^n.$$
Given $T > 0$, let
$$D = \{x_0 \in \mathbb{R}^n : \; x(t;x_0) \text{ exists for } 0 \le t - t_0 \le T\}$$
Define the mapping $a_T : D \to \mathbb{R}^n$ by
$$a_T(t_0, x_0) = x(t_0 + T; x_0).$$
The point x_0 is a fixed point of a_T if
$$a_T(t_0, x_0) = x_0 \quad \text{or} \quad x(t_0 + T; x_0) = x_0.$$

Example. *Consider the system*
$$\dot{x} = 2x + \sin t, \qquad \dot{y} = -y.$$
The general solutions starting for $t_0 = 0$, $(x(0), y(0)) = (x_0, y_0)$, are
$$x(t) = \left(x_0 + \frac{1}{5}\right) e^{2t} - \frac{1}{5}\cos t - \frac{2}{5}\sin t, \qquad y(t) = y_0\, e^{-t}.$$

- Let D be a square in the first quadrant. The mappings $a_1(0, x_0, y_0)$ and $a_{2\pi}(0, x_0, y_0)$ produce rectangles in the first quadrant which are the same qualitatively but not quantitatively. See figure 4.14. There are no fixed points in these two cases.
- Let D be the square with side length 1 and center $(0,0)$. The mapping $a_1(0, x_0, y_0)$ produce a contraction in the y-direction and an asymmetric expansion in the x-direction. See figure 4.15. There are no fixed points
- Let D be the square with side length 1 and center $(0,0)$. Again, the mapping $a_{2\pi}(0, x_0, y_0)$ produce a contraction in the y-direction and an asymmetric expansion in the x-direction. See figure 4.16.

  By inspection $(-1/5, 0)$ is a fixed point. Thus there exists one periodic solution which starts in $(-1/5, 0)$ with period 2π.

In a number of cases a fixed point of the mapping a_T corresponds with a T-periodic solution of the equation

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Lemma. *Consider the equation $\dot{x} = f(t, x)$ in $\mathbb{R}^n$ with a righthand side which is T-periodic $f(t + T, x) = f(t, x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^n$. The equation has a T-periodic solution if and only if the T-mapping a_T has a fixed point.*

Proof. It is clear that a T-periodic solution produces a fixed point of a_T. Suppose now that a_T has a fixed point x_0. Then we have $x(t_0 + T; x_0) = x_0$ for a certain solution $x(t; x_0)$ starting at $t = t_0$.

The vector function $x(t + T; x_0)$ will also be a solution as

$$\dot{x} = f(t, x) = f(t + T, x).$$

As x_0 is a fixed point of a_T, $x(t + T; x_0)$ has the same initial value x_0 as $x(t; x_0)$ since $x(t_0 + T; x_0) = x_0$. Because of the uniqueness we have $x(t + T; x_0) = x(t; x_0)$, $t \in \mathbb{R}$, so $x(t; x_0)$ is T-periodic.

Theorem. *(Brouwer's fixed point theorem) Consider a compact, convex set $V \subset \mathbb{R}^n$ which is not empty. Each continuous mapping of V into itself has at least one fixed point.*

Theorem. *(Application of Brouwer's fixed point theorem) Consider the equation $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ with positive invariant, compact, convex set $V \subset \mathbb{R}^n$. The equation has at least one critical point in V.*

Proof. For any $T > 0$, consider the T-mapping $a_T : V \to V$. It follows from Brouwer's theorem that there is at least one fixed point and so the equation contains a T-periodic solution.

Consider a sequence $\{T_m\} \to 0$ as $m \to \infty$ with corresponding fixed points $\{q_m\} \subset V$ of a_{T_m}: $x(T_m; q_m) = q_m$. Since the sequence is bounded, there is a convergent subsequence $\{p_m\}$. Let $p_m \to p$ as $m \to \infty$. We have the estimate

$$\|x(t; p) - p\| \leq \|x(t; p) - x(t; p_m)\| + \|x(t; p_m) - p_m\| + \|p_m - p\|.$$

Consider the right hand side. The last term $\to 0$ as $m \to \infty$. Because of the convergence and continuity, the first term $\to 0$ as $m \to \infty$. The second term is estimated as follows. The solution $x(t; p_m)$ is T_m-periodic. For any $t > 0$ we can find $N \in \mathbb{N}$ such that $t = (N + u)T_m$. Thus

$$x(t; p_m) = x((N + u)T_m; p_m) = x(uT_m; p_m), \qquad 0 \leq u < 1.$$

Also since $T_m \to 0$ as $m \to \infty$ we have

$$\|x(t; p_m) - p_m\| \underset{def}{=} \|x(uT_m; p_m) - x(0; p_m)\| \longrightarrow 0, \text{ for any } t.$$

Thus it follows from the estimate that

$$x(t;p) = p \text{ for any } t.$$

and thus p is a critical point. ■

Remark. *The methods of this section have serious limitations.*

- *Provide a lower bound only of the number of periodic solutions.*
- *Do not lead to a localization of the periodic solutions so explicit calculations are still needed.*

Another theorem based on Brouwer's fixed point theorem.

Theorem. *Consider the equation*

$$\dot{x} = A(t)\,x + g(t,x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n.$$

We assume the following:

- $A(t)$ *is continuous in* t.
- $g(t,x)$ *is continuous in* t *and Lipschitz continuous in* x.
- $A(t)$ *and* $g(t,x)$ *are* T*-periodic in* t.
- $g(t,x) = o(\|x\|)$ *as* $\|x\| \to \infty$ *uniformly in* t.
- *The equation* $\dot{y} = A(t)\,y$ *has no* T*-periodic solutions except the solution* $y = \dot{y} = 0$.

Then, the equation for x *has at least one* T*-periodic solution.*

See the reference in the book.

Example. *Consider the scalar equation*

$$\ddot{x} + \omega^2 x = \frac{1+x}{1+x^2}\cos t.$$

The equivalent system is

$$\dot{u} = v, \qquad \dot{v} = \frac{1+u}{1+u^2}\cos t - \omega^2 u.$$

For this system

$$A(t) = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \qquad g(t,u,v) = \begin{pmatrix} 0 \\ \frac{1+u}{1+u^2}\cos t \end{pmatrix}$$

The linearized system has the eigenvalues $\pm\omega i$. Then clearly the system has no 2π-periodic solution if $\omega^2 \neq 1$. By the above theorem, the equation for x has at least one 2π-periodic solution if $\omega^2 \neq 1$.
