2 Autonomous Equations

An equation is autonomous if t does not appear explicitly in the equation. The vector equation

$$\dot{x} = f(x)$$

is called autonomous.

The scalar equation of order n

$$x^{(n)} + F(x^{(n-1)}, \dots, x) = 0$$

is autonomous and can be written as an autonomous system.

2.1 Phase-space, orbits

Translation property

Lemma. *Suppose that $\phi(t)$ of $\dot{x} = f(x)$ in the domain $D \subset \mathbb{R}^n$, then $\phi(t - t_0)$ is also a solution.*

Proof. Let $\tau = t - t_0$. Then $d/dt = d/d\tau$ and the rhs does not change since t does not occur explicitly. Thus the autonomous equation does not change. So $\phi(\tau)$ is a solution of the transformed equation which is the same as the original equation. ■

Remark. *It follows from this lemma that:*

$$\begin{aligned} &\phi(t) \text{ is a solution of } \dot{x} = f(x), x(0) = x_0 \quad \Longrightarrow \\ &\phi(t - t_0) \text{ is a solution of } \dot{x} = f(x), x(t_0) = x_0. \end{aligned}$$

For example since $\sin t$ *is a solution of the equation $\ddot{x} + x = 0$, then* $\cos t$ *is also a solution since* $\sin(t - \pi/2) = \cos t$.

Remark.

- *Although the solutions $\phi(t)$ and $\phi(t-t_0)$ are different, we will see that these solutions correspond to the same orbital curves in phase space.*
- *The translation property is important for the study of periodic solutions and for the theory of dynamical systems.*

Definition. *Consider the equation* $\dot{x} = f(x)$ *with* $x \in D \subset \mathbb{R}^n$.

- D *is called* phase-space. *The space* $G = \mathbb{R} \times \mathbb{R}^n$ *is called the* solution space.
- *A point in the phase space with coordinates* $(x_1(t), x_2(t), \ldots, x_n(t))$ *for certain* t *is called a* phase point.
- *The solution curves in the phase-plane are called* orbits.
- *The motion of a set of points with* t *along the corresponding orbits is called the* phase flow.

Example. *Consider the harmonic equation*

$$\ddot{x} + x = 0.$$

which have the linearly independent solutions $\{\sin t, \cos t\}$.

To obtain the corresponding vector equation we put $x = x_1$ *and* $\dot{x} = x_2$ *to obtain*

$$\dot{x_1} = x_2, \quad \dot{x_2} = -x_1,$$

for which the solution is $\{x_1(t), x_2(t)\} = \{\sin t, \cos t\}$. *In the solution space the sketch of the solution is a spiraling curve centered around the line* $(x, \dot{x}, 0) = (0, 0, t)$. *The projection in the phase-plane, the* $x\dot{x}$*-plane, are circles centered at* $(0, 0)$ *and parameterized by* t.

Construction of solution curves

In general to obtain the solution curves in the phase-space we formulate a differential equation describing the behavior of the orbits as follows.

The components can be written as

$$\dot{x}_i = f_i(x), \qquad i = 1, \ldots, n.$$

We use one of the components x_i as independent variable if $f_i(x) \neq 0$. Then with the chain rule we obtain $(n-1)$ equations

$$\frac{dx_j}{dx_i} = \frac{f_j(x)}{f_i(x)}, \qquad j = 1, \ldots, n, \quad j \neq i.$$

The solutions of this system are the orbits.

The existence and uniqueness theorem applied to this system implies that the orbits in phase space will not intersect.

for this method to work we have to exclude the singularities of the rhs corresponding to $f_i(x) = 0$ by taking another independent variable.

There is a problem if there is a point $a = (a_1, \ldots, a_n)$ such that $f_1(a) = \cdots = f_n(a) = 0$, i.e. $f(a) = 0$. Such a point is called a critical point or equilibrium point.

Example. *Consider the system* $\dots x_1 = x_2$, $\dot{x}_2 = -x_1$ *corresponding to the harmonic oscillator equation* $\ddot{x} + x = 0$*. The phase space is* $\mathbb{R}^2$ *and the orbits are described by the equation*

$$\frac{dx_2}{dx_1} = -\frac{x_1}{x_2}.$$

Integration produces the family of circles

$$x_1^2 + x_2^2 = c.$$

The point $(0,0)$ *is a critical point and is called a* <u>*center*</u>

Example. *Consider the equation* $\ddot{x} - x = 0$*. The equation of the orbits is* $dx_2/dx_1 = x_1/x_2$*. Integration produces the family of hyperbolas*

$$x_1^2 - x_2^2 = 0.$$

$(0,0)$ *is critical point and is called a* <u>*saddle*</u>*.*

2.2 Critical points and linearization

Consider the equation $\dot{x} = f(x)$, $x \in D \subset \mathbb{R}^n$.

Definition. *The point* $x = a$ *with* $f(a) = 0$ *is called a critical point of equation* $\dot{x} = f(x)$*.*

- A critical point in phase-space can be considered as an orbit, degenerated into a point.
- The critical point $x(t) = a$ is a solution that satisfies the equation for all t (equilibrium solution).
- The existence and uniqueness implies that an equilibrium solution can never be reached in a finite time (otherwise two solutions would intersect).

Example.

$$\dot{x} = -x, \qquad t \geq 0.$$

$x = 0$ *is a critical point,* $x(t) = 0$, $t \geq 0$ *is an equilibrium solution. For all solutions* $x(t)$ *with* $x(0) \neq 0$ *we have* $\lim_{t\to\infty} x(t) = 0$ *(the solutions are* $x(t) = x(0)\, e^{-t}$ *).*

Example.

$$\dot{x} = -x^2, \qquad t \geq 0.$$

$x = 0$ is a critical point, $x(t) = 0$, $t \geq 0$ is an equilibrium solution. So the solutions $x(t)$ show qualitative and quantitative different behavior depending on the sign of $x(0)$.

If $x(0) > 0$ then the solutions is positive and decreasing (negative derivative). If $x(0) < 0$ then the solution is negative and decreasing.

More precisely, the exact solutions are

$$x(t) = \frac{1}{x_0^{-1} + t}, \qquad x_0 \neq 0.$$

If $x_0 < 0$, the solution becomes unbounded in a finite time.

Definition. *A critical point $x = a$ is called a positive (negative) attractor if there exists a neighborhood $\Omega_a \subset \mathbb{R}^n$ of $x = a$ such that $x(t_0) \in \Omega_a$ implies $x(t) \to a$ as $t \to \infty$ $(-\infty)$.*

Linearization

Suppose f has a Taylor series expansion at $x = a$. Then we can write

$$f(x) = f(a) + \frac{\partial f}{\partial x}(a)\,(x-a) + O\left[(x-a)^2\right].$$

If $x = a$ is a critical point of $\dot{x} = f(x)$ then we have

$$\dot{x} = \frac{\partial f}{\partial x}(a)\,(x-a) + O\left[(x-a)^2\right].$$

The linearized system is the system

$$\dot{y} = \frac{\partial f}{\partial y}(a)\,(y-a).$$

Putting $\bar{y} = y - a$ yields

$$\dot{\bar{y}} = \frac{\partial f}{\partial y}(a)\,\bar{y}.$$

Letting $A = \partial f/\partial y(a)$ and omitting the bar, the linearized system in a neighborhood of $x = a$ takes the form

$$\dot{y} = A\,y.$$

Example. *Consider the pendulum equation*

$$\ddot{x} + \sin x = 0, \qquad x \in [-\pi, \pi].$$

The corresponding autonomous system is

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1.$$

The critical points are $(x_1, x_2) = (0,0), (-\pi, 0), (\pi, 0)$

The linearized system in a neighborhood of $(0,0)$ *is*

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}.$$

This system is described above.

Example. *Consider the system*

$$\begin{aligned} \dot{x} &= ax - bxy \\ \dot{y} &= bxy - cy \end{aligned}$$

with $x, y \geq 0$ *and* a, b, c *positive constants.*

Critical points are $(0,0)$ and $(c/b, a/b)$.

$$\frac{\partial f(x,y)}{\partial(x,y)} = \begin{pmatrix} a - by & -bx \\ by & bx - c \end{pmatrix}.$$

In a neighborhood of $(0,0)$ the linearized system is

$$\begin{aligned} \dot{x} &= ax \\ \dot{y} &= -cy \end{aligned}$$

The solution of this system is $(x(t), y(t)) = (x(0)\, e^{at}, y(0)\, e^{-ct})$.

In a neighborhood of $(c/b, a/b)$ the linearized system is

$$\begin{aligned} \dot{x} &= -c\,(y - \tfrac{a}{b}) \\ \dot{y} &= a\,(x - \tfrac{c}{b}). \end{aligned}$$

The solutions of this system are combinations of $\cos(\sqrt{ac}\,t)$ and $\sin(\sqrt{ac}\,t)$.

2.3 Periodic solutions

Definition. *Suppose* $\phi(t)$ *is a solution of the equation* $\dot{x} = f(x)$, $x \in D \subset \mathbb{R}^n$. *Suppose there exists a positive number* T *such that* $\phi(t + T) = \phi(t)$ *for all* $t \in \mathbb{R}$. *Then* $\phi(t)$ *is a periodic solution of the equation with period* T.

Lemma.

Let $\phi(t)$ *be a solution of* $\dot{x} = f(x)$. *Then* $\phi(t)$ *is periodic* $\Longleftrightarrow$ *the corresponding orbit in phase space is closed.*

Proof. A periodic solution assumes the same values in $\mathbb{R}^n$ and thus produces a closed orbit.

Consider a closed orbit C traced by the solution $\phi(t)$. We want to show that there is $T > 0$ such that $\phi(t + T) = \phi(t)$ for all $t \in \mathbb{R}$.

By the uniqueness theorem, C can not contain a critical point. Thus $\|\dot{x}\| = \|f(x)\| > 0$. Thus if we start from a point $x_0 \in C$ then we return to the same point after a certain time T. This implies that there is $T > 0$ such that $\phi(0) = \phi(T)$. By uniqueness and translation theorem, $\phi(t) = \phi(t + T)$ for all $t \in \mathbb{R}$.

Example. *For the pendulum equation* $\ddot{x} + \sin x = 0$ *the phase plane contains a family of closed orbits corresponding with periodic solutions.*

Remark. *For non-autonomous equations of the form* $\dot{x} = f(t, x)$*, closed orbits do not necessarily correspond with periodic solutions since the translation property is not valid.*

Consider for example the system

$$\dot{x} = 2ty, \quad \dot{y} = -2tx,$$

with solution of the form

$$x(t) = \alpha \cos t^2 + \beta \sin t^2, \qquad y(t) = -\alpha \sin t^2 + \beta \cos t^2.$$

In the phase-plane we have closed orbits, the solutions are not periodic.

2.4 First integrals and integral manifolds

Consider the harmonic oscillator equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1.$$

Then the orbits satisfy the differential equation

$$\frac{dx_1}{dx_2} = -\frac{x_2}{x_1}.$$

The solution of this separable equation is

$$F(x_1, x_2) = \frac{x_1^2}{2} + \frac{x_2^2}{2} = E, \qquad E \geq 0 \text{ a constant.}$$

We call this expression the first integral of the harmonic oscillator equation.

In phase space, this relation with $E > 0$ corresponds to a manifold, a circle around the origin. Also we can think of the orbits as the level surfaces $F(x_1, x_2) = E$.

Moreover, on a solution (x_1, x_2) of the differential equation we have

$$\frac{d}{dt} F(x_1, x_2) = \frac{\partial F}{\partial x_1}\,\dot{x}_1 + \frac{\partial F}{\partial x_2}\,\dot{x}_2 = x_1\,x_2 + x_2\,(-x_1) = 0$$

In general we have the following definitions.

Definition. *Consider the differentiable function $F : \mathbb{R}^n \to \mathbb{R}$ and the vector function $x : \mathbb{R} \to \mathbb{R}^n$. The derivative L_t of the function F along the vector function x, parameterized by t, is*

$$L_t F := \frac{\partial F}{\partial x_1}\,\dot{x}_1 + \cdots + \frac{\partial F}{\partial x_n}\,\dot{x}_n = \nabla F(x) \cdot \dot{x}.$$

L_t is called the orbital derivative.

Definition. *Consider the equation $\dot{x} = f(x)$, $x \in D \subset \mathbb{R}^n$; the function $F(x)$ is called first integral of the equation if in D holds*

$$L_t F = 0$$

with respect to the vector function $x(t)$.

Remark. *It follows from the definition that*

- *The first integral $F(x)$ is constant along a solution. For this reason the first integrals are called "constants of motion".*
- *The level sets of $F(x)$ contain orbits of the equation.*
- *A level set defined by $F(x) = constant$ which consists of a family of orbits is called an integral manifold.*
- *Integral manifolds of an equation help in understanding the build-up of phase-space of this equation.*
- *The concept of integral manifold is related to the concept of invariant set.*

Definition. *Consider the equation $\dot{x} = f(x)$ in $D \subset \mathbb{R}^n$. The set $M \subset D$ is invariant if the solution $x(t)$ with $x(0) \in M$ for $-\infty < t < \infty$. If this property is valid only for $t \geq 0$ ($t \leq 0$) then M is called a positive (negative) invariant set.*

Clearly the equilibrium points and in general solutions which exist for all time are (trivial) examples of invariant sets.

Example. *The harmonic equation*

$$\ddot{x} + x = 0$$

has a first integral

$$F(x, \dot{x}) = \frac{1}{2}(\dot{x})^2 + \frac{1}{2}x^2.$$

The family of circles $F(\dot{x}, x) = E$, $E \geq 0$, is the corresponding set of integral manifolds.

Example. *Consider the second order equation*

$$\ddot{x} + f(x) = 0$$

with $f(x)$ sufficiently smooth. If we multiply both sides with $\dot{x}$, we can write the equation as

$$\frac{d}{dt}\left(\frac{1}{2}\dot{x}^2\right) + \frac{d}{dt}\int^x f(\tau)\, d\tau = 0.$$

So a first integral of the equation is given by

$$F(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \int^x f(\tau)\, d\tau.$$

The orbits correspond with the level curves $F(x, \dot{x}) = c$.

Example. *Consider the equation*

$$\ddot{x} + x - \frac{1}{2}x^2 = 0.$$

To find the critical points we write the equation in the form $\dot{z} = f(z)$ with

$$f(z) = \begin{pmatrix} z_2 \\ \frac{z_1^2}{2} - z_1 \end{pmatrix}.$$

Thus the critical points are $(0,0)$ and $(2,0)$.

For the linearized problem we have

$$\frac{\partial f}{\partial z} = \begin{pmatrix} 0 & 1 \\ z_1 - 1 & 0 \end{pmatrix}.$$

which at the critical points takes the values

$$\frac{\partial f}{\partial z}(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \textit{and} \qquad \frac{\partial f}{\partial z}(2,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Clearly $(0,0)$ is center point (stable but not asymptotically stable). The point $(2,0)$ is a saddle point.

For nonlinear characterization, we can use the first integral

$$F(x, \dot{x}) = \frac{\dot{x}^2}{2} + \frac{x^2}{2} - \frac{x^3}{6}.$$

The orbits corresponds with the level sets

$$\dot{x}^2 + x^2 - \frac{x^3}{3} = c.$$

The level surfaces and orbits are shown in Figure 2.10.

The level set that passes through $(2,0)$ is

$$\dot{x}^2 + x^2 - \frac{x^3}{3} = \frac{4}{3}.$$

Definition. *Consider $F : \mathbb{R}^n \to \mathbb{R}$ which is supposed to be C^∞. The point $x = a$ is called a critical point of F if $\nabla F(a) = 0$.*

A critical point $x = a$ of F is called non-degenerate if

$$\det \frac{\partial^2 F(a)}{\partial x^2} \neq 0$$

where $\partial^2 F(a)/\partial x^2$ is the $n \times n$ matrix with entries $\frac{\partial^2 F}{\partial x_i \partial x_j}$.

This implies that the vector function ∇F has a "non-degenerate linearization" in a neighborhood of $x = a$.

Example. *Consider the function*

$$F(x_1, x_2) = x_1^2 + x_2^2.$$

Then

$$\nabla F = (2x_1, 2x_2), \qquad \textit{and} \qquad \frac{\partial^2 F(x)}{\partial x^2} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Thus $(0, 0)$ is a nondegenerate critical point of F.

Definition. *If $x = a$ is a non-degenerate critical point of the C^∞ function $F(x)$, then $F(x)$ is called a Morse-function in a neighborhood of $x = a$.*

Remark. *The behavior of a Morse-function in a neighborhood of the critical point $x = a$ is determined by the quadratic part of the Taylor expansion of the function.*

Suppose $x = 0$ is a non-degenerate critical point of the Morse-function $F(x)$ with expansion

$$F(x) = F_0 - c_1 x_1^2 - c_2 x_2^2 - \cdots - c_k x^2 \quad + \quad c_{k+1} x_{k+1}^2 + \cdots + c_n x_n^2 + O(x^3),$$

with positive coefficients $c_1, \ldots, c_n$. k is called the index of the critical point.

There exists a transformation $x \to y$ in a neighborhood of the critical point such that $F(x) \to G(y)$ where $G(y)$ is a Morse-function with critical point $x = 0$, the same index k, and apart from $G(0)$ only has quadratic terms.

Lemma. *Consider the C^∞ function $F : \mathbb{R}^n \to \mathbb{R}$ with non-degenerate critical point $x = 0$ of index k. In a neighborhood of $x = 0$ there exists a diffeomorphism (one-to-one, unique C^1 transform) which transforms $F(x)$ to the form*

$$G(y) = G(0) - y_1^2 - y_2^2 - \cdots - y_k^2 + y_{k+1}^2 + \cdots + y_n^2.$$

If $k = 0$, then the level sets in a neighborhood of the critical point correspond with a positive definite quadratic form. The level sets are locally diffeomorphic to a sphere. In the case of dimension $n = 2$ we have in this case closed orbits around the critical point.

Example. *Consider the system*

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -f(x_1)$$

with $f(0) = 0$.

Then $(0,0)$ is a critical point. The linearization produces

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -f'0) & 0 \end{pmatrix} x$$

The eigenvalues are $\pm\sqrt{f'(0)}$. So $(0,0)$ is a center if $f'(0) > 0$ or is a saddle if $f'(0) < 0$ for the linearized equation. If $f'(0) = 0$ then we have the degenerate case which we do not consider.

A nonlinear characterization of the critical point can be given with the Morse lemma. From above the first integral is

$$F(x_1, x_2) = \frac{1}{2}x_2^2 + \int_0^{x_1} f(t)\, dt.$$

$F(x_1, x_2)$ is a Morse-function in a neighborhood of $(0,0)$ with the expansion

$$\begin{aligned} F(x_1, x_2) &= f(0,0) + \nabla F(0,0) \cdot (x_1, x_2) + h.o.t \\ &= \frac{1}{2}x_2^2 + \frac{1}{2}f'(0)x_1^2 + h.o.t \end{aligned}$$

If $f'(0) > 0$, there is a C^1 transformation to the quadratic form $y_2^2 + y_1^2$ (index $k = 0$) so for the nonlinear system we also have closed orbits (nonlinear center).

If $f'(0) < 0$, there is a C^1 transformation to the quadratic form $y_2^2 - y_1^2$ (index $k = 1$, saddle).

Example. *Consider the Volterra-Lotka equations*

$$\dot{x} = ax - bxy \qquad \dot{y} = bxy - cy, \qquad x, y \geq 0$$

with positive parameters a,b, and c.

Linearization produced periodic solutions in a neighborhood of the critical point (c/b,a/b).

For the complete nonlinear equations the orbits satisfy the separable differential equation

$$\frac{dx}{dy} = \frac{x}{y}\frac{a - by}{bx - c}.$$

If we integrate we obtain

$$bx - c \ln x + by - a \ln y = C.$$

So a first integral is

$$F(x, y) = bx - c \ln x + by - a \ln y.$$

Notice that

$$L_t F = b\dot{x} - c\frac{\dot{x}}{x} + b\dot{y} + a\frac{\dot{y}}{y} = \frac{\dot{x}}{x}(bx - c) + \frac{\dot{y}}{y}(by + a) = 0.$$

for any solution (x, y) of the system.

Now, $F(x, y)$ is a Morse-function in a neighborhood of $(c/b, a/b)$ with expansion

$$F(x, y) = F(c/b, a/b) + \frac{b^2}{2c}\left(x - \frac{c}{b}\right)^2 + \frac{b^2}{2a}\left(y - \frac{a}{b}\right)^2 + \dots$$

The critical point has index zero and using Morse lemma it follows that the orbits are closed.