3 Critical points

$$\dot{x} = f(x) \quad \underset{linearization}{\longrightarrow} \quad y = A\dot{y}$$

In this formulation the critical point has been translated to the origin. We also assume that the critical point is non-degenerate, i.e. $\det A \neq 0$.

The eigenvalues of the matrix A can be used to analyze the critical point of the linear system.

Using matrix notation, we use the eigenvalues to construct a linear transformation $y = Tz$ to simplify the system to

$$T\dot{z} = ATz \qquad \Longrightarrow \qquad \dot{z} = T^{-1}ATz.$$

Here $T^{-1}AT$ is the Jordan normal form which is so simple that we can integrate to find z and thus $y = Tz$.

3.1 Two-dimensional linear systems

Stable manifold. If the eigenvalues of A are real and have different sign then the critical point is a saddle point. The orbits of $\dot{z} = Az$ are given by

$$|z_1| = c|z_2|^{-|\lambda_1/\lambda_2|}.$$

The behavior of the orbits is hyperbolic.

Note that the coordinate axes correspond to five solutions: the critical point $(0,0)$ and the four half axes.

Two of these solutions approach the origin as $t \to \infty$ while the other two approach the origin as $t \to -\infty$.

The first two are called the stable manifolds of the saddle point, the other two the unstable manifolds

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3.2 Remarks on three-dimensional linear systems

If the eigenvalues are distinct then $T^{-1}AT$ is a diagonal matrix and we have

$$z(t) = \left(c_1 e^{\lambda_1 t}, c_2 e^{\lambda_2 t}, c_3 e^{\lambda_3 t}\right).$$

We consider the following cases.

a. Three real eigenvalues. If all have the same sign then we have a three-dimensional node otherwise we have a saddle-node

b. One real and two complex conjugate eigenvalues.

- If all lie in $\{z : Re\, z < 0\}$ then we have a positive attractor.
- If all lie in $\{z : Re\, z > 0\}$ then we have a negative attractor.
- Sign of the real part of the complex eigenvalues is different from the third real eigenvalue. Then we have (positive/negative) attraction in one direction and (attractor/expansion) in the other two directions.
- Two eigenvalues purely imaginary. Then there is only in one direction positive or negative attraction

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3.3 Critical points of nonlinear equations

Above we analyzed critical points of autonomous equations $\dot{x} = f(x)$ by linear analysis. We assume that the critical point has been translated to the origin and that we can write the equation in the form

$$\dot{x} = Ax + g(x),$$

with A a non-singular $n \times n$ matrix and that

$$\lim_{\|x\| \to 0} \frac{\|g(x)\|}{\|x\|} = 0.$$

Remark.

- *A sufficient condition for this assumption is that $f(x)$ is continuously differentiable in a neighborhood of $x = 0$.*
- *The analysis of nonlinear equations usually starts with a linear analysis as described here, after which one tries to draw conclusions about the original, nonlinear system.*
- *Some properties of the linearized system also hold for the nonlinear systems.*
- *Other properties do not carry over from linear to nonlinear systems.*
- *Nomenclature: node, saddle , focus, center for the nonlinear equation.*
- *For plane systems ($n = 2$) we can still extend this terminology to the nonlinear case.*
- *For systems with dimension $n > 2$ this nomenclature meets with many problems and so we characterize a critical point in that case with properties as attraction, eigenvalues, existence of (un)stable manifolds etc.*

Nonlinear equation attractors vs Linearized equation attractors

To prepare for the stability analysis later we introduce the following two theorems.

Theorem. *Consider the equation $\dot{x} = Ax + g(x)$; if $x = 0$ is a positive (negative) attractor for the linearized equation, then $x = 0$ is a positive (negative) attractor for the non-linear equation $\dot{x} = Ax + g(x)$.*

That is the origin remains an attractor when we add the perturbation terms.
This result is not true for nonautonomous equations, $A = A(t)$. See Bruer and Nohel p. 158.

Theorem. *Consider the equation $\dot{x} = Ax + g(x)$; if A has an eigenvalue with positive real part, then the critical point $x = 0$ is not a positive attractor for the nonlinear equation.*

In particular, if the critical point is a saddle, then the critical point can not be an attractor for the nonlinear equation.

Stable and unstable manifolds.
Consider the linear system

$$\dot{y} = Ay.$$

Let $E(\lambda)$ be the generalized eigenspace of the $n \times n$ matrix A for the eigenvalue λ.

Definition. *The stable manifold of the linear system $\dot{y} = Ay$ is defined as the linear subspace E_s of $\mathbb{R}^n$ which is equal to the sum over the generalized eigenspaces with eigenvalues λ which have negative real part.*
The unstable manifold E_u is the sum over the eigenvalues with positive real parts.

Note that since A is assumed to be nonsingular, 0 is not an eigenvalue.

Remark.

- *E_s and E_u are invariant sets of the linear system.*
- *Solutions are bounded for $t \to \infty$ only if $y(t_0) \in E_s$.*
- *Solutions starting in the complement of $E_s \cup E_u$ "come from infinity and run away to infinity".*
- *If non of the eigenvalues is pure imaginary, then we have*

$$\mathbb{R}^n = E_s \oplus E_u.$$

The following theorem contains a generalization to the nonlinear case.

Theorem. *Consider the equation*

$$\dot{x} = Ax + g(x), \qquad x \in \mathbb{R}^n$$

in which the constant $n \times n$ matrix A has n eigenvalues with nonzero real part; $g(z)$ is C^k in a neighborhood of $x = 0$ and

$$\lim_{\|x\| \to 0} \frac{\|g(x)\|}{\|x\|} = 0.$$

Then there exists a C^k manifold W_s, called the stable manifold of $x = 0$, with the following properties:

a1. $0 \in W_s$, *W_s has the same dimension as E_s and the tangent space of W_s at $x = 0$ is equal to E_s.*

a2. *If we have $x(t_0) \in W_s$ for a solution $x(t)$, then $x(t) \in W_s$ for all $t \geq t_0$ and* $\lim_{t \to \infty} x(t) = 0$

a3. *If $x(t_0) \notin W_s$ for a solution $x(t)$, then $\|x(t)\| \geq \delta$ for some real, positive δ, suitable $t_1 \geq t_0$ and $t \geq t_1$.*

Similarly, there exists a C^k manifold W_u, called the unstable manifold of $x = 0$ with the properties

b1. $0 \in W_u$, *W_u has the same dimension as E_u and the tangent space of W_u at $x = 0$ is equal to E_u.*

a2. *If we have $x(t_0) \in W_u$ for a solution $x(t)$, then $x(t) \in W_u$ for all $t \leq t_0$ and* $\lim_{t \to -\infty} x(t) = 0$

a3. *If $x(t_0) \notin W_s$ for a solution $x(t)$, then $\|x(t)\| \geq \delta$ for some real, positive δ, suitable $t_1 \leq t_0$ and $t \leq t_1$.*

Results.

Under the conditions of the above theorem, the phase-flows of the linear and the nonlinear equations are homeomorphic in a neighborhood of $x = 0$ (a homeomorphism is a continuous mapping which has a continuous inverse).

Example. *Consider the system in* $\mathbb{R}^2$

$$\dot{x} = -x, \qquad \dot{y} = 1 - x^2 - y^2.$$

The critical points are $(0,1)$ and $(0,-1)$. We have

$$\frac{\partial f}{\partial(x,y)}(x,y) = \begin{pmatrix} -1 & 0 \\ -2x & -2y \end{pmatrix}.$$

$$\frac{\partial f}{\partial(x,y)}(0,\pm 1) = \begin{pmatrix} -1 & 0 \\ 0 & \mp 2 \end{pmatrix}.$$

Thus by linear analysis, $(0,1)$ is a positive attractor while $(0,-1)$ is a saddle.

Remark. *It is easy to see that the conditions of the above theorem for the existence of stable and unstable manifolds have been satisfied.*

The stable manifold for the saddle separate the phase plane into two domains where the behavior of the orbits is qualitatively different. Such a manifold we call a <u>*separatrix*</u>.
