1 Introduction

1.1 Definitions and Notation

We consider differential equation of the form

$$\boxed{\dot{x} = f(t, x)} \tag{1}$$

$$\dot{x} = \frac{dx}{dt}, \qquad t \in \mathbb{R}, \qquad x \in \mathbb{R}^n, \qquad G \text{ open subset of } \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1},$$

$f : G \to \mathbb{R}^n$ is continuous in t and x, i.e. $f \in C(G)$.

Definition.
The vector function $x(t)$ is a solution of (1) on an interval $I \subset \mathbb{R}$ if $x : I \to \mathbb{R}^n$ is continuously differentiable and if $x(t)$ satisfies (1).

Remark.
Any general n^{th} order scalar equation

$$\frac{d^n x}{dt^n} = g\left(t, x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}\right),$$

with $g : \mathbb{R}^{n+1} \to \mathbb{R}$, can also be put into the form of (1).

Derivatives.
For $f(t, x)$ we have the following notation

$$\frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \end{pmatrix}$$

Smooth functions.
If no explicit assumption is made, we assume the function $f(t, x)$ to have a convergent Taylor expansion in the domain considered.

A vector function is smooth means that the function has a continuous first derivatives.

Norm.

$$\|f\| = \sum_{i=1}^{n} |f_i|, \qquad f \in \mathbb{R}^n.$$

$$\|A\| = \sum_{i,j=1}^{n} |a_{ij}|, \qquad A \text{ is } n \times n \text{ matrix.}$$

If $f(t,x)$ is a vector function for $t_0 \leq t \leq t_0 + T$ and $x \in D$ with D a bounded domain in $\mathbb{R}^n$; then the uniform norm is defined by:

$$\|f\|_{\sup} = \sup_{\substack{t_0 \leq t \leq t_0 + T \\ x \in D}} \|f\|$$

1.2 Existence and uniqueness

Lipschitz condition

Consider the function $f(t,x)$ with

$$f : \mathbb{R}^{n+1} \to \mathbb{R}^n, \qquad |t - t_0| \le a, \qquad x \in D \subset \mathbb{R}^n.$$

$f(t,x)$ satisfies the Lipschitz condition with respect to x if

$$\|f(t,x_1) - f(t,x_2)\| \le L\|x_1 - x_2\|, \qquad x_1, x_2 \in D,$$

and L a constant. L is called the Lipschitz constant. Also we can use the expression:

$$f(t,x) \text{ is Lipschitz continuous in } x.$$

Note that:

- Necessary condition: Lipschitz continuity in x implies continuity in x
- Sufficient condition: continuous differentiability implies Lipschitz continuity

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Equivalence of the Cauchy problem and the integral equation

We show first that the IVP

$$\dot{x}(t) = f(t,x), \qquad x(t_0) = x_0,$$

is equivalent to finding a continuous solution of the Volterra integral equation (VIE):

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

Theorem. *Suppose $f(t,x)$ is continuous in a domain $G \in \mathbb{R}^{n+1}$ and that $(t_0, x_0) \in G$.*

i. If $x(t)$ is a solution of the IVP on interval I, then $x(t)$ satisfies VIE on I.

ii. If $x(t)$ is a continuous solution of the VIE on some interval J containing t_0, then $x(t)$ satisfies the IVP on J.

Proof.

i. Follows clearly by integration.

ii. If $x(t)$ is a continuous solution of the VIE, then by the continuity of $f(t,x(t))$ the solution $x(t)$ is differentiable. Thus by the FTC applied to the VIE we have that $x(t)$ satisfies

$$\dot{x}(t) = f(t, x(t)), \qquad t \in J$$

and by substituting $t = t_0$, we obtain $x(t_0) = x_0$. ∎

Existence and uniqueness theorem

Theorem. *Consider the initial value problem*

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0,$$

with

$$x \in D \subset \mathbb{R}^n, \qquad |t - t_0| \leq a, \qquad D = \{x : \|x - x_0\| \leq d\},$$

a and b are positive constants. The vector function $f(t, x)$ satisfies the following conditions:

i. *$f(t, x)$ is continuous in $G = [t_0 - a, t_0 + a] \times D$;*

ii. *$f(t, x)$ is Lipschitz continuous in x:*

$$\|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\|, \qquad x_1, x_2 \in D,$$

Then the IVP has a unique solution for

$$|t - t_0| \leq \alpha = \min(a, d/M), \qquad M = \sup_G \|f\|.$$

Proof.

Consider the successive approximations $\{x_m(t)\} \in \mathbb{R}^n$ defined by

$$x_m(t) = x_0 + \int_{t_0}^{t} f(s, x_{m-1}(s))\, ds, \qquad x_0(t) = x_0.$$

Then for all $m = 1, 2, \ldots,$

$$\|x_m(t) - x_0(t)\| \leq \int_{t_0}^{t} \|f(s, x_m(s))\|\, ds \leq M|t - t_0| \leq Md/M = d.$$

Thus $\{x_m(t)\} \subset D$ for $|t - t_0| \leq \alpha$.

Next we show by induction that

$$\|x_{m+1}(t) - x_m(t)\| \leq \frac{ML^m|t - t_0|^{m+1}}{(m+1)!}, \qquad m = 0, 1, \ldots, \quad |t - t_0| < \alpha.$$

We consider the interval $[t_0, t_0 + a]$ since the same argument applies to the interval $[t_0 - a, t_0]$. From above we already have

$$\|x_1(t) - x_0(t)\| \leq M(t - t_0).$$

Suppose

$$\|x_m(t) - x_{m-1}(t)\| \leq \frac{ML^{m-1}(t - t_0)^m}{m!}.$$

Then

$$\begin{aligned}
\|x_{m+1}(t) - x_m(t)\| &\leq \int_{t_0}^{t} \|f(t, x_m(s)) - f(t, x_{m-1}(s))\|\, ds \\
&\leq L\int_{t_0}^{t} \|x_m(s) - x_{m-1}(s)\|\, ds \\
&\leq \frac{ML^m(t - t_0)^m}{m!} \int_{t_0}^{t} (s - s_0)^m\, ds = \frac{ML^m(t - t_0)^{m+1}}{(m+1)!},
\end{aligned}$$

as required. Thus we have

$$r_m(t) = \|x_{m+1}(t) - x_m(t)\| \le \frac{M(La)^{m+1}}{L(m+1)!}, \qquad m = 0, 1, \ldots, \quad |t - t_0| < \alpha.$$

This implies that

$$\sum_{m=0}^{\infty} r_m(t) \le \sum_{m=0}^{\infty} \frac{M(La)^{m+1}}{L(m+1)!} = \frac{M}{L} \sum_{m=0}^{\infty} \frac{(La)^{m+1}}{(m+1)!} = \frac{M}{L} \left(e^{La} - 1\right), \qquad |t - t_0| < \alpha.$$

$$\Longrightarrow \quad \sum_{m=0}^{\infty} r_m(t) \text{ converges (in fact, uniformly) on } |t - t_0| < \alpha.$$

$$\Longrightarrow \quad \sum_{m=0}^{\infty} [x_{m+1}(t) - x_m(t)] \text{ converges absolutely (and uniformly) on } |t - t_0| < \alpha.$$

Let

$$x(t) = \lim_{m\to\infty} x_m(t) = x_0(t) + \lim_{m\to\infty} \sum_{j=0}^{m-1} [x_{j+1}(t) - x_j(t)]$$

Since each term $x_{j+1}(t) - x_j(t)$ is continuous, then $x(t)$ is continuous on $|t - t_0| < \alpha$ being a uniform limit of continuous functions.

Moreover, $x(t)$ satisfies the VIE on $|t - t_0| < a$ since the uniform convergence of $x_m(t)$ allows us to take $m \to \infty$ in the definition of x_m.

To prove uniqueness, suppose that $\tilde{x}(t)$ is another solution. Then the function $z(t) = x(t) - \tilde{x}(t)$ satisfies the inequality

$$\begin{aligned} \|z(t)\| &= \left\| \int_{t_0}^{t} f(s, x(s)) - f(s, \tilde{x}(s))\, ds \right\| \\ &\le \int_{t_0}^{t} \|f(s, x(s)) - f(s, \tilde{x}(s))\|\, ds \\ &\le L \int_{t_0}^{t} \|x(s) - \tilde{x}(s)\|\, ds = L \int_{t_0}^{t} \|z(s)\|\, ds. \end{aligned}$$

Let $u(t) = \int_{t_0}^{t} \|z(s)\|\, ds$, then $u'(t) = \|z(t)\|$ and

$$u'(t) \le L u(t).$$

We multiply by $e^{-(t-t_0)L}$ and write the inequality in the form

$$\frac{d}{dt}\left[e^{-(t-t_0)L} u(t)\right] \le 0.$$

Integration from t_0 to t gives

$$e^{-(t-t_0)L} u(t) \le 0 \quad \Longrightarrow \quad \|z(t)\| = u(t) \le 0 \quad \Longrightarrow \quad z(t) \equiv 0.$$

Thus $x(t) = \tilde{x}(t)$.

Remark.

- *The solution of the IVP will be indicated by $x(t)$, or sometimes by $x(t; x_0)$ or $x(t; t_0, x_0)$.*
- *The theorem guarantees the existence of the solution in a neighborhood of $t = t_0$.*
- *The size of the neighborhood depends on the sup norm M of the vector function $f(t, x)$.*
- *Often the solution can be extended outside this neighborhood.*

Examples

Example. *Consider the IVP*

$$\dot{x} = x, \quad x(0) = 1, \quad t \geq 0.$$

To apply the theorem we have $f(t, x) = x$ which is continuous for all t and x. Also f is Lipschitz continuous in x with $L = 1$ in $D = \mathbb{R}$. Then the solution exists and unique for $0 \leq t \leq a$ with a an arbitrary positive constant. The solution can be continued for all positive t. The solution of this problem is $x(t) = e^t$ which exists for all $t \geq 0$.

Example. *Consider the IVP*

$$\dot{x} = x^2, \quad x(0) = 1, \quad t \geq 0.$$

To apply the theorem we have $f(t, x) = x^2$ which is continuous in $\mathbb{R} \times \mathbb{R}$. Let $D = \{x : |x - 1| \leq d\}$ and $G = \{(t, x) : 0 \leq t \leq a, x \in D\}$ for some arbitrary positive constant $a > 1$. Then

$$M = \sup_G \|f\| = (1 + d)^2.$$

Moreover,

$$|f(t, x_1) - f(t, x_2)| = |x_1^2 - x_2^2| \leq |x_1 + x_2||x_1 - x_2| \leq 2(1 + d)|x_1 - x_2|,$$

and thus f is Lipschitz continuous with $L = 2(1 + d)$. It follows from the theorem that the IVP has a unique solution for

$$0 \leq t \leq \alpha = \min\{a, d/M\} = \frac{d}{(1 + d)^2}.$$

Notice that $\alpha < 1$ for all d.

On the other hand, using the separation of variables method we obtain the solution

$$x(t) = \frac{1}{1 - t}.$$

This solution exists for $0 \leq t < 1$.

1.3 Gronwall Inequality

Theorem. *Let ψ and ϕ be continuous nonnegative functions on $t_0 \le t \le t_0 + a$ such that*

$$\phi(t) \le \delta_1 \int_{t_0}^{t} \psi(s)\phi(s)\, ds + \delta_3, \qquad \delta_1, \delta_3 > 0.$$

Then

$$\phi(t) \le \delta_3 \, e^{\delta_1 \int_{t_0}^{t} \psi(s)\, ds}, \qquad t_0 \le t \le t_0 + a.$$

Proof. Divide by the RHS, multiply by $\delta_1 \psi(t)$ and then integrate. ■

Theorem. *Let ϕ be continuous nonnegative functions on $t_0 \le t \le t_0 + a$ such that*

$$\phi(t) \le \delta_2(t - t_0) + \delta_1 \int_{t_0}^{t} \phi(s)\, ds + \delta_3, \qquad \delta_1 > 0,\ \delta_2, \delta_3 \ge 0.$$

Then

$$\phi(t) \le \left(\frac{\delta_2}{\delta_1} + \delta_3\right) e^{\delta_1(t-t_0)} - \frac{\delta_2}{\delta_1}, \qquad t_0 \le t \le t_0 + a.$$

Proof. *Note that $t - t_0 = \int_{t_0}^{t} ds$ and thus we can combine the first two terms in one integral. Let $\psi(t) = \phi(t) + \frac{\delta_2}{\delta_1}$, then the estimate becomes*

$$\psi(t) \le \delta_1 \int_{t_0}^{t} \psi(s) ds + \frac{\delta_2}{\delta_1} + \delta_3.$$

Application of the first theorem yields

$$\psi(t) \le \left(\frac{\delta_2}{\delta_1} + \delta_3\right) e^{\delta_1(t-t_0)}.$$

Replacing ψ produces the required result. ■

Remark. *In the special case $\delta_2 = \delta_3 = 0$ we have $\phi(t) = 0$ for $t_0 \le t \le t_0 + a$.*

Theorem. *Consider the equation*

$$\dot{x}(t) = f(t, x), \quad x \in R^n, \quad f : \mathbb{R}^{n+1} \to \mathbb{R}^n, \quad t \ge 0;$$

f is continuous in t and x and satisfies the Lipschitz condition. Let $x(t)$ and $\tilde{x}(t)$ be the solutions of the IVP's

$$\dot{x} = f(t, x), \quad x(0) = x_0,$$
$$\dot{x} = f(t, x), \quad x(0) = x_0 + \eta,$$

on interval I, respectively. Then if $\|\eta\| \le \varepsilon > 0$, then

$$\|x(t) - \tilde{x}(t)\| \le \epsilon e^{Lt} \text{ on } I.$$

Proof. The two IVP are equivalent to the integral equations:

$$x(t) = x_0 + \int_0^t f(\tau, x(\tau))\, d\tau, \quad \tilde{x}(t) = x_0 + \eta + \int_0^t f(\tau, \tilde{x}(\tau))\, d\tau$$

Subtracting,

$$\|x(t) - \tilde{x}(t)\} \le \|\eta\| \int_0^t \|f(\tau, x(\tau)) - f(\tau, \tilde{x}(\tau))\| d\tau \le \varepsilon + L \int_0^t \|x(\tau) - \tilde{x}(\tau)\| d\tau.$$

The result follows from Gronwall's inequality. ■