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The Galerkin method for singular nonoscillatory two-point boundary value problems

Beg, Ghulam Kabir, Ph.D.

King Fahd University of Petroleum and Minerals (Saudi Arabia), 1994

U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

The Galerkin Method for Singular Nonoscillatory Two-point Boundary Value Problems

BY

Ghulam Kabir Beg

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This dissertation, written by *GHULAM KABIR BEG* under the direction of his Dissertation Advisor and approved by his Dissertation Committee, has been presented to and accepted by the Dean of the College of Graduate Studies, in partial fulfillment of the requirements for the degree of *DOCTOR OF PHILOSOPHY*.

Dissertation Committee

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Dr. Ala H. Al-Rabeh

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Contents

DISSERTATION ABSTRACT

NAME OF STUDENT	:	Ghulam Kabir Beg
TITLE OF STUDY	:	**The Galerkin Method for Singular Nonoscillatory Two-point Boundary Value Problems**
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The Galerkin method with special patch basis is used for the approximation of the solution of a general class of second order singular two-point boundary value problems. Convergence is analysed in several norms. Higher order convergence in the uniform norm are obtained for special data. The results are new both for special and general data. The class of problems treated in this dissertation extends the class of problems treated in the literature in many directions. The existence, uniqueness and the regularity of the solutions are not assumed; these are studied in proper places as needed. Both linear and nonlinear problems are treated in this dissertation.

DOCTOR OF PHILOSOPHY DEGREE

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JANUARY, 1994.

خلاصة الرسالة

اسم الطالب : غلام كبير بيغ .

عنوان الرسالة : طريقة غالركن لمسائل القيم الحدية ذات النقطتين الشاذة غيرالمتذبذبة

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استعملنا طريقة غالركن مع أساسات مقطعة خاصة لتقريب الحل لطبقة عامة من معادلات القيم الحدية من الرتبة الثانية ذات النقطتين الشاذة .

حللنا التقارب في عدة معيارات . حصلنا على تقارب برتب أعلى في المعيار المنتظم تحت بيانات خاصة . النتائج التي حصلنا عليها جديدة لكل من البيانات الخاصة و العامة . طبقة المسائل التي تعاملنا معها في هذه االأطروحة هي توسيع لتلك التي عوملت في المراجع في عدة اتجاهات . لم نفترض وجود ، أحادية أو انتظام الحل و إنما بحثنا هذه المسائل في أماكنها المناسبة كلما دعت الحاجة .

عاملنا مسائل خطية ولا خطية في هذه الأطروحة .

دكتوراة الفلسفة

جامعة الملك فهد للبترول والمعادن .

الظهران ، المملكة العربية السعودية

يناير ١٩٩٤

Chapter 1

Introduction

Various problems in physics and engineering lead to an ordinary differential equation with the coefficient of the highest derivative vanishing at certain points. Such an equation, in general, is called degenerate or singular and the zero of the leading coefficient is called the singular point. According to Naimark [29] if the reciprocal of this coefficient is not integrable in the whole domain then the equation is said to be singular. In section 1.1 we give a description of a class of second order boundary value problems with singularity at one end point.

There is a growing literature on the existence, uniqueness and the numerical approximation of the solution of such problems. A brief review to the literature is given in section 1.2. The objective of this dissertation is to apply the Galerkin method to approximate the solution of a general class of singular equations and obtain error estimation in various norms. To achieve this goal it is necessary to address carefully the problems of existence, uniqueness, regularity, variational formulation, weak solutions and strong solutions for our chosen class of singular equations. As we will see shortly, the class of problems we treated in this dissertation extends class of singular problems treated so far in many directions.

1.1 Second order boundary value problems with one singular end-point and their classification

Consider the differential equation:

$$\begin{aligned} (p(x)u'(x))' &= f(x, u(x))w(x), \qquad 0 < x < 1 \\ \lim_{x \to 0+} p(x)u'(x) &= 0 \\ u(1) &= 0 \end{aligned}$$

where $p(x)$, $w(x) \geq 0$, $\int_0^1 w(x)dx < \infty$ and $f(x,u)$ is continuous in u (linear or nonlinear) such that for any real number u, $\int_0^1 f(x,u)w(x)dx < \infty$. This equation is formally written as

$$\begin{aligned} -\frac{1}{w}(pu')' + f(x,u) &= 0, \qquad 0 < x < 1 \\ \lim_{x \to 0+} pu' &= 0 \\ u(1) &= 0 \end{aligned}$$

and is said to be *regular* (at both end points $x = 0$ and $x = 1$) if $p^{-1} \in L^1(0,1)$. It is *singular* at $x = 0$ (but regular at $x = 1$) if $p^{-1} \in L^1_{loc}(0,1]$ but $p^{-1} \notin L^1(0,1)$. We will consider this kind of singularity throughout this dissertation. A particular example is:

$$\begin{aligned} -\frac{1}{x^\beta}(x^\alpha u')' + f(x,u) &= 0, \qquad 0 < x < 1 \\ \lim_{x \to 0^+} x^\alpha u'(x) &= 0 \\ u(1) &= 0 \end{aligned}$$

where $\alpha \geq 1$ and $\beta > -1$.

Such kind of singular boundary value problems appear in many areas of applied mathematics. They occur, for example, in transport processes [2], the study of electrohydrodynamics [23], in the theory of thermal explosions [4], Gaussian processes [12], separation of variables in PDE's [32], etc.

According to Weyl's classification [37,27] the singular point $x = 0$ must fall into one of the following two mutually exclusive categories:

Limit Circle case: All solutions of

$$-\frac{1}{w}(pu')' = \lambda u \tag{1.1}$$

are in $L^2_w(0,1)$ for all λ, real or nonreal. In particular when $\lambda = 0$, the two linearly independent solutions, $u(x) = 1$ and $u(x) = \int_x^1 \frac{1}{p}$ are in $L^2_w(0,1)$. Clearly by the assumptions on $w(x)$, $1 \in L^2_w(0,1)$. Therefore, the singular point is of limit circle case iff $\int_x^1 \frac{1}{p} \in L^2_w(0,1)$.

Limit Point case: For all nonreal λ there is exactly one solution of (1.1) in $L^2_w(0,1)$. For any real λ atmost one solution is in $L^2_w(0,1)$. So for $\lambda = 0$ atmost one of $u = 1$ or $u = \int_x^1 \frac{1}{p}$ is in $L^2_w(0,1)$. Already $1 \in L^2_w(0,1)$, therefore $\int_x^1 \frac{1}{p} \notin L^2_w(0,1)$. Hence $x = 0$ is of limit point case iff $\int_x^1 \frac{1}{p} \notin L^2_w(0,1)$.

The term limit circle and limit point arise from the use of a certain sequence of nested circles that Weyl made in his proof, the sequence converging in the limit to either a circle or a point.

1.2 Literature survey

In this literature survey we mention two different aspects seperately in two subsections. One is the literature regarding the existence and uniqueness of the solution of these types of problems from a numerical analyst's point of view. The second one is the numerical approximations to the solution. In most of the literature concerning the numerical treatment of such problems the existence of the solution is assumed. In many of them smoothness of the solution is also assumed for the purpose of error estimation.

1.2.1 About the existence and uniqueness

Russel and Shampine [36] considered the problem:

$$-\frac{1}{x^\alpha}(x^\alpha u'(x))' = f(x,u(x)), \quad 0 < x < b \tag{1.2}$$

$$u'(0) = 0 \tag{1.3}$$

$$u(b) = B \tag{1.4}$$

with $\alpha = 1$ or 2. They first considered the linear case

$$-\frac{1}{x^\alpha}(x^\alpha u'(x))' + Ku(x) = f(x)$$

with the same boundary conditions and proved that, for $\alpha = 1$ and $K > -j_0^2$ (where $j_0 = 2.40483$ is the first positive zero of the Bessel's function of order zero), the problem (1.2) has a unique solution. They also proved that, for $\alpha = 2$ and $K > -\pi^2$, this linear equation also has a unique solution. The proof is based on the explicit constuction of the Green's function associated with the linear operator. For the nonlinear case they constructed a sequence of linear problems and proved that under suitable assumptions on $f(x,u)$ and the starter u_0, the sequence of solutions to these problems converges to the solution of the real problem.

Similar iterative techniques were also used by Chawla et. al. [7] and Fink [16] in their proof of the existence and uniqueness of the solution. Chawla et. al.

extended the work of Russel and Shampine to any $\alpha \geq 1$. They assumed their data $f(x,u)$ to be continuous and having a continuous partial derivative w.r.t. u.

Fink et. al. [16] considered the same problem with $\alpha = n-1$. They introduced a singularity in $f(x,u)$ at $u=0$. They assumed $f(x,u)$ positive and continuous on $[0,1)\times(0,\infty)$ and strictly decreasing in u. They regularized f by replacing $f(x,u)$ by $f(x,u+\epsilon)$. Under some more assumption on f they proved the existence of a positive solution u as a uniform limit of the solutions u_ϵ as $\epsilon \to 0$. They also proved the uniqueness of the solution. Numerical illustrations were given in their paper for $f(x,u) = u^{-p}$, $p = \frac{1}{9}, \frac{1}{3}$, with different values of ϵ.

Remark 1.1 *It is observed that in all these works the special case* $p(x) = w(x) = x^\alpha$, $\alpha \geq 1$ *was only considered. Continuity of* $f(x,u)$ *in both* x *and* u *was assumed. In some cases even continuity of* $\frac{\partial f}{\partial u}$ *was also required. Proofs were based on the iterative procedure using a succession of linear boundary value problems. In this regard we want to mention here that the problems of this dissertation are more general and the proofs are based on the results of linear and nonlinear functional analysis.*

1.2.2 About the numerical approximations

Eriksson and Thomee [15] used the Galerkin method with piecewise polynomials (as a basis) for the linear problem:

$$\begin{aligned} -\frac{1}{x^\alpha}(x^\alpha u'(x))' + qu &= f, \quad 0 < x < 1 \\ u'(0) = u(1) &= 0 \end{aligned}$$

where $\alpha > 0$ and q is a bounded nonnegative function. They assumed that these problems admit a unique and sufficiently smooth solution. For $\alpha > 1$ they obtained the error estimation

$$\|u^G - u\|_{L^\infty} \leq Ch^r \|u^{(r)}\|_{L^\infty}$$

where $u^{(r)}$ denotes the r-th derivative of u. This work is an improvement of the work of Jesperson [21] who derived earlier the following error estimate for the model problem with $q = 0$:

$$\|u^G - u\|_{L^\infty} \leq C \left(ln \frac{1}{h} \right)^{\overline{r}} h^r \|u^{(r)}\|_{L^\infty}$$

where $\overline{r} = 1$ if $r = 2$ and $\overline{r} = 0$ if $r > 2$. He demonstrated with an example that the logarithmic term cannot be removed for $r = 2$. Both the error estimations of Jesperson and Eriksson & Thomee are dependent on the assumption that the solution is smooth. We notice also that the Galerkin method used by Eriksson and Thomee is not symmetric.

Chawla et. al. [5] used a finite difference method for the problem

$$-\frac{1}{x^\alpha}(x^\alpha u'(x))' + f(x,u) = 0, \quad 0 < x < 1$$

$$u'(0) = u(1) = 0$$

where $\alpha \geq 1$. They assumed f and $\frac{\partial f}{\partial u}$ to be continuous in $[0,1] \times (-\infty, \infty)$ and $\frac{\partial f}{\partial u} \geq 0$. They derived an $O(h^2)$ error under the assumptions that $|f'| \leq C_1$ and $x|f''| \leq C_2$ for $0 < x \leq 1$ (where f' denotes the derivative of $f(t, u(t))$). Using a similar approach Chawla et. al. in [6] obtained a fourth order method under more smoothness assumptions on f, namely, $|f'''| \leq C_1$ and $x|f^{(4)}| \leq C_2$.

Recenty Abu-Zaid [1] considered the linear problem:

$$-\frac{1}{p}(pu')' + qu = f, \quad 0 < x < 1$$

$$\lim_{x \to 0^+} pu' = u(1) = 0.$$

He assumed $q, f \in C[0,1]$ and $q \geq 0$. In addition to the singularity condition on p he also assumed that p' is bounded, p is increasing in a neighborhood of 0 and $\int_x^1 \frac{1}{p} \in L^2(0,1)$. Under these conditions he proved that a generalization of the finite difference scheme of Chawla is to be of order h^2.

Remark 1.2 *It is observed that in all these works either the particular case $p(x) = w(x) = x^\alpha$, $\alpha \geq 1$ was treated or a more general one: $p(x) = w(x)$ with assumptions on p was considered. In the linear case f and q were considered to be at least continuous with $q \geq 0$. Similar assumptions were made for the* *nonlinear case also. In addition to the generalization $p = w$ (with aforementioned conditions on p) Abu-Zaid also removed the smoothness condition on f (upto continuity) and still obtained $O(h^2)$ convergence through the finite difference method. All others assumed directly or indirectly smoothness of f. It is also noted that for the model case $q = 0$ the matrix obtained by Abu-Zaid in terms of p (in his finte difference discretization) is the same matrix obtained by Ciarlet et. al. [8] when Galerkin method was used with the patch basis in terms of p. Ciarlet et. al. took the case $\int_0^1 \frac{1}{p} < \infty$ which is no more singular according to our definition. In this dissertation the patch basis in terms of p (like Ciarlet et. al.) to an extended class of singular problems (which generalizes all the above cases in many ways) has been used and different order of covergences (in terms p and w and the partition) are obtained for different cases. For example, a covergence of $O(h^2)$ (in the uniform norm is obtained) for the case $p = w$, p monotone increasing, and $f, q \in L_w^\infty(0,1)$ with $q \geq 0$ (for a similar condition for the nonlinear case see chapter 6). We like to mention that this result in this special case is also a new one.*

1.3 Objective of this dissertation

As mentioned earlier our goal is to treat a wider class of problems in terms of the functions w, p and q, f ($f(x,u)$ for the nonlinear case). This study therefore includes the linear case:

$$-\frac{1}{w}(pu')' + qu = f, \quad 0 < x < 1 \tag{1.5}$$

$$\lim_{x\to 0+} p(x)u'(x) = 0 \tag{1.6}$$

$$u(1) = 0 \tag{1.7}$$

with any general p and w (satisfying the following LC or LP1 condition). The relaxed condition $w \geq 0$ (allowing w to vanish on any subset of $[0,1]$) is not excluded. f is taken to be any function in $L^2_w(0,1)$. The function q is in $L^2_w(0,1)$ with certain conditions depending on the limit circle or limit point cases. The limit circle (LC) case:

$$\int_x^1 \frac{1}{p} \in L^2_w(0,1) \tag{1.8}$$

is completely studied while the limit point case (LP):

$$\int_x^1 \frac{1}{p} \notin L^2_w(0,1) \tag{1.9}$$

is considered in the case:

$$\int_x^1 \frac{1}{p} \in L^1_w(0,1). \tag{1.10}$$

We call this case as *limit point one* (LP1).

The study is then extended to the nonlinear case:

$$-\frac{1}{w}(pu')' + f(x,u) = 0, \quad 0 < x < 1 \tag{1.11}$$

$$\lim_{x \to 0^+} p(x)u'(x) = 0 \quad (1.12)$$

$$u(1) = 0 \quad (1.13)$$

with same general p, w covering both LP and LC1 cases. Here $f(x,u)$ is a nonlinear forcing term which is continuous in u and is in $L^2_w(0,1)$ for any fixed real u (No smoothness condition in terms of both the variables is assumed, even the continuity w.r.t. x is not assumed). Conditions (with an eye on keeping the relaxed conditions) on $f(x,u)$ are imposed for the existence and uniqeness of the solution to this equation. The Galerkin method with patch basis (in terms of p) is applied to both the linear and nonlinear cases.

This dissertation is written into 7 chapters. The first chapter is an introductory one. The literature review is done in this chapter.

In chapter 2 we give the preliminary material to be used in the subsequent chapters. Specially some materials on the Galerkin method and the maximal monotone operators are prepared for the use in chapter 5 and chapter 6.

In chapter 3 we begin our study with the model problem with $q = 0$. The regularity of the solution to this model problem is studied in detail. Both limit circle and limit point (one) cases are studied seperately. The behavior of the solution along with its derivative is studied in both cases. Other related results about the operator invoked are also studied for the use in the subsequent chapters. A class of examples is studied as a particular case. The limit circle case with

$q \in L^\infty_w(0,1)$ is also studied in this chapter.

In chapter 4 a Hilbert space V is defined which contains the solution space of the model problem of chapter 3 and is contained in the original space $L^2_w(0,1)$. Considering $q \neq 0$, a variational boundary value problem is defined. The existence and uniqueness of the solution of this problem in this space V is studied. The equivalence of this problem with the original boundary value problem is then studied. The properties of the related operators are also studied. The study covers both the LC and LP1 cases with relaxing w as mentioned above.

In chapter 5 we apply the Galerkin method with the patch basis (depending on p) for the solution of the variational boundary value problem defined in chapter 4. Interpolation error for the solution in the space V with respect to this basis is estimated (both in V-norm and uniform norm). The error of the Galerkin approximation is then studied with respect to these norms. Optimal order of convegence (with respect to both norms) is studied for several cases with particular emphasis on finding higher order accuracy for special classes (which are important in applications). The existence of the solution of the variational boundary value problem is also reflected through the analyses of this chapter (giving an alternate proof for the existence theorem).

In chapter 6 all the studies of chapter 4 and chapter 5 are extended to the nonlinear case. The results are applied to a rather large class of nonlinear problems. Comparative studies are also done with the problems available in the literature.

New examples are developed. The existence and uniqueness of the solution of the finite dimensional nonlinear system arising from the Galerkin method is also studied.

In chapter 7 the validation of the analyses is demonstrated through numerical examples. Comparative studies for the approximations are done. Numerical explorations for further extensions are made.

Chapter 2

Preliminaries

In this chapter we include the preliminaries which will be used in the subsequent chapters. The main definitions and theorems are stated. The proofs of the theorems are omitted, but the references are included. We have tailored the results towards our application and written them as remarks and corollaries with proofs. Due to the same reason we have written them in the Hilbert space settings although in some references they are given in a more general setting. Also we have omitted the standard definitions which can be found in any standard introductory book of functional analysis or linear operator theory. The preliminary is arranged into five sections. The material in section 2.1 will be used in chapters 3, 4 and 6 while that in sections 2.2, 2.3, and 2.4 will be used in chapters 4, 5, and 6 respectively. The first part of section 2.5 will be used in chapter 5 and the rest of it will be used in chpater 7.

2.1 Compact and self-adjoint operators

Suppose H is a real seperable Hilbert space and T is a linear operator defined in it. By the notation $D(T)$ and $R(T)$ we mean the domain and range of the operator T. All the Hilbert spaces we consider in this dissertation are real and seperable.

Theorem 2.1 *If T_1 is compact and T_2 is bounded on H then T_1T_2 is compact on H. On the other hand, if T_2 is compact on H and T_1 is bounded on $R(T_2)$ then T_1T_2 is also compact on H.*

Theorem 2.2 *The spectrum $\sigma(T)$ of a bounded self-adjoint operator T consists of only point and continuous spectrum (i.e., the residual spectrum $\sigma_r(T)$ is empty) and is contained in the closed interval $[m, M]$ where*

$$m = \inf_{u \neq 0} \frac{\langle Tu, u \rangle}{\langle u, u \rangle} \text{ and } M = \sup_{u \neq 0} \frac{\langle Tu, u \rangle}{\langle u, u \rangle}.$$

Moreover, m and M are also spectral values of T.

(The proof is given in Kreyszig [24], pp 459-469)

Theorem 2.3 *If T is a self-adjoint compact operator then the set of all eigenfunctions of T (including those corresponding to the zero eigenvalue) forms a basis for H. The set of normalized eigenfunctions corresponding to nonzero eigenvalues forms a basis for H if and only if $\lambda = 0$ is not an eigenvalue of T.*

(For a proof see Stakgold [37], pp 372-374)

Definition 2.1 *Let $T : D(T) \longrightarrow H$ be a (possibly unbounded) densely defined linear operator. Then the* Hilbert-adjoint operator $T^* : D(T^*) \longrightarrow H$ *of T is defined as follows. The domain $D(T^*)$ consists of all $v \in H$ such that there is a $v^* \in H$ satisfying*

$$\langle Tu, v\rangle = \langle u, v^*\rangle$$

for all $u \in D(T)$. For each such $v \in D(T^*)$ the Hilbert-adjoint operator T^* is then defined in terms of v^* by

$$v^* = T^*u.$$

In other words,

$$D(T^*) = \{v \in H \quad : \quad u \mapsto \langle Tu, v\rangle \text{ is continuous on } D(T)\}.$$

Definition 2.2 *Let $T : D(T) \longrightarrow H$ be a (possibly unbounded) densely defined linear operator. Then T is called a* symmetric linear operator *if for all $u, v \in D(T)$*

$$\langle Tu, v\rangle = \langle u, Tv\rangle.$$

Lemma 2.1 *A densely defined linear operator in H is symmetric if and only if*

$$T \subset T^*.$$

Definition 2.3 *Let* $T : D(T) \longrightarrow H$ *be a (possibly unbounded) densely defined linear operator. Then* T *is called a* self-adjoint linear operator *if*

$$T = T^*.$$

Remark 2.1 *It is therefore clear that a symmetric linear operator defined on the whole space* H *is self-adjoint. In other words if* $D(T) = H$ *then the concepts of symmetry and self-adjointness are identical.*

Theorem 2.4 (Hellinger-Toeplitz theorem) *A symmetric linear operator defined on the whole space* H *is bounded. In other words, a self-adjoint operator defined on the whole space is bounded.*

Theorem 2.5 *If a self-adjoint operator* $T : D(T) \rightarrow H$ *is injective then (a)* $\overline{R(T)} = H$ *and (b)* T^{-1} *is self-adjoint.*

(see Kreyszig [24], page 535)

2.2 The Lax-Milgram theorem

Let V be a real seperable Hilbert space. Let $B(u, v)$ be a *bilinear form* in $V \times V$. The bilinear form is called *continuous* if there is a constant C s.t.

$$|B(u, v)| \leq C\|u\|\|v\|$$

for all $u, v \in V$, and it is called *V-elliptic* if there is a constant $\alpha > 0$ s.t.

$$B(u,u) \geq \alpha \|u\|^2$$

for all $u \in V$.

Let $u \in V$. Then

$$B(u,.) : V \longrightarrow \mathbf{R}$$

is a continuous linear functional in V. By the Riesz-representation theorem there exists a unique element $\tilde{u} \in V$ such that

$$B(u,v) = \langle \tilde{u}, v \rangle \quad \text{for every } v \in V.$$

This gives a correspondence $A : V \rightarrow V$ by

$$A(u) = \tilde{u}.$$

Remark 2.2 *A is symmetric if the bilinear form is symmetric, since*

$$\langle Au, v \rangle = B(u,v) = B(v,u) = \langle Av, u \rangle$$

Theorem 2.6 (Lax-Milgram) *Let $B(u,v)$ be a bilinear form continuous in V such that $B(u,v)$ is V-elliptic. Then $A : V \rightarrow V$ is bijective.*

(For a proof see Huet [19], page 22)

2.3 Galerkin method for the variational problem $B(u,v) = l(v)$

In this section we describe the Galerkin method for solving the problem

$$B(u,v) = l(v) \qquad \forall v \in V \tag{2.1}$$

where V is a real seperable Hilbert space, $B(u,v)$ is a symmetric, continuous and V-elliptic bilinear form in $V \times V$ and $l(v)$ is a bounded linear functional on V. The existence and uniqeness of the solution is thus assured by the Lax-Milgram theorem. We will discuss the Galerkin approximation to this solution. The material in this section is an adaption of results known in the literature. We refer Reddy [34] and Zeidler [40].

Let V_n be a subspace of V spaned by n linearly independent elements of V. These elements form a basis for the Galerkin method. This basis is to be choosen in such a way that

$$\lim_{n\to\infty} \operatorname{dist}(u, V_n) = 0 \quad \forall u \in V$$

where

$$\operatorname{dist}(u, Y) := \inf_{v\in Y} \|u - v\|_V.$$

This is the main theme of the Galerkin method.

Since $B(u,v)$ is symmetric and coercive we can define an inner product

$$\langle u, v\rangle_B := B(u,v) \tag{2.2}$$

in V. The corresponding norm is called the *energy norm.* We now describe the Galerkin method in operator form with the help of this inner product.

Definition 2.4 *The Galerkin method for (2.1) is an orthogonal projection operator P_n of V onto V_n with respect to the inner product (2.2), and the Galerkin approximation of the solution u of (2.1) is $v^G = P_n u$.*

For the sake of simplicity of notation we denote P_n by P only.

Theorem 2.7 *The error of the Galerkin approximation u^G denoted as $e = u - u^G$ satisfies*

$$B(e, v_n) = 0, \quad \forall \ v_n \in V_n \tag{2.3}$$

and the Galerkin approximation u^G satisfies

$$B(u^G, v_n) = l(v_n), \quad \forall \ v_n \in V_n \tag{2.4}$$

Proof: By the above definition, for any $v_n \in V_n$

$$B(e, v_n) = B(u - u^G, v_n) = \langle u - Pu, Pv_n \rangle_B = 0. \tag{2.5}$$

Since equation (2.1) is satisfied for all $v \in V$ it is also satisfied for all $v_n \in V_n$:

$$B(u, v_n) = l(v_n). \tag{2.6}$$

Subtracting (2.5) from (2.6) we obtain

$$B(u^G, v_n) = l(v_n).$$

This completes the proof.

Corollary 2.1 *Let $\{r_i\}, i = 1, ..., n$ be a basis for V_n. Then (2.3) and (2.4) are satisfied for $v_n = r_i,\ i = 1, ..., n$. Thus*

$$B(e, r_i) = 0, \quad i = 1, ..., n \tag{2.7}$$

and the Galerkin approximation u^G satisfies

$$B(u^G, r_i) = l(r_i), \quad i = 1, ..., n. \tag{2.8}$$

Since $u^G \in V_n$ and $\{r_i\}$ is a basis for V_n, the Galerkin approximation u^G can be written as

$$u^G = \sum_{j=1}^{n} \alpha_j r_j$$

and substituting this into (2.8) we obtain

$$\sum_{j=1}^{n} B(r_j, r_i)\alpha_j = l(r_i), \quad i = 1, ..., n \tag{2.9}$$

i.e.,

$$A\alpha = \mathbf{b} \tag{2.10}$$

where $A = (a_{ij})$ is given by $a_{ij} = B(r_j, r_i), \quad i, j = 1, ...n$ and $b_i = l(r_i), \quad i = 1, .., n$.

Remark 2.3 *The matrix A is symmetric and positive definite.*

Proof: A is symmetric because

$$a_{ij} = B(r_i, r_j) = B(r_j, r_i) = a_{ji}.$$

To prove the positive definiteness, let $\mathbf{y}$ be any nonzero element in R^n. Then

$$\begin{aligned}
\mathbf{y}^T A\mathbf{y} &= \sum_{i=1}^{n} y_i \sum_{j=1}^{n} a_{ij} y_j \\
&= \sum_{i=1}^{n} y_i \sum_{j=1}^{n} B(r_i, r_j) y_j \\
&= \sum_{i=1}^{n} y_i B(r_i, \sum_{j=1}^{n} y_j r_j) \\
&= B(\sum_{i=1}^{n} y_i r_i, \sum_{j=1}^{n} y_j r_j) \\
&= B(v_n, v_n) \\
&= \|v_n\|_B^2 \ > \ 0.
\end{aligned}$$

where

$$v_n = \sum_{i=1}^{n} y_i r_i.$$

This completes the proof.

Remark 2.4 *The solution of (2.10) determines u^G uniquely.*

Theorem 2.8

$$\|e\|_V = \|u - u^G\|_V \leq \frac{C}{\alpha} \|u - v_n\|_V \tag{2.11}$$

for any $v_n \in V_n$.

Proof:

$$\begin{aligned}\alpha\|e\|_V^2 = \alpha\|u-u^G\|_V^2 &\leq B(u-u^G, u-v_n-u^G+v_n)\\ &= B(e,u-v_n) - B(e,u^G-v_n)\\ &= B(e,u-v_n) \;\; [using \;\; (2.3)]\\ &\leq C\|e\|_V\|u-v_n\|_V\end{aligned}$$

which completes the proof.

2.4 Maximal monotone operators

Let V be a real seperable Hilbert space and $A : D(A) \subset V \rightarrow V$ be an operator (possibly nonlinear). Let $\langle .,. \rangle$ be the inner product in V.

Definition 2.5 *The operator A is said to be* monotone *if*

$$\langle Au - Av, u - v\rangle \geq 0 \qquad \forall \;\; u, v \in D(A). \tag{2.12}$$

A *is* strictly monotone *if strict inequality holds in (2.12) whenever* $u \neq v$ *and* uniformly monotone *or* strongly monotone *if there is an* $\alpha > 0$ *so that*

$$\langle Au - Av, u - v\rangle \geq \alpha\|u-v\|^2 \quad \forall \;\; u, v \in D(A). \tag{2.13}$$

Remark 2.5 *If A is uniformly monotone then it is an injection.*

Proof: Suppose $Au = Av$ and $u \neq v$ for some $u, v \in D(A)$. Then

$$0 = \langle Au - Av, u - v \rangle \geq \alpha \|u - v\|^2 > 0$$

a contradiction. This completes the proof.

Definition 2.6 *The opeartor A is called* maximal monotone *if it is monotone and the set $\{(u, Au) : u \in D(A)\}$ is not properly contained in the set $\{(u, Bu) : u \in D(B)\}$ for any other monotone operator B in V.*

Remark 2.6 *For any $\alpha > 0$ the operator αI is a maximal and uniformly monotone operator.*

Proof: The proof is trivial.

Definition 2.7 *The operator A is called* coercive *if for all $u_n \in D(A)$,*

$$\lim_{n\to\infty} \|u_n\| = \infty \quad \Rightarrow \quad \lim_{n\to\infty} \frac{\langle Au_n, u_n \rangle}{\|u_n\|} = \infty.$$

Remark 2.7 *If A is uniformly monotone then it is also coercive.*

Proof: Let $u_n \in D(A)$ such that $\lim_{n\to\infty} \|u_n\| = \infty$. Then putting $u = u_n$ and $v = 0$ in (2.13) we obtain,

$$\langle Au_n, u_n \rangle \geq \alpha \|u_n\|^2$$

$$\Rightarrow \quad \frac{\langle Au_n, u_n\rangle}{\|u_n\|} \geq \alpha\|u_n\|$$

$$\to \infty \quad as \quad \|u_n\| \to \infty.$$

Definition 2.8 *A is said to be* hemicontinuous *on V if*

$$\lim_{t\to 0}\langle A(u+t\tilde{u}) - Au, v\rangle = 0$$

$\forall u, \tilde{u}, v \in V$.

Remark 2.8 *If V is finite dimensional and B is monotone and hemicontinuous then B is continuous on V.*

(For a proof see Barbu [3], lemma 1.1, page 35)

We now restate (in Hilbert space setting) the main theorem of this section from Barbu [3]. The proof can be found therein (pp 33-48).

Theorem 2.9 *Let V be a real Hilbert space and B be* monotone, *everywhere defined and* hemicontinuous *from V to V. Let A be a* maximal monotone *operator in V. Then $A+B$ is maximal monotone. Moreover, if $A+B$ is* coercive *then* $R(A+B) = V$.

The following corollary will be used in chapter 6.

Corollary 2.2 *Suppose $\alpha > 0$ and B is monotone and hemicontinuous on V. Then $\alpha I + B$ is maximal monotone on V. It is also uniformly monotone. Moreover,*

$$\alpha I + B \,:\, V \longrightarrow V$$

is bijective.

Proof: Since αI is maximal monotone, then by the above theorem $\alpha I + B$ is maximal monotone. It is also uniformly monotone because,

$$\begin{aligned} & \langle(\alpha I + B)u - (\alpha I + B)v, u - v\rangle \\ = & \langle\alpha(u - v) + (Bu - Bv), u - v\rangle \\ = & \alpha\|u - v\|^2 + \langle Bu - Bv, u - v\rangle \\ \geq & \alpha\|u - v\|^2. \end{aligned}$$

So it is an injection. Also it is coercive. Therefore by the above theorem it is a surjection. This completes the proof.

2.5 Finite dimensional operators

In this section we study the nature of the finite dimensional operators (linear and nonlinear) which are obtained when the Galerkin method is applied to our problems. These operators will be needed for the error estimation and for the numerical computations. We refer Stoer & Bulirsch [38] and Ortega [31] for the origin of the material of this section.

2.5.1 Linear operators

We consider the matrix operator A on R^n.

Definition 2.9 *The (directed)* graph *of a matrix A denoted as $G(A)$ consists of n vertices $P_1, \ldots\ldots, P_n$ and there is an (oriented) arc from P_i to P_j in $G(A)$ precisely if $a_{ij} \neq 0$.*

Definition 2.10 *A is* irreducible *if and only if the graph $G(A)$ is* connected *in the sense that for each pair of vertices (P_i, P_j) in $G(A)$ there is an oriented path from P_i to P_j.*

Definition 2.11 *A matrix A is called* diagonally dominant *if*

$$\sum_{j=1, j\neq i}^{n} |a_{ij}| \leq |a_{ii}|, \quad i = 1, \ldots\ldots, n.$$

Definition 2.12 *We write*

$$A \leq B \;\; if\ and\ only\ if \;\; a_{ij} \leq b_{ij} \quad \forall i, j = 1, \ldots., n.$$

Definition 2.13 *A is called an M*-matrix *if $a_{ij} \le 0$ for all $i,j = 1, ..., n,\ i \ne j$ and A is invertible with $A^{-1} \ge 0$.*

Theorem 2.10 *Let A be irreducible and diagonally dominant such that $a_{ii} > 0,\ i = 1, ..., n$ and $a_{ij} \le 0,\ i \ne j$. Then A is an M-matrix.*

Theorem 2.11 *A symmetric M-matrix is positive definite.*

Theorem 2.12 *Let A be an M-matrix with offdiagonal part A_1. Let Q be a nonnegative matrix with offdiagonal part Q_1. If $Q_1 \le -A_1$ then $A + Q$ is an M-matrix and $(A+Q)^{-1} \le A^{-1}$. Furthermore, if A and Q are symmetric then $A + Q$ is positve definite.*

(For a proof see Ortega [31], pp 54-55)

Consider now the equation

$$(A + Q)\mathbf{x} = \mathbf{b}$$

such that $(A + Q)$ is a symmetric M-matrix. Then by theorem 2.11 it is positive definite and so any direct method for solving this system will be stable. We will deal with one such a system in chapter 7.

2.5.2 Nonlinear operators

Consider the nonlinear operator $F : R^n \to R^n$.

Definition 2.14 *F is a* homeomorphism *of R^n onto R^n if F is one-to-one and F and F^{-1} are continuous.*

Theorem 2.13 (Uniform monotonicity theorem) *If F is continuous and uniformly monotone, then F is a homeomorphism of R^n onto R^n.*

(For a proof see Ortega [31], pp 165-167)

It is therefore clear that, under these assumptions on F, the equation

$$F\mathrm{x} = 0$$

has a unique solution. For the numerical computation of this solution we use a suitable iteration method. The literature on the iteration methods for such a problem is extensive. The appropriate method which assures the convergence is completely dependent on the type of the problem. We therefore confine ourselves to discussing two methods for our problems in chapter 7. One is globally convergent (usually slow) and the other one is locally convergent but faster (usually quadratic or superlinear). In our case $F = A + G$, where A is a symmetric M-matrix and G is nonlinear. The corresponding equation becomes:

$$A\mathrm{x} + G\mathrm{x} = 0.$$

The Picards iteration:

$$(A + \gamma I)\mathrm{x}^{k+1} = \gamma \mathrm{x}^k - G\mathrm{x}^k, \quad k = 0, 1, ...,$$

where $\gamma > 0$ depends on G, converges to the solution $\mathbf{x}$ starting with any $\mathbf{x}^0 \in R^n$. For the proof of this global convergence result we refer to Ortega [31], pp 387-388.

The secant method:

$$\mathbf{x}^{k+1} = \mathbf{x}^k - (A + J_k)^{-1}(A\mathbf{x}^k + G\mathbf{x}^k)$$

where J_k is a matrix dependent on G which is updated at each step k, is convergent to the solution $\mathbf{x}$ if $\mathbf{x}^0$ is sufficiently close to it (i.e. locally convergent). For the rate of convergence we again refer to Ortega [31], pp 355-365.

The constant γ and the matrix J_k will be given explicitly in chapter 7. We will also see that the matrix $(A + J_k)$ will be positive definitie at each iteration.

Chapter 3

The Regularity of the Solution

In this chapter we first investigate the nature of the solution and its derivatives for both the limit circle (LC) and limit point (LP1) cases. We also study the operator invoked therein. We begin with the particular linear case when $q(x) = 0$. In section 3.1 the limit circle case is studied. In section 3.2 the limit point LP1 case is considered. In section 3.3 a class of examples is given to illustrate the above two cases. We then consider the case LC for $q \in L^\infty_w(0,1)$ in section 3.4 . We will consider both limit circle and limit point cases for a more general q in chapter 4.

Consider the problem (1.5)-(1.7) with $q = 0$ i.e.,

$$Lu \equiv -\frac{1}{w(x)}(p(x)u')' \;=\; f(x), \quad 0 < x < 1 \tag{3.1}$$

$$\lim_{x \to 0^+} p(x)u'(x) \;=\; 0 \tag{3.2}$$

$$u(1) \;=\; 0 \tag{3.3}$$

with $w, p \geq 0, \frac{1}{p} \in L^1_{loc}(0,1], \frac{1}{p} \notin L^1(0,1)$ and $\int_0^1 w < \infty$.

3.1 The linear limit circle case with q=0

We cosider the limit circle condition

$$\int_0^1 \left(\int_x^1 \frac{1}{p(t)}dt\right)^2 w(x)dx \;<\; \infty \tag{3.4}$$

We also assume that

$$f \in L^2_w(0,1) \tag{3.5}$$

3.1.1 The behavior of the solution

The following lemmas will be helpful.

Lemma 3.1 *For $x > 0$*

$$\int_0^x \left(\int_x^1 \frac{1}{p(t)}dt\right) |f(s)|w(s)ds \leq \int_0^x \left(\int_s^1 \frac{1}{p(t)}dt\right) |f(s)|w(s)ds.$$

Proof: Since for $x > 0$, $\int_x^1 \frac{1}{p(t)}dt$ exists and, since $p(t)$ is nonnegative, it follows that for $0 < s < x$

$$\int_x^1 \frac{1}{p(t)}dt \leq \int_s^1 \frac{1}{p(t)}dt.$$

The result, therefore, follows because $|f(s)|w(s)$ is nonnegative.

Let

$$K = \|f\|_{L^2_w(0,1)} \left(\int_0^1 \left(\int_x^1 \frac{1}{p(t)} dt \right)^2 w(x) dx \right)^{1/2} < \infty.$$

Lemma 3.2 *There exists a monotone increasing function $M(x)$ defined on $[0,1]$ with $M(0) = 0$ and $M(1) = K$ such that for $0 < x \leq 1$*

$$\left| \int_0^x f(s)w(s)ds \int_x^1 \frac{1}{p(t)} dt \right| \leq M(x).$$

Proof:

$$\left| \int_0^x f(s)w(s)ds \int_x^1 \frac{1}{p(t)} dt \right|$$

$$\leq \int_0^x |f(s)|w(s)ds \int_x^1 \frac{1}{p(t)} dt$$

$$= \int_0^x \left(\int_x^1 \frac{1}{p(t)} dt \right) |f(s)|w(s)ds$$

$$\leq \int_0^x \left(\int_s^1 \frac{1}{p(t)} dt \right) |f(s)|w(s)ds \qquad (3.6)$$

[*by lemma* 3.1]

$$\leq \left(\int_0^x f^2(s)w(s)ds \right)^{1/2} \left(\int_0^x \left(\int_s^1 \frac{1}{p(t)} dt \right)^2 w(s)ds \right)^{1/2}$$

[*by using Cauchy – Schwarz inequality*]

$$= \quad M(x)$$

where

$$M(x) = \left(\int_0^x f^2(s)w(s)ds\right)^{1/2} \left(\int_0^x \left(\int_s^1 \frac{1}{p(t)}dt\right)^2 w(s)ds\right)^{1/2}$$

Clearly this function is nonnegative, monotone, increasing and $M(0) = 0$, $M(1) = K$.

Corollary 3.1

$$\lim_{x\to 0} \left(\int_0^x f(s)w(s)ds\right) \left(\int_x^1 \frac{1}{p(t)}dt\right) = 0.$$

Lemma 3.3 *There exists a monotone decreasing function $N(x)$ defined on the interval $[0,1]$ with $N(0) = K$ and $N(1) = 0$ such that for all $x \in [0,1]$*

$$\left|\int_x^1 \left(\int_s^1 \frac{1}{p(t)}dt\right) f(s)w(s)ds\right| \leq N(x).$$

Proof:

$$\left|\int_x^1 \left(\int_s^1 \frac{1}{p(t)}dt\right) f(s)w(s)ds\right|$$

$$\leq \int_x^1 \left(\int_s^1 \frac{1}{p(t)}dt\right) |f(s)|w(s)ds \tag{3.7}$$

$$\leq \left(\int_x^1 f^2(s)w(s)ds\right)^{1/2} \left(\int_x^1 \left(\int_s^1 \frac{1}{p(t)}dt\right)^2 w(s)ds\right)^{1/2}$$

[*using the Cauchy – Schwarz inequality*]

$$= \quad N(x)$$

where

$$N(x) = \left(\int_x^1 f^2(s)w(s)ds\right)^{1/2} \left(\int_x^1 \left(\int_s^1 \frac{1}{p(t)}dt\right)^2 w(s)ds\right)^{1/2}$$

Clearly this function is nonnegative, monotone, decreasing and $N(0) = K$, $N(1) = 0$.

Lemma 3.4 *The solution $u(x)$ of (3.1)-(3.3) satisfies:*

$$|u(x)| \leq K$$

for any $x \in [0,1]$.

Proof: From equation (3.1)-(3.3) we obtain

$$u(x) = \int_x^1 \frac{1}{p(t)} \left(\int_0^t f(s)w(s)ds\right) dt \tag{3.8}$$

Integrating by parts we obtain

$$\begin{aligned} u(x) &= \left[\left(\int_0^t f(s)w(s)ds\right)\left(-\int_t^1 \frac{1}{p(s)}ds\right)\right]_x^1 + \int_x^1 f(s)w(s)\int_s^1 \frac{1}{p(t)}dtds \\ &= \left(\int_0^x f(s)w(s)ds\right)\left(\int_x^1 \frac{1}{p(t)}dt\right) + \int_x^1 \left(\int_s^1 \frac{1}{p(t)}dt\right) f(s)w(s)ds. \end{aligned} \tag{3.9}$$

Taking absolute values of both sides and using (3.6) and (3.7) we obtain,

$$\begin{aligned}
|u(x)| &\leq \int_0^x \int_s^1 \frac{1}{p(t)} dt |f(s)| w(s) ds + \int_x^1 \int_s^1 \frac{1}{p(t)} dt |f(s)| w(s) ds \\
&= \int_0^1 \int_s^1 \frac{1}{p(t)} dt |f(s)| w(s) ds \\
&\leq \left(\int_0^1 f^2(s) w(s) ds \right)^{1/2} \left(\int_0^1 \left(\int_s^1 \frac{1}{p(t)} dt \right)^2 w(s) ds \right)^{1/2} \\
&= K.
\end{aligned}$$

This completes the proof.

Corollary 3.2

$$u(0) = \int_0^1 \left(\int_s^1 \frac{1}{p(t)} dt \right) f(s) w(s) ds$$

and if f is positive then

$$0 < u(x) \leq u(0) \leq K \quad \forall x \in [0, 1]$$

Corollary 3.3 *The function $u = L^{-1} f$ is an indefinite integral.*

Corollary 3.4 *For any $x \in [0, 1]$*

$$\int_x^1 \frac{1}{p(t)} \int_0^t |f(s)| w(s) ds dt \leq K.$$

Proof: Replacing f by $|f|$ in the proof of lemma 3.4 we get the desired result.

Theorem 3.1 *$L^{-1}f \in C[0,1]$ for any $f \in L^2_w(0,1)$.*

Proof: Let $u = L^{-1}f$. Take $x < z$. Then by (3.8)

$$\begin{aligned} u(x) - u(z) &= \int_x^1 \frac{1}{p(t)} \int_0^t f(s)w(s)dsdt - \int_z^1 \frac{1}{p(t)} \int_0^t f(s)w(s)dsdt \\ &= \int_x^z \frac{1}{p(t)} \int_0^t f(s)w(s)dsdt. \end{aligned} \tag{3.10}$$

Taking absolute values of both sides we obtain

$$|u(x) - u(z)| \leq \int_x^z \frac{1}{p(t)} \int_0^t |f(s)|w(s)dsdt. \tag{3.11}$$

By corollary 3.4 this integral exists and therefore, $u = L^{-1}f$ is continuous on [0,1]. This completes the proof.

Theorem 3.2 *For any $f \in L^2_w(0,1)$ the function $u = L^{-1}f$ is absolutely continuous on $[0,1]$.*

Proof: Let $\{(x_i, x_i')\}$ be a finite collection of nonoverlapping intervals in $[0,1]$. Then by (3.11)

$$\begin{aligned} \sum_{i=1}^n |u(x_i') - u(x_i)| &\leq \sum_{i=1}^n \int_{x_i}^{x_i'} \frac{1}{p(t)} \int_0^t |f(s)|w(s)dsdt \\ &= \int_A \frac{1}{p(t)} \int_0^t |f(s)|w(s)dsdt, \end{aligned}$$

where

$$A = \bigcup_{i=1}^n (x_i, x_i').$$

Therefore, given any $\epsilon > 0$, there is a $\delta > 0$ such that for such collection of intervals with

$$mA = \sum_{i=1}^{n} |x_i' - x_i| < \delta$$

we have

$$\int_A \frac{1}{p(t)} \int_0^t |f(s)|w(s)dsdt < \epsilon,$$

$$i.e. \quad \sum_{i=1}^{n} |u(x_i') - u(x_i)| < \epsilon.$$

This completes the proof.

We have, in fact, proved an important and well known result of real analysis: *every indefinite integral is absolutely continuous.*

Corollary 3.5 *For any $f \in L_w^2(0,1)$ the function $u = L^{-1}f$ has the following properties:*

1. *u is of bounded variation on $[0,1]$.*
2. *$u'(x)$ exists for almost all x in $[0,1]$.*

3.1.2 Compactness of the inverse operator

Theorem 3.3 *$L^{-1} : L_w^2(0,1) \rightarrow C[0,1]$ is compact.*

Proof: Let B be a bounded set in $L^2_w(0,1)$ with a bound M. We want to show that $L^{-1}B$ is relatively compact and to do this it is enough to show that the set of functions $L^{-1}B$ is equicontinuous and equibounded (Arzela-Ascoli theorem).

Equicontinuity:

Let $f \in B$ and $y = L^{-1}f$. Then from (3.10), for $x < z$

$$u(x) - u(z) = \int_x^z \frac{1}{p(t)} \int_0^t f(s)w(s)dsdt$$

By lemma 3.4 this integral exists. Therefore, by Fubini's theorem, changing the order of integration we get

$$u(x) - u(z) = \int_0^x f(s)w(s)ds \int_x^z \frac{1}{p(t)}dt + \int_x^z f(s)w(s) \int_s^z \frac{1}{p(t)}dtds \qquad (3.12)$$

Case 1 $x = 0$:

Using corollary 3.1 in (3.12) and taking absolute values of both sides we obtain

$$\begin{aligned}
|u(0) - u(z)| &= \left| \int_0^z f(s)w(s) \int_s^z \frac{1}{p(t)}dtds \right| \\
&\leq \left(\int_0^z f^2(s)w(s)ds \right)^{1/2} \left(\int_0^z \left(\int_s^z \frac{1}{p(t)}dt \right)^2 w(s)ds \right)^{1/2} \\
&\leq \|f\|_{L^2_w(0,1)} \left(\int_0^z \left(\int_s^z \frac{1}{p(t)}dt \right)^2 w(s)ds \right)^{1/2} \\
&\leq M \left(\int_0^z \left(\int_s^z \frac{1}{p(t)}dt \right)^2 w(s)ds \right)^{1/2} . \qquad (3.13)
\end{aligned}$$

Case 2 $x > 0$:

Taking absolute values of both sides of (3.12) we obtain

$$\begin{aligned}
|u(x) - u(z)| &\leq \left|\int_x^z \frac{1}{p(t)}dt\right| \left|\int_0^x f(s)w(s)ds\right| + \left|\int_x^z f(s)w(s)\int_s^z \frac{1}{p(t)}dtds\right| \\
&\leq \left(\int_x^z \frac{1}{p(t)}dt\right)\left(\int_0^x f^2(s)w(s)ds\right)^{1/2}\left(\int_0^x w(s)ds\right)^{1/2} \\
&\quad + \left(\int_x^z f^2(s)w(s)ds\right)^{1/2}\left(\int_x^z\left(\int_s^z \frac{1}{p(t)}dt\right)^2 w(s)ds\right)^{1/2} \\
&\leq \left(\int_0^1 f^2(s)w(s)ds\right)^{1/2}\left(\int_0^x w(s)ds\right)^{1/2}\left(\int_x^z \frac{1}{p(t)}dt\right) \\
&\quad + \left(\int_0^1 f^2(s)w(s)ds\right)^{1/2}\left(\int_x^z\left(\int_s^z \frac{1}{p(t)}dt\right)^2 w(s)ds\right)^{1/2} \\
&\leq M\left(\left(\int_x^z \frac{1}{p(t)}dt\right)\left(\int_0^x w(s)ds\right)^{1/2} + \left(\int_x^z\left(\int_s^z \frac{1}{p(t)}dt\right)^2 w(s)ds\right)^{1/2}\right).
\end{aligned} \tag{3.14}$$

If $0 < z < x$ we interchange x & z.

It is now clear from (3.13) and (3.14) that $L^{-1}B$ is equicontinuous.

Equiboundedness:

By lemma 3.4

$$\begin{aligned} |u(x)| &\leq K \\ &= \|f\|_{L^2_w(0,1)} \left(\int_0^1 \left(\int_s^1 \frac{1}{p(t)} dt \right)^2 w(s) ds \right)^{1/2} \\ &\leq M \left(\int_0^1 \left(\int_s^1 \frac{1}{p(t)} dt \right)^2 w(s) ds \right)^{1/2}. \end{aligned}$$

Thus $L^{-1}B$ is also equibounded. This completes the proof.

Corollary 3.6 $L^{-1} : C[0,1] \to C[0,1]$ *is compact.*

Corollary 3.7 $L^{-1} : L^2_w(0,1) \to L^2_w(0,1)$ *is also compact.*

Proof: Let B be a bounded set in $L^2_w(0,1)$. Then by theorem 3.3 $L^{-1}B$ is relatively compact in $C[0,1]$. But any set which is relatively compact in $C[0,1]$ is also relatively compact in $L^2_w(0,1)$. This completes the proof.

An Alternate Approach to show that $L^{-1} : L^2_w(0,1) \to L^2_w(0,1)$ is compact:

Here we like to add that the compactness of L^{-1} from $L^2_w(0,1)$ to $Lw2$ can be proved alternately through Hilbert-Schimdt kernel. From (3.9) $u(x)$ can be written as

$$u(x) = \int_0^1 k(x,s) f(s) w(s) ds \tag{3.15}$$

where $k(x,s)$ is a kernel given by:

$$k(x,s) = \begin{cases} \int_x^1 \frac{1}{p(t)}dt, & s \le x \\ \int_s^1 \frac{1}{p(t)}dt, & s > x \end{cases} . \tag{3.16}$$

Lemma 3.5 *The kernel $k(x,s)$ is a* Hilbert-Schimdt kernel.

Proof:

$$\begin{aligned}
& \int_0^1 \int_0^1 |k(x,s)|^2 w(x)w(s)dxds \\
= & \int_0^1 \int_0^s \left(\int_s^1 \frac{1}{p(t)}dt\right)^2 w(x)w(s)dxds + \int_0^1 \int_s^1 \left(\int_x^1 \frac{1}{p(t)}dt\right)^2 w(x)w(s)dxds \\
= & \int_0^1 \left(\int_s^1 \frac{1}{p(t)}dt\right)^2 w(s)\left(\int_0^s w(x)dx\right) ds + \int_0^1 \int_s^1 \left(\int_x^1 \frac{1}{p(t)}dt\right)^2 w(s)w(x)dxds \\
\le & \int_0^1 \left(\int_s^1 \frac{1}{p(t)}dt\right)^2 w(s)\left(\int_0^s w(x)dx\right) ds + \int_0^1 \left(\int_s^1 \frac{1}{p(t)}dt\right)^2 w(s)\left(\int_s^1 w(x)dx\right) ds \\
= & \left(\int_0^1 \left(\int_s^1 \frac{1}{p(t)}dt\right)^2 w(s)ds\right)\left(\int_0^1 w(x)dx\right) \\
< & \infty.
\end{aligned}$$

We have thus proved

Theorem 3.4 *The integral operator $L^{-1} : L^2_w(0,1) \to L^2_w(0,1)$ defined by*

$$u(x) = L^{-1}f = \int_0^1 k(x,s)f(s)w(s)ds$$

is a Hilbert-Schimdt operator *and, therefore, is compact.*

3.1.3 The derivative of the solution

Theorem 3.5 *If* $f \in L^2_w(0,1)$ *then* pu' *is absolutely continuous in* $[0,1]$ *where* $u = L^{-1}f$.

Proof: From (3.1) & (3.2) we have

$$p(x)u'(x) = -\int_0^x f(s)w(s)ds$$

Using Cauchy-Scharz inequality it can be easily shown that the integral exists for any $x \in [0,1]$. Hence pu' being an indefinite integral is absolutely continuous.

Theorem 3.6 *If* $f \in L^2_w(0,1)$ *then* $u' \in L^2_p(0,1)$ *where* $u = L^{-1}f$.

Proof: We use the integration by parts in the second step in the following.

$$\begin{aligned}
\int_0^1 p(x)|u'(x)|^2dx &= \int_0^1 (p(x)u'(x))\, u'(x)dx \\
&= [p(x)u'(x)u(x)]_0^1 - \int_0^1 (p(x)u'(x))'\, u(x)dx \\
&= -\int_0^1 (p(x)u'(x))'\, u(x)dx \\
&= \int_0^1 f(x)w(x)u(x)dx \\
&\leq \|f\|_{L^2_w(0,1)}\|u\|_{L^2_w(0,1)} \\
&< \infty.
\end{aligned}$$

This completes the proof.

Remark 3.1 *Suppose $p(x)$ is continuous in a neighborhood of 0. Then in this neighborhood*

$$u'(x) = -\frac{1}{p(x)}\int_0^x f(s)w(s)ds,$$

Suppose $f \in L^2_w(0,1)$. Then

$$|u'(x)| \leq \|f\|_{L^2_w(0,1)}\frac{1}{p(x)}\left(\int_0^x w(s)ds\right)^{1/2}.$$

So if $\lim_{x\to 0^+}\frac{1}{p(x)}\left(\int_0^x w(s)ds\right)^{1/2}$ exists and is bounded then $\lim_{x\to 0^+} u'(x)$ exists and is bounded. If $\lim_{x\to 0^+}\frac{1}{p(x)}\left(\int_0^x w(s)ds\right)^{1/2} = 0$, then $\lim_{x\to 0^+} u'(x) = 0$. Moreover, if $f \in L^\infty(0,1)$, then

$$|u'(x)| \leq \|f\|_{L^\infty_w(0,1)}\frac{1}{p(x)}\int_0^x w(s)ds.$$

Therefore, if $\lim_{x\to 0^+}\frac{1}{p(x)}\int_0^x w(s)ds$ exists and is bounded then $\lim_{x\to 0^+} u'(x)$ exists and is bounded. In particular, if $\lim_{x\to 0^+}\frac{1}{p(x)}\int_0^x w(s)ds = 0$, then $\lim_{x\to 0^+} u'(x) = 0$. We also note that if $p(x) = w(x)$ and $p(x)$ monotone increasing in a neighborhood of 0 then for any x in this neighborhood we have

$$\begin{aligned}
|u'(x)| &\leq \|f\|_{L^\infty_w(0,1)}\frac{1}{p(x)}\int_0^x p(s)ds \\
&\leq \|f\|_{L^\infty_w(0,1)}\frac{1}{p(x)}p(x)\int_0^x ds \\
&= \|f\|_{L^\infty_w(0,1)}\int_0^x ds
\end{aligned}$$

and therefore we also have $\lim_{x\to 0^+} u'(x) = 0$.

3.1.4 The self-adjointness of the operator L

Let us define the domain of L as the following:

$$D(L) = \left\{ u \in H : u, pu' \in AC_{loc}(0,1], Lu \in L^2_w(0,1), u(1) = 0, \lim_{x \to 0^+} pu' = 0 \right\}.$$

We mean $D(L)$ to be this throughout this dissertation.

Theorem 3.7 *$D(L)$ is dense in $L^2_w(0,1)$ and L is self-adjoint.*

Proof:

$$L^{-1} : L^2_w(0,1) \to D(L) \subset L^2_w(0,1)$$

is a symmetric operator defined on the whole space. Therefore it is self-adjoint. It is also injective, since L is injective. Therefore by theorem 2.5, $D(L) = R(L^{-1})$ is dense in $L^2_w(0,1)$ and L is self-adjoint. This completes the proof.

Corollary 3.8 *The spectrum of L is purely discrete, positive and can be arranged in a sequence*

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots$$

with $\lambda_n \to \infty$. Furthermore, the normalized eigenvalues $\{u_n\}$ form a basis in $L^2_w(0,1)$.

Proof: The proof follows from the fact that L^{-1} is a selfadjoint compact operator (theorem 2.3).

3.2 The linear limit point case LP1 ($q = 0$)

We recall the limit point LP1 condition

$$\int_0^1 \left(\int_x^1 \frac{1}{p(t)} dt \right)^2 w(x)dx \ = \ \infty \tag{3.17}$$

and

$$\int_0^1 \left(\int_x^1 \frac{1}{p(t)} dt \right) w(x)dx \ < \ \infty. \tag{3.18}$$

We will show that if the data $f \in L_w^\infty(0,1)$ then the solution $u(x)$ is absolutely continuous on $[0,1]$. We will also show that if $f \in L_w^2(0,1)$ then the solution is absolutely continuous in $(0,1]$. This means that the solution $u(x)$ may be unbounded at $x = 0$.

Theorem 3.8 *Let $f \in L_w^\infty(0,1)$. Then the solution $u = L^{-1}f$ is absolutely continuous on $[0,1]$. Furthermore, the operator*

$$L^{-1} : L_w^\infty(0,1) \to C[0,1]$$

is compact.

Proof: The proof is similar to the limit circle case. In this case we redefine K by

$$K = \|f\|_{L_w^\infty(0,1)} \int_0^1 \left(\int_s^1 \frac{1}{p} \right) w \ < \ \infty$$

and in the proofs instead of using Cauchy-Schuartz inequality we use Holder's inequality.

Remark 3.2 *As a consequence of the results of chapter 5 (see section 5.4) we see that, also in the LP1 case the operator* $L^{-1} : L^2_w(0,1) \to L^2_w(0,1)$ *is compact.*

Theorem 3.9 *If* $f \in L^2_w(0,1)$ *then the solution* $u = L^{-1}f \in L^2_w(0,1)$ *and is absolutely continuous in* $(0,1]$.

Proof: Let $f \in L^2_w(0,1)$. Then from (3.9) it is clear that $u = L^{-1}f$ is absolutely continuous in $(0,1]$. To show that u is in $L^2_w(0,1)$ we show that both the terms of (3.9) are in $L^2_w(0,1)$.

Step1: To show that the first term of (3.9) is in $L^2_w(0,1)$ i.e., $\int_0^x fw \int_x^1 \frac{1}{p} \in L^2_w(0,1)$:

$$\int_0^1 \left(\int_0^x fw\right)^2 \left(\int_x^1 \frac{1}{p}\right)^2 w$$

$$\leq \|f\|^2_{L^2_w(0,1)} \int_0^1 \left(\int_0^x w\right) \left(\int_x^1 \frac{1}{p}\right)^2 w$$

$$\leq \|f\|^2_{L^2_w(0,1)} \int_0^1 \left(\int_0^x w \int_s^1 \frac{1}{p}\right) \left(\int_x^1 \frac{1}{p}\right) w$$

$$\leq \|f\|^2_{L^2_w(0,1)} \left(\int_0^1 w \int_s^1 \frac{1}{p}\right) \int_0^1 \left(\int_x^1 \frac{1}{p}\right) w$$

$$\leq \infty.$$

Step2: To show that the second term of (3.9) is in $L^2_w(0,1)$ i.e., $\int_x^1 \left(\int_s^1 \frac{1}{p}\right) fw \in L^2_w(0,1)$:

$$\int_0^1 \left(\int_x^1 \left(\int_s^1 \frac{1}{p}\right) fw\right)^2 w$$

$$\leq \|f\|^2_{L^2_w(0,1)} \int_0^1 \left(\int_x^1 \left(\int_s^1 \frac{1}{p} \right)^2 w \right) w$$

$$\leq \|f\|^2_{L^2_w(0,1)} \int_0^1 \left(\int_x^1 \left(\int_s^1 \frac{1}{p} \right) w \right) \int_x^1 \frac{1}{p} w$$

$$\leq \|f\|^2_{L^2_w(0,1)} \int_0^1 \left(\int_s^1 \frac{1}{p} \right) w \int_0^1 \left(\int_x^1 \frac{1}{p} \right) w$$

$$\leq \infty.$$

This completes the proof.

Remark 3.3 *If $f \in L^2_w(0,1)$ and not in $L^\infty_w(0,1)$ then the solution $u = L^{-1}f$ may be unbounded at $x = 0$.*

Proof: Consider the example:

$$p(x) = x^3,$$

$$w(x) = x^{3/2},$$

and

$$f(x) = \frac{1}{x}.$$

Then

$$u(x) = (L^{-1}f)(x) = \frac{1}{\sqrt{x}} - 1$$

which is clearly unbounded at $x = 0$.

Theorem 3.10 *For any $f \in L^2_w(0,1)$ the derivative of the solution $u' \in L^2_p(0,1)$.*

Proof: The proof is exactly same as the limit circle case.

3.2.1 The self-adjointness of the operator L

We note that also in the limit point case the operator L is self-adjoint.

Theorem 3.11 *$D(L)$ is dense in $L^2_w(0,1)$ and L is self-adjoint.*

Proof: The proof is exactly same as the proof of theorem 3.7 .

Remark 3.4 *For the proof of the self-adjointness of the operator L in the limit point case, the boundary condition $\lim_{x \to 0^+} p(x)u'(x) = 0$ is not needed, but it naturally holds (see El-Gebeily et. al. [17], page 350, equation (3.15)).*

Corollary 3.9 *L^{-1} is bounded on $L^2_w(0,1)$.*

Proof: This follows from theorem 2.4.

We will also see a direct proof of this in corollary 4.2 in chapter 4.

Remark 3.5 *We define the solution $u(x)$ to be* oscillatory *at the singular point $x = 0$ if $u(x)$ has a zero in every interval $(0, \alpha), 0 < \alpha < 1$ and $\lim_{x \to 0^+} u(x)$ does not exist. Otherwise it is nonoscillatory. We see that in the LC and in the LP1 cases the solution is nonoscillatory.*

Remark 3.6 *It is possible by changing w to trade a limit point case for a limit circle one. However, this trade off will have to be balanced by a more restricted class of data functions.*

3.3 Examples

In this section we illustrate the results of section 3.1 and section 3.2 with a class of examples. We take the problem (3.1)-(3.3) with

$$p(x) = x^{\alpha}, \quad \alpha \geq 1,$$

$$w(x) = x^{\beta}, \quad \beta > -1.$$

It is in the limit circle case if

$$\beta - 2\alpha + 3 \ > \ 0.$$

It is in the limit point case with $\int_x^1 \frac{1}{p} \in L^1_w(0,1)$ if

$$\beta - 2\alpha + 3 \ \leq \ 0$$

and

$$\beta - \alpha + 2 \ > \ 0.$$

It is also checked that if $f \in L^2_w(0,1)$ and

$$\beta - 2\alpha - 1 \ > \ 0$$

then $u'(0) = 0$. Similarly if $f \in L^\infty_w(0,1)$ and

$$\beta - \alpha + 1 \ > \ 0$$

then $u'(0) = 0$.

To show that there are examples where $u'(0)$ may not be equal to zero even in the limit circle case, consider $\alpha = 1$ and $\beta = 0$ (which is a particular example of the limit circle case). Let $f(x) = lnx$ which is in $L^2_w(0,1)$. The solution in this case is given by $u(x) = -2 - xlnx + 2x$. Therefore, $u(0) = -2$ and $\lim_{x \to 0^+} u'(x) = \infty$. But if we take $f(x) = 2cos(\pi x)$ then $\lim_{x \to 0^+} u'(x) = -2$.

The eigenvalues of L are the positive zeros of the Bessel's function

$$J_{\frac{\alpha-1}{\beta-\alpha+2}}\left(\frac{2\sqrt{\lambda}}{\beta-\alpha+2}\right) \ = \ 0 \tag{3.19}$$

with the corresponding eigenfunctions (upto a multiplication by a constant)

$$u(x;\lambda) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(r+k+1)} \left(\frac{\sqrt{\lambda}}{\beta-\alpha+2}\right)^{2k} x^{(\beta-\alpha+2)k}$$

where $r = (\alpha-1)/(\beta-\alpha+2)$. Using the bound (4.4) of chapter 4 it is verified that the first eigenvalue, $\lambda_1(\alpha,\beta)$ is bounded below by:

$$0 < (\beta+1)(\beta-\alpha+2) \ \leq \ \lambda_1(\alpha,\beta). \tag{3.20}$$

For $\alpha = \beta$ equation (3.19) becomes

$$J_{\frac{\alpha-1}{2}}\left(\sqrt{\lambda}\right) = 0$$

and (3.20) becomes

$$0 < 2(\alpha + 1) \;\leq\; \lambda_1(\alpha). \tag{3.21}$$

In this special case Chawla et. el. [7] showed that the first positive zero $\lambda_1(\alpha)$ is an increasing function of α. We copy the list for $\lambda_1(\alpha)$ for different values of α from this paper:

α	1.0	1.5	2.0	2.5	3.0	5.0	10.0
$\lambda_1(\alpha)$	5.781	7.730	9.865	12.179	14.670	26.338	66.721

The bounds (3.20),(3.21) is sufficient for application in many examples.

3.4 The linear limit circle case LC with $q \in L_w^\infty(0,1)$

In this section we study a more general linear case i.e., when $q(x)$ is not necessarily equal to zero. We assume that $q \in L_w^\infty(0,1)$. We extend the results of section 3.1 to this case. We recall the equation:

$$L_q \equiv Lu + q(x)u \;=\; f(x), \quad 0 < x < 1 \tag{3.22}$$

$$\lim_{x \to 0^+} p(x)u'(x) \;=\; 0 \tag{3.23}$$

$$u(1) \;=\; 0 \tag{3.24}$$

with the same conditions on p, w and f for the limit circle LC.

It is observed that the domain of L_q is same as the domain of L.

Lemma 3.6 $qL^{-1} : L_w^2(0,1) \to L_w^2(0,1)$ *is compact.*

Proof: Since $q \in L_w^\infty(0,1)$, the multiplication operator

$$q : D(L) \to L_w^2(0,1)$$

is bounded. But L^{-1} is compact. Hence qL^{-1} is compact.

Corollary 3.10 *The operator*

$$I + qL^{-1} : L_w^2(0,1) \to L_w^2(0,1)$$

is bounded. The spectrum is discrete with 1 as the point of accumulation.

Assumption: In addition to $q \in L_w^\infty(0,1)$ we also assume that

$$0 \notin \sigma(I + qL^{-1}) \tag{3.25}$$

Theorem 3.12 *The solution* $u = (L+q)^{-1}f$ *exists and is in* $D(L)$ *for any* $f \in L_w^2(0,1)$ *and the operator*

$$(L+q)^{-1} : L_w^2(0,1) \to L_w^2(0,1)$$

is compact.

Proof: By the assumption (3.25) 0 is in the resolvent set of the operator $I + qL^{-1}$ and so it's inverse is bounded. Since L^{-1} is compact, then

$$(L+q)^{-1} = L^{-1}(I + qL^{-1})^{-1}$$

exists and is compact.

Corollary 3.11 *The spectrum of $L+q$ is purely discrete and the assumption (3.25) is equivalent to the assumption*

$$0 \notin \sigma(L+q) \tag{3.26}$$

Corollary 3.12 *$L+q$ is self-adjoint.*

Remark 3.7 *We mention here some of the sufficient conditions on q which will gurantee the above assumption (3.26). We define*

$$\text{ess inf } q \doteq \sup\{M : \mu\{x : q(x) < M\} = 0\}$$

and

$$\text{ess sup } q \doteq \inf\{M : \mu\{x : q(x) > M\} = 0\}$$

where μ is the w-measure: $\mu(E) = \int_E w$. Since $q \in L_w^\infty(0,1)$ then both ess sup q *and* ess inf q *are finite.*

1. *If $\lambda_1 + \text{ess inf } q > 0$ where λ_1 is the smallest eigenvalue of L then (3.26) is satisfied. In particular, if $q(x) \geq 0$, this condition is automatically satisfied.*

Proof: For any $u \in D(L)$ with $u \neq 0$ we have

$$\begin{aligned} \langle (L+q)u, u\rangle_w &= \langle Lu, u\rangle_w + \langle qu, u\rangle_w \\ &\geq \langle Lu, u\rangle_w + \text{ess inf } q \ \langle u, u\rangle_w \end{aligned}$$

This gives

$$\begin{aligned}\frac{\langle (L+q)u,u\rangle_w}{\langle u,u\rangle_w} &\geq \frac{\langle Lu,u\rangle_w}{\langle u,u\rangle_w} + \text{ess inf } q \\ &\geq \inf_{u\in D(L)} \frac{\langle Lu,u\rangle_w}{\langle u,u\rangle_w} + \text{ess inf } q \\ &= \lambda_1 + \text{ess inf } q \qquad (3.27) \\ &> 0.\end{aligned}$$

Thus all the eigenvalues of $L+q$ are positive.

2. *If q satisfies*

$$\lambda_i \ < \ \text{ess inf } (-q) \ < \ \text{ess sup } (-q) \ < \ \lambda_{i+1}$$

for some i where λ_i's are the eigenvalues of L then (3.26) is satisfied.

To see this let $k = \frac{c+d}{2}$ where $c = \text{ess inf } (-q)$ and $d = \text{ess sup } (-q)$. Suppose

$$(L+q)u = 0$$

Then

$$Lu = -qu$$

and so

$$(L-k)u = (-q-k)u.$$

Clearly $(L-k)^{-1}$ exists and therefore

$$u = (L-k)^{-1}(-q-k)u.$$

Taking norm of both sides we obtain

$$\begin{aligned}
\|u\|_{L^2_w(0,1)} &\leq \|(L-k)^{-1}\|_{L^2_w(0,1)}\|(-q-k)u\|_{L^2_w(0,1)} \\
&\leq \|(L-k)^{-1}\|_{L^2_w(0,1)}(\text{ess sup }(-q-k))\|u\|_{L^2_w(0,1)} \\
&= \|(L-k)^{-1}\|_{L^2_w(0,1)}\left(\frac{d-c}{2}\right)\|u\|_{L^2_w(0,1)} \\
&= \alpha\|u\|_{L^2_w(0,1)}
\end{aligned}$$

where

$$\alpha = \|(L-k)^{-1}\|_{L^2_w(0,1)}\left(\frac{d-c}{2}\right).$$

But

$$\begin{aligned}
\|(L-k)^{-1}\|_{L^2_w(0,1)} &= \max\{(k-\lambda_i)^{-1}, (\lambda_{i+1}-k)^{-1}\} \\
&= \frac{1}{\min\{k-\lambda_i, \lambda_{i+1}-k\}} \\
&< \frac{1}{(d-c)/2} = \frac{2}{d-c}.
\end{aligned}$$

So $\alpha < 1$ and hence $u \equiv 0$. Thus 0 is not an eigenvalue of $L+q$.

Lemma 3.7 *$L+q$ is bounded below by*

$$\lambda_1 + ess\ inf\ q \geq \lambda_1 - \|q\|_{L^\infty_w(0,1)}$$

where λ_1 is the smallest and positive eigenvalue of L.

Proof: From (3.27) we have,

$$\frac{\langle (L+q)u, u\rangle_w}{\langle u, u\rangle_w} \geq \lambda_1 + ess\ inf\ q \geq \lambda_1 - \|q\|_{L_w^\infty(0,1)}.$$

This completes the proof.

Theorem 3.13 *The spectrun of $(L+q)$ i.e. of (3.22)-(3.24) is purely discrete and can be listed as the sequence*

$$\mu_1 < \mu_2 < \mu_3 < \cdots < \mu_n < \cdots$$

with $\lim_{n\to\infty} \mu_n = +\infty$ *(thus there are atmost finitely many negative eigenvaluse). The corresponding normalized eigenfuntions $\{u_n\}$ form an orthonormal basis in* $L_w^2(0,1)$.

Proof: Follows from theorem 2.3.

Theorem 3.14 $u' \in L_p^2(0,1)$

Proof: Since $u \in D(L)$, and $R(L) = L_w^2(0,1)$ the proof is same as the proof of theorem 3.6.

Chapter 4

The Linear Variational Boundary Value Problem

Consider the linear boundary value problem (1.5)-(1.7) rewritten as:

$$(L+q)u \equiv Lu + qu = f. \tag{4.1}$$

with both limit circle and limit point one (i.e. LP with $\int_s^1 \frac{1}{p} \in L^1_w(0,1)$) cases. Also consider f to be in $L^2_w(0,1)$. The assumptions on q will be made systematically in proper places.

In section 4.1 a new Hibert space V is introduced which contains the domain of L. We show that $D(L)$ is also dense in this space V.

In section 4.2 a linear variational boundary value problem (VBVP) is defined. This variational problem contains the function q. A sufficient but as relaxed as possible assumption on q is made so that this VBVP has a unique solution. A detailed discussion on this assumption is also given seperately in a subsection.

In section 4.3, under a regularity assumption on q (i.e., $q \in L^4_w(0,1)$ for the limit circle case and $q \in L^\infty_w(0,1)$ for the limit point case), it is shown that the solution of the VBVP is indeed the classical solution of the linear boundary value problem.

Finally, in section 4.4, with the same assumptions on q it is shown that the operator $L+q$ is self-adjoint and in the limit circle case the inverse of this operator is compact and the solution of the BVP is in $AC[0,1]$. Also it is shown that in the limit point one case, for $f, q \in L^\infty_w(0,1)$, the solution lies in $AC[0,1]$.

4.1 The Hilbert space V

Let us denote the Hilbert space :

$$H = L^2_w(0,1)$$

with the usual inner product:

$$\langle u, v\rangle_H = \langle u, v\rangle_w = \int_0^1 uvw.$$

Let V be the space defined by:

$$\begin{aligned} V &= \left\{u : u' \in L^2_p(0,1) \textit{ and } u(1) = 0\right\} \\ &= \left\{u : u(x) = \int_x^1 v(s)ds \textit{ where } v \in L^2_p(0,1)\right\} \end{aligned}$$

Clearly $V \subset AC(0,1]$ where by $AC(0,1]$ we mean the set of functions which are absolutely continuous in $(0,1]$ and may or may not be continuous at $x = 0$. It is also clear from chapter 3 that $D(L) \subset V$. Also $V \subset H$ which is a direct consequence of the following theorem. We define the following inner product in V:

$$\langle u, v\rangle_V = \int_0^1 pu'v'$$

Clearly V is a real Hilbert space.

Theorem 4.1 *V is continuously embedded in H.*

Proof: Let $u \in V$. Then $u = \int_s^1 v$ for some $v \in L_p^2(0,1)$. Now

$$\begin{aligned}
\|u\|_H^2 = \int_0^1 u^2 w &= \int_0^1 \left(\int_s^1 v\right)^2 w \\
&= \int_0^1 \left(\int_s^1 \frac{1}{\sqrt{p}}\sqrt{p}v\right)^2 w \\
&\leq \int_0^1 \left(\int_s^1 \frac{1}{p}\right)\left(\int_s^1 pv^2\right) w \\
&\leq \|v\|_{L_p^2(0,1)}^2 \int_0^1 \left(\int_s^1 \frac{dt}{p(t)}\right) w(s)ds \\
&= \|u\|_V^2 \int_0^1 \left(\int_s^1 \frac{dt}{p(t)}\right) w(s)ds = C\|u\|_V^2
\end{aligned}$$

This completes the proof.

Corollary 4.1 *Let*

$$\Lambda := \inf_{u \in V} \frac{\|u\|_V^2}{\|u\|_H^2} \tag{4.2}$$

Then $\Lambda \geq \frac{1}{C}$ *where* $C = \int_0^1 \left(\int_s^1 \frac{dt}{p(t)} \right) w(s) ds.$

Since $D(L) \subset V \subset H$ and $D(L)$ is dense in H, then V is also dense in H.

Corollary 4.2 $L^{-1} : H \longrightarrow D(L) \subset H$ *is bounded.*

Proof: Let $u \in D(L)$. Then by the previous theorem and by the proof of theorem 3.6

$$\begin{aligned} \|u\|_H^2 &\leq C\|u\|_V^2 \\ &= C\langle Lu, u\rangle_H \\ &\leq C\|Lu\|_H \|u\|_H \end{aligned}$$

Thus $\|u\|_H \leq C\|Lu\|_H$. This completes the proof.

Lemma 4.1 *If* $u \in D(L)$ *and* $v \in V$ *then*

$$pu'v \mid_0^1 = 0.$$

Proof: For $u \in D(L)$ there exists $f \in H$ s.t. $pu' = -\int_0^x fw$ and we have

$$|pu'v| = \left| \int_0^x fw \right| \left| \int_x^1 v' \right|$$

$$
\begin{aligned}
&\leq \int_0^x |f|w \int_x^1 |v'| \\
&\leq \int_0^x |f|w \int_s^1 |v'|ds \\
&\leq \int_0^x |f|w \left(\int_s^1 \frac{1}{p}\right)^{1/2} \left(\int_s^1 pv'^2\right)^{1/2} \\
&\leq \|v\|_V \int_0^x |f|w \left(\int_s^1 \frac{1}{p}\right)^{1/2} \\
&\leq \|v\|_V \left(\int_0^x |f|^2 w\right)^{1/2} \left(\int_0^x w \int_s^1 \frac{1}{p}\right)^{1/2} \\
&\longrightarrow 0 \; as \; x \to 0.
\end{aligned}
$$

Since $p(1)u'(1) = -\int_0^1 fw$ which clearly exists and $v(1) = 0$ so $p(1)u'(1)v(1) = 0$. This completes the proof.

A suficient condition for $D(L)$ to be dense in V is given in the following theorem.

Theorem 4.2 *If the measure generated by p is absolutely continuous with respect to the measure generated by w i.e., $\int_E p = 0$ whenever $\int_E w = 0$, then $D(L)$ is dense in V.*

Proof: Let $v \in V$ be such that

$$\langle u, v\rangle_V = 0$$

for all $u \in D(L)$. Thus

$$\int_0^1 pu'v' = 0$$

$$\Rightarrow \quad pu'v\,|_0^1 - \int_0^1 (pu')'v = 0$$

[*by lemma* 4.1]

$$\Rightarrow \quad -\int_0^1 (pu')'v = 0$$

$$\Rightarrow \quad \langle Lu, v\rangle_H = 0$$

$$\Rightarrow \quad \langle f, v\rangle_H = 0$$

for all $f \in H$. This implies that $v = 0$ *a.e.* with respect to the measure generated by w. Then by the assumption of the theorem $v = 0$ *a.e.* with respect to the measure generated by p. Hence $D(L)$ is dense in V. This completes the proof.

Let

$$\lambda_1 := \inf_{u \in D(L)} \frac{\langle Lu, u\rangle_H}{\|u\|_H^2}. \tag{4.3}$$

Then we have the inequality:

$$0 < \frac{1}{\int_0^1 \left(\int_s^1 \frac{dt}{p(t)}\right) w(s)ds} \leq \Lambda \leq \lambda_1 \tag{4.4}$$

Remark 4.1 *If $D(L)$ is dense in V then $\Lambda = \lambda_1$.*

Proof: $v \mapsto \frac{\|v\|_H^2}{\|v\|_V^2}$ is a continuous and bounded functional on V satisfying:

$$0 < \frac{\|v\|_H^2}{\|v\|_V^2} \leq C$$

where

$$C = \int_0^1 \left(\int_s^1 \frac{dt}{p(t)} \right) w(s) ds.$$

Therefore, if $D(L)$ is dense in V

$$\sup_{v \in D(L)} \frac{\|v\|_H^2}{\|v\|_V^2} = \sup_{v \in V} \frac{\|v\|_H^2}{\|v\|_V^2}$$

$$\Rightarrow \quad \frac{1}{\inf_{v \in D(L)} \frac{\|v\|_V^2}{\|v\|_H^2}} = \frac{1}{\inf_{v \in V} \frac{\|v\|_V^2}{\|v\|_H^2}}$$

$$\Rightarrow \quad \frac{1}{\lambda_1} = \frac{1}{\Lambda}.$$

This completes the proof.

4.2 The Variational Boundary Value Problem

Assumptions on q:

Let us assume that q satisfies the following two conditions:

$$\int_0^1 |q(t)| w(t) \left(\int_t^1 \frac{1}{p(s)} ds \right) dt := C_q < \infty \tag{4.5}$$

and

$$\inf_{u \in V} \frac{\int_0^1 q u^2 w}{\|u\|_V^2} := \gamma > -1 \tag{4.6}$$

We want to mention here that under various conditions on p, w, and q these assumptions are satisfied. We give more details on this (for the clarification as well as the implementation) at the end of this section.

We first establish the following lemma.

Lemma 4.2 *Under the assumption (4.5) we have*

$$\begin{aligned}\int_0^1 quvw &\leq \|u\|_V\|v\|_V \int_0^1 |q(t)|w(t)\int_t^1 \frac{1}{p(s)}dsdt \\ &= C_q\|u\|_V\|v\|_V\end{aligned}$$

Proof:

$$\begin{aligned}&\int_0^1 q(t)w(t)u(t)v(t)dt \\ =\ &\int_0^1 q(t)w(t)\left(\int_t^1 u'(s)ds\right)\left(\int_t^1 v'(s)ds\right)dt \\ =\ &\int_0^1 q(t)w(t)\left(\int_t^1 \frac{1}{\sqrt{p(s)}}\sqrt{p(s)}u'(s)ds\right)\left(\int_t^1 \frac{1}{\sqrt{p(s)}}\sqrt{p(s)}v'(s)ds\right) \\ \leq\ &\int_0^1 |q(t)|w(t)\left(\int_t^1 \frac{1}{p(s)}\right)^{1/2}\left(\int_t^1 p(s)u'^2(s)ds\right)^{1/2}\left(\int_t^1 \frac{1}{p(s)}\right)^{1/2}\left(\int_t^1 p(s)v'^2(s)ds\right)^{1/2} \\ \leq\ &\|u\|_V\|v\|_V\int_0^1 |q(t)|w(t)\left(\int_t^1 \frac{1}{p(s)}ds\right)dt \\ =\ &C_q\|u\|_V\|v\|_V \text{ [by assumption (4.5)]}\end{aligned}$$

This completes the proof.

Corollary 4.3 *If $q \equiv 1$ then*

$$\langle u,v\rangle_H = \langle u,v\rangle_w \leq \|u\|_V\|v\|_V \int_0^1 \left(\int_s^1 \frac{1}{p(s)}\right) w = C\|u\|_V\|v\|_V$$

Corollary 4.4 *Theorem 4.1 also directly follows from the corollary 4.3.*

We now define the bilinear form in $V \times V$ as follows:

$$B(u,v) := \langle u,v\rangle_V + \int_0^1 quvw \tag{4.7}$$

It is clear that $B(u,v)$ is symmetric. We define the variational boundary value problem (VBVP) in the following way.

Given $f \in H$ find $u \in V$ such that for all $v \in V$ the following holds:

$$B(u,v) = \langle f,v\rangle_H \tag{4.8}$$

The solution of (4.8), if any, is called the generalized solution or the weak solution of (4.1). We will show that under the above assumptions on q this problem has a unique solution.

Lemma 4.3 *The bilinear form $B(u,v)$ defined by (4.7) is continuous in V.*

Proof:

$$\begin{aligned}
|B(u,v)| &= \left|\langle u,v\rangle_V + \int_0^1 quvw\right| \\
&\leq \|u\|_V\|v\|_V + \|u\|_V\|v\|_V \int_0^1 |q(t)|w(t)\int_t^1 \frac{1}{p(s)}dsdt
\end{aligned}$$

[*by lemma* 4.2]

$$\begin{aligned}
&= \left(1+\int_0^1 |q(t)|w(t)\int_t^1 \frac{1}{p(s)}dsdt\right)\|u\|_V\|v\|_V \\
&= (1+C_q)\|u\|_V\|v\|_V
\end{aligned}$$

This completes the proof.

Lemma 4.4 *Under the assumption (4.6) $B(u,v)$ is V-elliptic.*

Proof:

$$\begin{aligned}
B(u,u) &= \|u\|_V^2 + \int_0^1 qu^2w \\
&\geq \alpha\|u\|_V^2,
\end{aligned}$$

by assumption (4.6), where $\alpha = 1+\gamma > 0$. Thus the bilinear form is V-elliptic.

We are now ready to show that under the assumptions (4.5) and (4.6) on q the VBVP (4.8) has a unique solution in V. This is proved in the following theorem.

Theorem 4.3 *For any $f \in H$ the variational boundary value problem*

$$B(u,v) = \langle f, v\rangle_H \quad \forall v \in V$$

has a unique solution in V.

Proof: Define the functional $l(v) = \langle f, v\rangle_H$ on V. Clearly it is linear. We show that it is bounded:

$$|l(v)| = |\langle f, v\rangle_H| \leq \|f\|_H\|v\|_H \leq \sqrt{C}\|f\|_H\|v\|_V$$

by theorem (4.1) or by corollary (4.3) where $C = \int_0^1 \left(\int_s^1 \frac{dt}{p(t)}\right) w(s)ds$. By lemma (4.3) and lemma (4.4) the bilinear form $B(u,v)$ is continuous and V-elliptic. Hence by the Lax-Milgram theorem there exists a unique solution $u \in V$. This completes the proof.

4.2.1 Remarks on the assumptions on q

Remark 4.2 *If $q \in L^\infty_w(0,1)$ then assumption (4.5) holds.*

Remark 4.3 *If q is Lebesgue integrable in [0,1] and*

$$w(t)\int_t^1 \frac{1}{p(s)}ds \in L^\infty(0,1) \tag{4.9}$$

then assumption (4.5) is also satisfied.

Example 4.1 *Let $p(x) = x^\alpha$, and $w(x) = x^\beta$,$\beta > \alpha - 1$. Then (4.9) of remark 4.3 is satisfied.*

Remark 4.4 *If $q \in L^2_w(0,1)$ and p ,w satisfy the limit circle condition then assumption (4.5) is also satisfied.*

Proof:

$$\begin{aligned}
& \int_0^1 |q(t)|w(t) \left(\int_t^1 \frac{1}{p(s)} ds\right) dt \\
= & \int_0^1 |q(t)|\sqrt{w(t)} \left(\int_t^1 \frac{1}{p(s)} ds\right) \sqrt{w(t)} dt \\
\leq & \left(\int_0^1 |q(t)|^2 w(t) dt\right)^{1/2} \left(\int_0^1 \left(\int_t^1 \frac{1}{p(s)} ds\right)^2 w(t) dt\right)^{1/2} \\
< & \infty
\end{aligned}$$

Remark 4.5 *If $C_q := \int_0^1 |q(t)|w(t) \int_t^1 \frac{1}{p(s)} ds dt < 1$, then assumption (4.6) holds automatically.*

Proof: Let $u \in V$, then

$$\begin{aligned}
-\int_0^1 qu^2 w &= \int_0^1 -q(t) \left(\int_t^1 u'(s) ds\right)^2 w(t) dt \\
&= \int_0^1 -q(t) \left(\int_t^1 \frac{1}{\sqrt{p(s)}} \sqrt{p(s)} u'(s) ds\right)^2 w(t) dt
\end{aligned}$$

$$\leq \int_0^1 |q(t)| \left(\int_t^1 \frac{1}{p(s)} ds \right) w(t) dt \|u\|_V^2$$

$$= C_q \|v\|_V^2$$

Thus

$$\frac{\int_0^1 qu^2 w}{\|u\|_V^2} \geq -C_q > -1.$$

This completes the proof.

Remark 4.6 *If* $\int_0^1 qu^2 w > 0$ *assumption (4.6) is satisfied. In particular if* $q > 0$ *then this is clearly satisfied.*

Remark 4.7 *Let* $D(L)$ *be dense in* V*. Then since the functional (not linear)*

$$v \mapsto \frac{\int_0^1 qv^2 w}{\|v\|_V^2}$$

is continuous on $V - \{0\}$ *(readily checked by using lemma 4.2) assumption (4.6) is equivalent to :*

$$\inf_{u \in D(L)} \frac{\int_0^1 qu^2 w}{\|u\|_V^2} := \gamma > -1 \tag{4.10}$$

Remark 4.8 *Suppose*

$$\lambda_1 + \inf_{u \in D(L)} \frac{\int_0^1 qu^2 w}{\|u\|_H^2} > 0$$

where λ_1 *is given by (4.3) then assumption (4.10) is satisfied.*

Proof: Let

$$\Gamma := \inf_{u \in D(L)} \frac{\int_0^1 qu^2 w}{\|u\|_H^2}. \tag{4.11}$$

Then $\Gamma > -\lambda_1$ and we have

$$\begin{aligned}
\frac{\int_0^1 qu^2w}{\|u\|_V^2} &= \frac{\int_0^1 qu^2w}{\|u\|_H^2}.\frac{\|u\|_H^2}{\|u\|_V^2} \\
&\geq \Gamma.\frac{\|u\|_H^2}{\|u\|_V^2} \\
&\geq \min(0,\Gamma).\frac{\|u\|_H^2}{\|u\|_V^2}
\end{aligned}$$

Taking infimum of both sides we obtain

$$\begin{aligned}
\inf_{u\in D(L)} \frac{\int_0^1 qu^2w}{\|u\|_V^2} &\geq \inf_{u\in D(L)} \left[\min(0,\Gamma).\frac{\|u\|_H^2}{\|u\|_V^2}\right] \\
&= \min(0,\Gamma).\sup_{u\in D(L)} \frac{\|u\|_H^2}{\|u\|_V^2} \\
&= \min(0,\Gamma).\frac{1}{\inf_{u\in D(L)} \frac{\|u\|_V^2}{\|u\|_H^2}} \\
&= \min(0,\Gamma).\frac{1}{\lambda_1} \\
&> -1.
\end{aligned}$$

This completes the proof.

Example 4.2 *If $\lambda_1 + ess\ inf\ q > 0$ then assumption (4.10) is satisfied.*

Proof:

$$\lambda_1 + \inf_{u\in D(L)} \frac{\int_0^1 qu^2w}{\langle u,u\rangle_H}$$

$$
\begin{aligned}
&= \lambda_1 + \inf_{u \in D(L)} \frac{\int_0^1 qu^2 w}{\int_0^1 u^2 w} \\
&\geq \lambda_1 + ess\ inf\ q \\
&> 0.
\end{aligned}
$$

4.3 The relation between the classical solution and the generalized solution

We term the solution of the boundary value problem (4.1) as the classical solution and the solution of the variational boundary value problem (4.8) as the generalized or the weak or the variational solution. We want to show that under a little more restriction on q these two solutions are identical.

Additional assumption on q:

In addition to the assumptions (4.5) and (4.6) let us also assume that q satisfies:

$$u \in V \Rightarrow qu \in H \tag{4.12}$$

Remark 4.9 *In the limit circle case if $q \in L_w^4(0,1)$ then both (4.5) and (4.12) hold.*

Proof: The first part is clear. For the second part we see that

$$\int_0^1 (qu)^2 w = \int_0^1 q^2 uuw$$

$$\leq \int_0^1 q^2 \left(\int_s^1 \frac{1}{p}\right) w.\|u\|_V^2$$

[*by lemma* (4.2)]

$$\leq \|q\|^2_{L^4_w(0,1)} \left(\int_0^1 \left(\int_s^1 \frac{1}{p}\right)^2 w\right)^{1/2} .\|u\|_V^2$$

$$< \infty.$$

Remark 4.10 *In the limit point case with* $\int_s^1 \frac{1}{p} \in L^1_w(0,1)$ *if* $q \in L^\infty_w(0,1)$ *then both (4.5) and (4.12) hold.*

Proof: The first part is direct. For the second one we follow

$$\int_0^1 (qu)^2 w$$

$$\leq \int_0^1 q^2 \left(\int_s^1 \frac{1}{p}\right) w.\|u\|_V^2$$

[*by lemma* (4.2)]

$$\leq \|q\|^2_{L^\infty_w(0,1)} \cdot \int_0^1 \left(\int_s^1 \frac{1}{p}\right) w.\|u\|_V^2$$

$$< \infty.$$

Definition 4.1 *Let S be an operator defined from V to H by*

$D(S) := \{\ u \in V$ *s.t.* $v \mapsto B(u,v)$ is continuous in the topology of $H\ \}$,

$$B(u,v) = \langle Su, v\rangle_H \ \forall v \in V.$$

Remark 4.11 *V is dense in H. So for each $u \in D(S)$ the correspondence $v \mapsto B(u,v)$, which is a continuous linear functional on V in the topology of H, can be extended to a continuous linear functional $G(v)$ on H. Therefore, there exists an element (unique) $f \in H$ such that $Su = f$ and*

$$G(v) = B(u,v) = \langle Su, v\rangle_H = \langle f, v\rangle_H \ \forall v \in V.$$

It is clear that S is symmetric since $B(u,v)$ is symmetric.

Lemma 4.5 *Under the assumption (4.12),*

$$D(L) \subset D(S)$$

and $Su = (L+q)u$ for all $u \in D(L)$.

Proof: Let us fix $u \in D(L)$. Then for any $v \in V$ we have,

$$\begin{aligned} B(u,v) &= \int_0^1 pu'v' + \int_0^1 quvw \\ &= (pu')\,v|_0^1 - \int_0^1 (pu')'v + \int_0^1 quvw \\ &= \langle Lu, v\rangle_H + \langle qu, v\rangle_H \end{aligned}$$

[*by lemma* 4.1]

$$= \langle (L+q)u, v\rangle_H$$

$$= \langle f, v\rangle_H$$

where $f := (L+q)u \in H$.

Therefore,

$$|B(u,v)| \leq \|f\|_H \|v\|_H$$

and for each $u \in D(L)$ the mapping $v \mapsto B(u,v)$ is continuous in V in the topology of H. Hence $D(L) \subset D(S)$. Also for any $u \in D(L)$

$$\langle Su, v\rangle_H = B(u,v) = \langle (L+q)u, v\rangle_H \quad \forall v \in V,$$

and since V is dense in H, so $Su = (L+q)u$. This completes the proof.

Theorem 4.4 *Under the assumption (4.12), $D(S) \subset D(L)$ and $S = L + q$.*

Proof: Let $u \in D(S)$. Then $v \mapsto \langle qv, u\rangle_H = \langle qu, v\rangle_H$ is continuous on $D(L)$ (in the topology of H), since $qu \in H$. Clearly $v \mapsto B(v,u)$ is also continuous on $D(L)$. Therefore,

$$v \mapsto \langle v, u\rangle_V = B(v,u) - \langle qv, u\rangle_H$$

is continuous on $D(L)$. But since $v \in D(L)$ and $u \in D(S) \subset V$, then by lemma (4.1) we see that

$$\langle v, u\rangle_V = \langle Lv, u\rangle_H.$$

Thus $v \mapsto \langle Lv, u\rangle_H$ is continuous on $D(L)$. This implies that $u \in D(L^*)$ (see definition (2.1). But $L = L^*$. Hence $u \in D(L)$. This completes the proof.

The following theorem shows the equivalence of the BVP and VBVP.

Theorem 4.5 *Let $f \in H$. Then the following statements are equivalent:*

(i) $u \in D(L+q)$ *and* $(L+q)u = f$.

(ii) $u \in V$ *and* $B(u,v) = \langle f, v\rangle_H$ *for all* $v \in V$.

Proof: The proof of (i) $\Rightarrow$ (ii) is direct.

(ii) $\Rightarrow$ (i):

Let $u \in V$ such that

$$B(u,v) = \langle f, v\rangle_H \quad \forall v \in V.$$

Then $v \mapsto B(u,v)$ is continuous in V in the topology of H. Thus by the definition of S and by the previous two lemmas $u \in D(S) = D(L+q) = D(L)$ and

$$\langle (L+q)u, v\rangle_H = \langle Su, v\rangle_H = B(u,v) = \langle f, v\rangle_H \quad \forall v \in V.$$

But since V is dense in H, then,

$$(L+q)u = f.$$

This completes the proof.

4.4 Properties of the operator $L + q$ and its inverse

In this section we also assume that q satisfies (4.5), (4.6) and (4.12). We have the following results.

Theorem 4.6 *The operator*

$$L + q : D(L + q) \subset H \longrightarrow H$$

is bijective and $(L + q)^{-1}$ *is bounded on* H.

Proof:

1. $L + q$ *is injective and* $(L + q)^{-1}$ *is continuous:*

$$\begin{aligned} \|u\|_H^2 &\le C_q \|u\|_V^2 \quad [\text{ by lemma 4.2 }] \\ &= \frac{C_q}{\alpha} \alpha \|u\|_V^2, \text{ where } \alpha = 1 + \gamma \\ &\le \frac{C_q}{\alpha} |B(u, u)| \\ &= \frac{C_q}{\alpha} |\langle (L + q)u, u \rangle_H| \\ &\le \frac{C_q}{\alpha} \|(L + q)u\|_H \|u\|_H \end{aligned}$$

2. $L+q$ *is surjective:*

Let $f \in H$. Then by theorem (4.5) it is sufficient to prove that there exists an $u \in V$ such that

$$B(u,v) = \langle f, v\rangle_H \quad \forall v \in H.$$

But this follows from theorem (4.3). This completes the proof.

Theorem 4.7

(i) $(L+q)^{-1}$ *is self-adjoint.*

(ii) $L+q$ *is self-adjoint.*

Proof: (i) It is enough to show that $(L+q)^{-1}$ is symmetric. For any $f, g \in H$ let $(L+q)^{-1}f = u$ and $(L+q)^{-1}g = v$. Then $u, v \in D(L)$ and

$$\begin{aligned}
\langle (L+q)^{-1}f, g\rangle_H &= \langle u, g\rangle_H \\
&= \langle u, (L+q)v\rangle_H \\
&= \langle (L+q)u, v\rangle_H \\
&= \langle f, (L+q)^{-1}g\rangle_H.
\end{aligned}$$

(ii) Since $(L+q)^{-1}$ is self-adjoint and injective, the result follows from theorem (2.5).

Theorem 4.8 *For the limit circle case*

$$(L+q)^{-1} \; : \; H \longrightarrow D(L) \subset H$$

is compact.

Proof:

$$(L+q)^{-1} = L^{-1}(I+qL^{-1})^{-1}.$$

Clearly qL^{-1} is compact. So $I+qL^{-1}$ has discrete spectrum with 1 as the only point of accumulation. We need to show that 0 is not an eigenvalue of $I+qL^{-1}$. Suppose it is. Then there exists $f \neq 0$ in H s.t.

$$(I+qL^{-1})f = 0$$

$$\Rightarrow \; f + qL^{-1}f = 0$$

$$\Rightarrow \; Lu + qu = 0 \quad \textit{where } L^{-1}f = u$$

$$\Rightarrow \; u = 0 \quad [\, \textit{since } L+q : D(L) \rightarrow H \textit{ is bijective} \,]$$

$$\Rightarrow \; f = 0,$$

a contradiction. Therefore, $(I+qL^{-1})^{-1}$ is bounded. Hence $(L+q)^{-1}$ is compact.

We now give a similar theorem for the limit pont (one) case. In this case we take q in $L^2_w(0,1)$ and we work in the space $L^\infty_w(0,1)$. The following theorem also tells about the solution u when $q, f \in L^\infty_w(0,1)$.

Theorem 4.9 *For the limit point (one) case, for any f and q in $L_w^\infty(0,1)$, the solution of $Lu + qu = f$ is in $AC[0,1]$. Futhermore,*

$$(L+q)^{-1} \; : \; L_w^\infty(0,1) \longrightarrow C[0,1]$$

is compact.

Proof: The argument of the proof is same as that of the above theorem. Define

$$I + qL^{-1} : L_w^\infty(0,1) \to C[0,1].$$

By theorem 3.8, $qL^{-1} : L_w^\infty(0,1) \to L_w^\infty(0,1)$ is compact. So 1 is the only point of accumulation. We show that 0 is not an eigenvalue. Suppose it is. Then there is $f \in L_w^\infty(0,1)$ so that $f \neq 0$ and $(I + qL^{-1})f = 0$. But $L^{-1}f = v \in AC[0,1]$ and $Lv + qv = 0$. Since $L + q$ is injective, $v = 0$ and so $f = 0$, a contradiction. This proves that $I + qL^{-1}$ is injective and

$$(I + qL^{-1})^{-1} : L_w^\infty(0,1) \to L_w^\infty(0,1)$$

is bounded. Hence for any $f \in L_w^\infty(0,1)$, the solution

$$u = (L+q)^{-1}f = L^{-1}(I + qL^{-1})^{-1}f \in AC[0,1]$$

and $(L+q)^{-1} : L_w^\infty(0,1) \to C[0,1]$ is compact. This completes the proof.

Chapter 5

Approximation by Galerkin method

In this chapter we approximate the solution of the variational boundary value problem (VBVP) defined by (4.8). Both the LC and LP1 cases are considered with the same assumptions on q as was made in chapter 4 (for the existence and uniqueness of the solution of VBVP as well as the coincidence of this solution to the solution of the original boundary value problem BVP). An approximation subspace V_n of the space V is defined in section 5.1. In section 5.2 we start the study of the convergence of the Galerkin approximation in this subspace. To do this we first find a relation between the Galerkin error and the interpolation error in terms of the V-norm. In section 5.3 we give the estimate of the interpolation error for various cases in terms of V-norm and the uniform norm. Error estimation of the Galerkin approximation in terms of the V-norm is shown in section 5.4. In this section we also deduce a uniform norm estimation. The $L^2_w(0,1)$-norm estimate is immediate from V-norm estimate because V is continuously embedded in $L^2_w(0,1)$ (theorem 4.1). In case when p is monotone increasing it can be shown that V is also continuously embedded in $L^2_p(0,1)$ and thus in this case the $L^2_p(0,1)$-norm of the Galerkin error also follows from the V-norm estimate. In section 5.5 both for LC and LP1 cases higher order accuracies in terms of the uniform norm are obtained for special data.

Since the bilinear form $B(u,v)$ is symmetric, continuous in V and is V-elliptic, it induces an inner product $\langle u,v\rangle_B = B(u,v)$ in the space V and the corresponding norm (B-norm) is equivalent to the V-norm. We recall that the Galerkin approximation u^G in a approximation subspace V_n is the orthogonal projection of the solution u to this subspace w.r.t. the inner product $\langle .,.\rangle_B$. We will see that the interpolation u^I of the solution u in the subspce V_n of section 5.1 is an orthogonal projection of u w.r.t. the inner product $\langle .,.\rangle_V$.

5.1 Approximation subspaces

Let P_n denote an arbitrary partition of [0,1]:

$$0 = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = 1.$$

For $i = 1, 2, \cdots n$ define the patch functions

$$r_i(x) = \begin{cases} r_i^-(x) & \text{if } x_{i-1} \le x \le x_i, \\ r_i^+(x) & \text{if } x_i \le x \le x_{i+1}, \\ 0 & \text{otherwise,} \end{cases} \quad (5.1)$$

where

$$r_1^-(x) = 1, \quad (5.2)$$

$$r_i^-(x) = \frac{\int_{x_{i-1}}^{x} \frac{1}{p(s)} ds}{\int_{x_{i-1}}^{x_i} \frac{1}{p(s)} ds}, \quad i = 2, 3, \cdots, n \quad (5.3)$$

and

$$r_i^+(x) = \frac{\int_{x}^{x_{i+1}} \frac{1}{p(s)} ds}{\int_{x_i}^{x_{i+1}} \frac{1}{p(s)} ds}, \quad i = 1, 2, \cdots, n. \quad (5.4)$$

It is observed that $r_i(x) \in V$ and

$$r_i(x_j) = \delta_{ij}.$$

Therefore, $\{r_i\}_{i=1}^n$ is a linearly independent subset of V. Let V_n be the subspace of V generated by the subset $\{r_i\}_{i=1}^n$.

As we have seen in section 2.3 that the Galerkin approximation $u^G = Pu \in V_n$ can be written as

$$u^G = \sum_{j=1}^{n} \alpha_j r_j \quad (5.5)$$

where α_j's are determined uniquely by the system

$$\tilde{A}\alpha = \mathbf{b}. \quad (5.6)$$

The entries of $\tilde{A}$ are given by

$$\begin{aligned}\tilde{a}_{ij} &= B(r_j, r_i)\\ &= B(r_i, r_j)\\ &= \langle r_i, r_j\rangle_V + \langle q r_i, r_j\rangle_H\\ &= a_{ij} + q_{ij}.\end{aligned}$$

Therefore, (5.6) becomes

$$(A+Q)\alpha = \mathbf{b} \tag{5.7}$$

where $A = (a_{ij}) = (\langle r_i, r_j\rangle_V)$ and $Q = (q_{ij}) = (\langle q r_i, r_j\rangle_H)$ are symmetric and tridiagonal matrices given by

$$a_{11} = \frac{1}{\int_{x_1}^{x_2} \frac{1}{p(s)} ds}, \tag{5.8}$$

$$a_{ii} = \frac{1}{\int_{x_{i-1}}^{x_i} \frac{1}{p(s)} ds} + \frac{1}{\int_{x_i}^{x_{i+1}} \frac{1}{p(s)} ds}, \quad i = 2, \cdots, n, \tag{5.9}$$

$$a_{i,i+1} = -\frac{1}{\int_{x_i}^{x_{i+1}} \frac{1}{p(s)} ds}, \quad i = 1, \cdots, n-1 \tag{5.10}$$

and

$$q_{11} = \int_{x_0}^{x_1} q(s)w(s)ds + \frac{\int_{x_1}^{x_2} q(s)\left(\int_s^{x_2} \frac{1}{p(t)}dt\right)^2 w(s)ds}{\left(\int_{x_1}^{x_2} \frac{1}{p(s)}ds\right)^2} \tag{5.11}$$

$$q_{ii} = \frac{\int_{x_{i-1}}^{x_i} q(s)\left(\int_{x_{i-1}}^{s} \frac{1}{p(t)}dt\right)^2 w(s)ds}{\left(\int_{x_{i-1}}^{x_i} \frac{1}{p(s)}ds\right)^2} + \frac{\int_{x_i}^{x_{i+1}} q(s)\left(\int_s^{x_{i+1}} \frac{1}{p(t)}dt\right)^2 w(s)ds}{\left(\int_{x_i}^{x_{i+1}} \frac{1}{p(s)}ds\right)^2},$$

$$i = 2, \cdots, n, \tag{5.12}$$

$$q_{i,i+1} = \frac{\int_{x_i}^{x_{i+1}} q(s) \left(\int_{x_i}^{s} \frac{1}{p(t)} dt\right) \left(\int_{s}^{x_{i+1}} \frac{1}{p(t)} dt\right) w(s) ds}{\left(\int_{x_i}^{x_{i+1}} \frac{1}{p(s)} ds\right)^2}, \quad i = 1, \cdots, n-1 \tag{5.13}$$

The vector $\mathbf{b} = (b_i) = (l(r_i)) = \left(\int_0^1 f(s) r_i(s) w(s) ds\right)$ is given by

$$b_1 = \int_{x_0}^{x_1} f(s) w(s) ds + \frac{\int_{x_1}^{x_2} f(s) \left(\int_{s}^{x_2} \frac{1}{p(t)} dt\right) w(s) ds}{\left(\int_{x_1}^{x_2} \frac{1}{p(s)} ds\right)}$$

$$b_i = \frac{\int_{x_{i-1}}^{x_i} f(s) \left(\int_{x_{i-1}}^{s} \frac{1}{p(t)} dt\right) w(s) ds}{\left(\int_{x_{i-1}}^{x_i} \frac{1}{p(s)} ds\right)} + \frac{\int_{x_i}^{x_{i+1}} f(s) \left(\int_{s}^{x_{i+1}} \frac{1}{p(t)} dt\right) w(s) ds}{\left(\int_{x_i}^{x_{i+1}} \frac{1}{p(s)} ds\right)}, \quad i = 2, \cdots, n.$$

The matrix A is irreducible and diagonally dominant with $a_{ii} > 0$, $i = 1, \cdots, n$ and $a_{ij} \leq 0$, $i \neq j$. Therefore by theorem 2.10 A is an M-matrix. A is also positive definite.

For the error analysis of the Galerkin approximation of the solution u of VBVP (4.8) we define the V_n-interpolate of the solution u (uniquely) by

$$u^I = \sum_{i=1}^{n} u_i r_i(x) \tag{5.14}$$

where $u_i = u(x_i)$ $i = 1, \cdots, n$. Clearly $u^I \in V_n$. We show that u^I is the orthogonal projection of u on V_n w.r.t. the original inner product of V. This is seen from the following lemma.

Lemma 5.1 *Let u^I be the V_n-interpolate of the solution u of the VBVP. Then for any $v_n \in V_n$*

$$\langle u - u^I, v_n \rangle_V = 0.$$

Proof: Let $v_n(x) = \sum_{i=1}^n c_i r_i(x)$. Then

$$\begin{aligned}
\langle u - u^I, v_n \rangle_V &= \int_0^1 p(u - u^I)' v_n' \\
&= \sum_{i=1}^n c_i \int_0^1 p r_i'(u - u^I)' \\
&= \sum_{i=1}^n c_i \int_{x_{i-1}}^{x_{i+1}} p r_i'(u - u^I)' \\
&= \sum_{i=1}^n c_i \left[\int_{x_{i-1}}^{x_i} p(r_i^-)'(u - u^I)' + \int_{x_i}^{x_{i+1}} p(r_i^+)'(u - u^I)' \right] \\
&= -c_1 \frac{\int_{x_1}^{x_2} (u - u^I)'}{\int_{x_1}^{x_2} \frac{1}{p}} + \sum_{i=2}^n c_i \left[\frac{\int_{x_{i-1}}^{x_i} (u - u^I)'}{\int_{x_{i-1}}^{x_i} \frac{1}{p}} - \frac{\int_{x_i}^{x_{i+1}} (u - u^I)'}{\int_{x_i}^{x_{i+1}} \frac{1}{p}} \right] \\
&= 0.
\end{aligned}$$

5.2 Relation between the Galerkin error and the interpolation error in the V-norm

From (2.11) we have

$$\|u - u^G\|_V \leq \frac{1 + C_q}{\alpha} \|u - u^I\|_V$$

where C_q and α are the constants of lemma 4.3 and lemma 4.4. This relation is independent of the choice of the basis. We show that for our basis we have a better relation (in the case α is very small compared to 1). For deriving such a relation we need to prove two lemmas. These lemmas will also be useful in the subsequent sections.

Lemma 5.2 *For any $v_n \in V_n$*

$$B(u^G - u^I, v_n) = \langle q(u - u^I), v_n \rangle_H.$$

Proof:

$$\begin{aligned} B(u^G - u^I, v_n) &= B(u - u^I, v_n) - B(u - u^G, v_n) \\ &= B(u - u^I, v_n) \quad [by \ (2.3)] \\ &= \langle u - u^I, v_n \rangle_V + \langle q(u - u^I), u_n \rangle_H \\ &= \langle q(u - u^I), u_n \rangle_H. \quad [by \ lemma \ 5.1] \end{aligned}$$

Lemma 5.3 *For any $v_n \in V_n$*

$$\langle u^G - u^I, v_n \rangle_V = \langle q(u - u^G), v_n \rangle_H.$$

Proof: By lemma 5.2

$$\langle u^G - u^I, v_n\rangle_V + \langle q(u^G - u^I), v_n\rangle_H = \langle q(u - u^I), v_n\rangle_H$$

$$\Rightarrow \quad \langle u^G - u^I, v_n\rangle_V = \langle q(u - u^G), v_n\rangle_H.$$

This completes the proof.

Now we have the following relation between the two errors (the Galerkin error and the interpolation error) with respect to our basis. We write this as a theorem.

Theorem 5.1 *Let u^G be the Galerkin approximation and u^I be the V_n-interpolate of the solution u of the VBVP. Then*

$$\|u - u^G\|_V \leq \left(1 + \frac{C_q}{\alpha}\right) \|u - u^I\|_V \tag{5.15}$$

where C_q and α are given by lemma 4.3 and lemma 4.4.

Proof: In lemma 5.2 put $v_n = u^G - u^I$, then

$$B(u^G - u^I, u^G - u^I) = \langle q(u - u^I), u^G - u^I\rangle_H$$

$$\Rightarrow \quad \alpha\|u^G - u^I\|_V^2 \leq C_q\|u - u^I\|_V\|u^G - u^I\|_V$$

(*by lemma* 4.4 *and lemma* 4.2)

$$\Rightarrow \quad \|u^G - u^I\|_V \leq \frac{C_q}{\alpha}\|u - u^I\|_V.$$

Therefore,

$$\begin{aligned}\|u-u^G\|_V &\leq \|u-u^I\|_V + \|u^I-u^G\|_V \\ &\leq \left(1+\frac{C_q}{\alpha}\right)\|u-u^I\|_V.\end{aligned}$$

We notice that for $q = 0$, the inner product $\langle .,. \rangle_B$ coincides with the inner product $\langle .,. \rangle_V$ and the Galerkin approximation u^G coincides with the interpolate u^I.

5.3 Interpolation error estimation in the V-norm and the uniform norm

Our purpose in this section is to find estimates for the error of the V_n-interpolation of the solution u and then use them for finding the estimates of the error for the Galerkin approximation. We estimate the interpolation error in both the V-norm and the uniform norm. We first consider the ideal case $q = 0$ then we derive the results for $q \neq 0$.

Lemma 5.4 *Let u be the solution of the VBVP (4.8) for $q = 0$. Then for any $x \in [x_i, x_{i+1}]$, $i = 1, \cdots, n$,*

$$\begin{aligned}u(x) - u^I(x) &= r_i^+(x)\int_{x_i}^{x}\int_{x_i}^{s}\frac{dt}{p(t)}f(s)w(s)ds \\ &\quad + r_{i+1}^-(x)\int_{x}^{x_{i+1}}\int_{s}^{x_{i+1}}\frac{dt}{p(t)}f(s)w(s)ds. \qquad (5.16)\end{aligned}$$

Proof: For any $x \in [x_i, x_{i+1}]$, $i = 1, \cdots, n$

$$u^I(x) = u_i r_i^+(x) + u_{i+1} r_{i+1}^-(x).$$

Since $r_i^+(x) + r_{i+1}^-(x) = 1$ we have

$$\begin{aligned} u(x) - u^I(x) &= r_i^+(x)\left\{u(x) - u(x_i)\right\} - r_{i+1}^-(x)\left\{u(x_{i+1}) - u(x)\right\} \\ &= r_i^+(x) \int_{x_i}^{x} \frac{1}{p(s)} \int_0^s f(t)w(t)dtds \\ &\quad - r_{i+1}^-(x) \int_x^{x_{i+1}} \frac{1}{p(s)} \int_0^s f(t)w(t)dtds. \end{aligned}$$

Integrating by parts and then simplifying we get the desired result.

Now we give the estimates of the interpolation error in V-norm.

Theorem 5.2 *Let $q = 0$ and u be the solution of the VBVP. Then*

$$\|u - u^I\|_V \leq 2\|f\|_{L^2_w(0,1)} \max_{0 \leq i \leq n} \left(\int_{x_i}^{x_{i+1}} \left(\int_s^{x_{i+1}} \frac{dt}{p(t)} \right) w(s)ds \right)^{1/2}. \tag{5.17}$$

Proof: Differentiating both sides of (5.16) and then simplifying we obtain

$$\left(u(x) - u^I(x)\right)' = \begin{cases} \frac{\frac{1}{p(x)}}{\int_{x_i}^{x_{i+1}} \frac{1}{p}} \left[\int_{x_i}^{x} \left(\int_{x_i}^{s} \frac{1}{p} \right) fw - \int_x^{x_{i+1}} \left(\int_s^{x_{i+1}} \frac{1}{p} \right) fw \right], & x \in [x_i. x_{i+1}], i \geq 1, \\ -\frac{1}{p(x)} \int_0^x f(s)w(s)ds, & x \in [0. x_1]. \end{cases} \tag{5.18}$$

Now

$$\|u - u^I\|_V^2 = \int_0^1 | \left(u(x) - u^I(x)\right)' |^2 p(x)dx$$

$$= \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} | \left(u(x) - u^I(x)\right)' |^2 p(x) dx$$

$$= \int_0^{x_1} \frac{1}{p(x)} \left(\int_0^x f(s)w(s)ds\right)^2 +$$

$$\sum_{i=1}^{n} \left(\int_{x_i}^{x_{i+1}} \frac{1}{p}\right)^{-2} \int_{x_i}^{x_{i+1}} \frac{1}{p} \left[\int_{x_i}^{x} \left(\int_{x_i}^{s} \frac{dt}{p(t)}\right) fwds - \int_{x}^{x_{i+1}} \left(\int_{s}^{x_{i+1}} \frac{dt}{p(t)}\right) fwds\right]^2 dx$$

$$\leq \int_0^{x_1} \frac{1}{p(x)} \left(\int_0^x f^2(s)w(s)ds\right) \left(\int_0^x w(s)ds\right) dx +$$

$$\sum_{i=1}^{n} 2 \left(\int_{x_i}^{x_{i+1}} \frac{ds}{p(s)}\right)^{-2} \int_{x_i}^{x_{i+1}} \frac{1}{p(x)} \left(\int_{x_i}^{x} \int_{x_i}^{s} \frac{dt}{p(t)} f(s)w(s)ds\right)^2 dx +$$

$$\sum_{i=1}^{n} 2 \left(\int_{x_i}^{x_{i+1}} \frac{ds}{p(s)}\right)^{-2} \int_{x_i}^{x_{i+1}} \frac{1}{p(x)} \left(\int_{x}^{x_{i+1}} \int_{s}^{x_{i+1}} \frac{dt}{p(t)} f(s)w(s)ds\right)^2 dx.$$

But the term under the first summation equals

$$\left(\int_{x_i}^{x_{i+1}} \frac{ds}{p(s)}\right)^{-2} \int_{x_i}^{x_{i+1}} \frac{1}{p(x)} \left(\int_{x_i}^{x} \left(\int_{x_i}^{s} \frac{dt}{p(t)}\right) f(s)w(s)ds\right)^2 dx$$

$$\leq \left(\int_{x_i}^{x_{i+1}} \frac{ds}{p(s)}\right)^{-2} \left(\int_{x_i}^{x_{i+1}} \frac{dt}{p(t)}\right)^2 \int_{x_i}^{x_{i+1}} \frac{1}{p(x)} \left(\int_{x_i}^{x} f(s)w(s)ds\right)^2 dx$$

$$= \int_{x_i}^{x_{i+1}} \frac{1}{p(x)} \left(\int_{x_i}^{x} f(s)w(s)ds\right)^2 dx$$

$$\leq \int_{x_i}^{x_{i+1}} \frac{1}{p(x)} \left(\int_{x_i}^{x} f^2(s)w(s)ds\right) \left(\int_{x_i}^{x} w(s)ds\right) dx$$

$$\leq \left(\int_{x_i}^{x_{i+1}} f^2(s)w(s)ds\right) \int_{x_i}^{x_{i+1}} \frac{1}{p(x)} \left(\int_{x_i}^{x} w(s)ds\right) dx$$

$$= \left(\int_{x_i}^{x_{i+1}} f^2(s)w(s)ds\right)\int_{x_i}^{x_{i+1}}\left(\int_x^{x_{i+1}}\frac{ds}{p(s)}\right)w(x)dx$$

[*by integration by parts*],

and the term under the second summation equals

$$\left(\int_{x_i}^{x_{i+1}}\frac{ds}{p(s)}\right)^{-2}\int_{x_i}^{x_{i+1}}\frac{1}{p(x)}\left(\int_x^{x_{i+1}}\left(\int_s^{x_{i+1}}\frac{dt}{p(t)}\right)f(s)w(s)ds\right)^2 dx$$

$$\leq \left(\int_{x_i}^{x_{i+1}}\frac{ds}{p(s)}\right)^{-2}\int_{x_i}^{x_{i+1}}\frac{1}{p(x)}\left(\int_x^{x_{i+1}} f^2(s)w(s)ds\right)\left(\int_x^{x_{i+1}}\left(\int_s^{x_{i+1}}\frac{dt}{p(t)}\right)^2 w(s)ds\right)dx$$

$$\leq \left(\int_{x_i}^{x_{i+1}} f^2(s)w(s)ds\right)\left(\int_{x_i}^{x_{i+1}}\frac{ds}{p(s)}\right)^{-1}\int_{x_i}^{x_{i+1}}\frac{1}{p(x)}\int_x^{x_{i+1}}\left(\int_s^{x_{i+1}}\frac{dt}{p(t)}\right)w(s)dsdx$$

$$\leq \left(\int_{x_i}^{x_{i+1}} f^2(s)w(s)ds\right)\left(\int_{x_i}^{x_{i+1}}\frac{ds}{p(s)}\right)^{-1}\int_{x_i}^{x_{i+1}}\left(\int_s^{x_{i+1}}\frac{dt}{p(t)}\right)w(s)ds\left(\int_{x_i}^{x_{i+1}}\frac{1}{p(x)}dx\right)$$

$$= \left(\int_{x_i}^{x_{i+1}} f^2(s)w(s)ds\right)\int_{x_i}^{x_{i+1}}\left(\int_s^{x_{i+1}}\frac{dt}{p(t)}\right)w(s)ds.$$

Therefore,

$$\begin{aligned}
\|u-u^I\|_V^2 &\leq \int_0^{x_1} f^2(s)w(s)ds\int_0^{x_1}\left(\int_x^{x_1}\frac{ds}{p(s)}\right)w(x)dx + \\
&\quad 4\sum_{i=1}^{n}\left(\int_{x_i}^{x_{i+1}} f^2(s)w(s)ds\right)\int_{x_i}^{x_{i+1}}\left(\int_x^{x_{i+1}}\frac{ds}{p(s)}\right)w(x)dx \\
&\leq 4\left(\max_{0\leq i\leq n}\int_{x_i}^{x_{i+1}}\left(\int_x^{x_{i+1}}\frac{ds}{p(s)}\right)w(x)dx\right)\sum_{i=0}^{n}\int_{x_i}^{x_{i+1}} f^2(s)w(s)ds \\
&= 4\|f\|^2_{L^2_w(0,1)}\max_{0\leq i\leq n}\int_{x_i}^{x_{i+1}}\left(\int_x^{x_{i+1}}\frac{ds}{p(s)}\right)w(x)dx.
\end{aligned}$$

This completes the proof.

Theorem 5.3 *If q satisfies (4.5),(4.6) and (4.12) (both for LC and LP1) then*

$$\|u-u^I\|_V \le 2\|f-qu\|_{L^2_w(0,1)} \left(\max_{0\le i\le n} \int_{x_i}^{x_{i+1}} \left(\int_s^{x_{i+1}} \frac{dt}{p(t)}\right) w(s)ds\right)^{1/2}. \quad (5.19)$$

Proof: The solution u is fixed and $qu \in L^2_w(0,1)$ (by assumption (4.12)). So $f-qu$ is fixed in $L^2_w(0,1)$. Therefore, replacing f by $f-qu$ in (5.17) we obtain the desired result.

From the above theorem we notice that the order of the interpolation error in the V-norm is

$$\max_{0\le i\le n} \left(\int_{x_i}^{x_{i+1}} \left(\int_s^{x_{i+1}} \frac{dt}{p(t)}\right) w(s)ds\right)^{1/2}.$$

We now give the interpolation error in uniform norm.

Lemma 5.5 *For $q=0$ and for any $x \in [x_i, x_{i+1}]$, $i = 0, 1, \cdots, n$*

$$u(x) - u^I(x) \le \int_{x_i}^{x_{i+1}} |f(s)| \left(\int_s^{x_{i+1}} \frac{dt}{p(t)}\right) w(s)ds. \quad (5.20)$$

Proof: For $i=0$ i.e., for $x \in [0, x_1]$ we have

$$\begin{aligned} u(x) - u^I(x) &= u(x) - u(x_1) \\ &= \int_x^{x_1} \frac{1}{p(s)} \int_0^s f(t)w(t)dt \end{aligned}$$

$$
\begin{aligned}
&= \int_x^{x_1} \frac{ds}{p(s)} \int_0^x f(s)w(s)ds + \int_x^{x_1} f(s)w(s) \int_s^{x_1} \frac{dt}{p(t)} ds \\
&\le \int_0^x |f(s)|w(s) \int_s^{x_1} \frac{dt}{p(t)} ds + \int_x^{x_1} |f(s)|w(s) \int_s^{x_1} \frac{dt}{p(t)} ds \\
&= \int_0^{x_1} |f(s)| \int_s^{x_1} \frac{dt}{p(t)} w(s)ds.
\end{aligned}
$$

For $i = 1, \cdots, n$, by lemma 5.4 we have

$$
\begin{aligned}
u(x) - u^I(x) &= \frac{\int_x^{x_{i+1}} \frac{ds}{p(s)}}{\int_{x_i}^{x_{i+1}} \frac{ds}{p(s)}} \int_{x_i}^x \int_{x_i}^s \frac{dt}{p(t)} f(s)w(s)ds \\
&\quad + \frac{\int_{x_i}^{x} \frac{ds}{p(s)}}{\int_{x_i}^{x_{i+1}} \frac{ds}{p(s)}} \int_x^{x_{i+1}} \int_s^{x_{i+1}} \frac{dt}{p(t)} f(s)w(s)ds \\
&\le \int_x^{x_{i+1}} \frac{ds}{p(s)} \int_{x_i}^x |f(s)|w(s)ds + \int_x^{x_{i+1}} \int_s^{x_{i+1}} \frac{dt}{p(t)} |f(s)|w(s)ds \\
&\le \int_{x_i}^x |f(s)|w(s) \int_s^{x_{i+1}} \frac{dt}{p(t)} ds + \int_x^{x_{i+1}} \int_s^{x_{i+1}} \frac{dt}{p(t)} |f(s)|w(s)ds \\
&= \int_{x_i}^{x_{i+1}} |f(s)| \int_s^{x_{i+1}} \frac{dt}{p(t)} w(s)ds.
\end{aligned}
$$

This completes the proof.

Corollary 5.1 *For $q = 0$ if $f \in L_w^2(0,1)$ then for any $x \in [x_i, x_{i+1}]$, $i = 0, \cdots, n$*

$$
u(x) - u^I(x) \le \|f\|_{L_w^2(0,1)} \left(\int_{x_i}^{x_{i+1}} \left(\int_s^{x_{i+1}} \frac{dt}{p(t)} \right)^2 w(s)ds \right)^{1/2}.
$$

Corollary 5.2 *For $q = 0$ if $f \in L^\infty_w(0,1)$ then for any $x \in [x_i, x_{i+1}]$, $i = 0, \cdots, n$*

$$u(x) - u^I(x) \leq \|f\|_{L^\infty_w(0,1)} \int_{x_i}^{x_{i+1}} \int_s^{x_{i+1}} \frac{dt}{p(t)} w(s)ds.$$

The uniform norm estmations of the interpolation error for the solution in the general case now follow.

Theorem 5.4 *For LC case with assumptions (4.5),(4.6) and (4.12) we have*

$$\|u - u^I\|_\infty \leq \|f - qu\|_{L^2_w(0,1)} \max_{0 \leq i \leq n} \left(\int_{x_i}^{x_{i+1}} \left(\int_s^{x_{i+1}} \frac{dt}{p(t)} \right)^2 w(s)ds \right)^{1/2}. \quad (5.21)$$

Theorem 5.5 *For $q \in L^\infty_w(0,1)$ such that (4.6) holds and $f \in L^\infty_w(0,1)$ (for both LC and LP1) we have the estimate*

$$\|u - u^I\|_\infty \leq \|f - qu\|_{L^\infty_w(0,1)} \max_{0 \leq i \leq n} \int_{x_i}^{x_{i+1}} \left(\int_s^{x_{i+1}} \frac{dt}{p(t)} \right) w(s)ds. \quad (5.22)$$

Remark 5.1 *$f - qu$ can be replaced by $(1 + qL^{-1})^{-1} f$ throughout this chapter.*

5.4 Galerkin error estimation in the V-norm and the uniform norm

We combine the results of the previous two sections to get the error in terms of the V-norm.

Theorem 5.6 *If $f \in L^2_w(0,1)$ and q satisfies (4.5),(4.6) and (4.12) (LC or LP1) then*

$$\|u - u^G\|_V \leq C \max_{0 \leq i \leq n} \left(\int_{x_i}^{x_{i+1}} \left(\int_s^{x_{i+1}} \frac{dt}{p(t)} \right) w(s)ds \right)^{1/2}$$

where

$$C = 2\left(1 + \frac{C_q}{\alpha}\right) \|f - qu\|_{L^2_w(0,1)}. \tag{5.23}$$

Proof: The result follows by combining (5.15) and (5.19).

Corollary 5.3 *Also in the LP1 case*

$$L^{-1} : H \to H$$

is compact.

Proof: Suppose $q = 0$ and $f \in L^2_w(0,1)$. Then by theorem 5.6 and theorem 4.1

$$\|u - u^G\|_H \leq \epsilon_n \|f\|_H.$$

Define

$$\langle T_n u, v \rangle_H = B(u, v) \quad \forall u, v \in V_n.$$

Then

$$\langle T_n u^G, v \rangle_H = \langle f, v \rangle_H \quad \forall v \in V_n$$

$$\Rightarrow \quad T_n u^G - P_n f = 0$$

$$\Rightarrow \quad u^G = T_n^{-1} P_n f$$

where P_n is the orthogonal projection operator w.r.t. the H-norm. Therefore,

$$\|L^{-1}f - T_n^{-1}P_nf\|_H \leq \epsilon_n\|f\|_H$$

$$\Rightarrow \quad \frac{\|L^{-1}f - T_n^{-1}P_nf\|_H}{\|f\|_H} \leq \epsilon_n$$

$$\Rightarrow \quad \|L^{-1} - T_n^{-1}P_n\| \leq \epsilon_n.$$

Hence L^{-1} is compact.

Remark 5.2 *With the assumptions of theorem 5.6, both for the LC and LP1 cases the operator* $(L+q)^{-1} : H \to H$ *is also compact. (For the LC case we have already seen it before.)*

Now we proceed to find estimates in the uniform norm. Unlike the regular case, the difficulty arises due to the fact that in the singular case the space $V \cap C[0,1]$ is not continuously embedded in $(V, \|.\|_V)$. Therefore, to get an uniform estimate we work directly with the system of equations obtained from the Galerkin method.

Lemma 5.6 *Let* $u_i = u(x_i)$ *and* α_i *be given by (5.5),* $i = 1, \cdots, n$, *then*

$$\max_{1\leq i\leq n} |\alpha_i - u_i| \leq 2\int_0^1 |q(s)(u(s) - u^G(s))| \int_s^1 \frac{dt}{p(t)} w(s)ds. \tag{5.24}$$

Proof: In lemma 5.3, taking $v_n = r_i$ for $i = 1, \cdots, n$, we obtain

$$\langle u^G - u^I, r_i\rangle_V = \langle q(u - u^G), r_i\rangle_H$$

$$\Rightarrow \sum_{j=1}^{n}(\alpha_j - u_j)\langle r_j, r_i\rangle_V = d_i.$$

This gives a system of equations

$$A\mathbf{e} = \mathbf{d}$$

where A the tridaigonal matrix given by (5.8)-(5.10), $\mathbf{e} = (e_i) = (\alpha_i - u_i)$ and $\mathbf{d} = (d_i)$ is given by

$$d_1 = \int_{x_0}^{x_1} g(s)w(s)ds + \frac{\int_{x_1}^{x_2} g(s)w(s)\int_s^{x_2}\frac{dt}{p(t)}ds}{\int_{x_1}^{x_2}\frac{dt}{p(t)}}$$

and

$$d_i = \frac{\int_{x_{i-1}}^{x_i} g(s)w(s)\int_{x_{i-1}}^{s}\frac{dt}{p(t)}ds}{\int_{x_{i-1}}^{x_i}\frac{dt}{p(t)}} + \frac{\int_{x_i}^{x_{i+1}} g(s)w(s)\int_s^{x_{i+1}}\frac{dt}{p(t)}ds}{\int_{x_i}^{x_{i+1}}\frac{dt}{p(t)}}$$

where $g(s)$ stands for $q(s)(u(s) - u^G(s))$. The inverse of the matrix A denoted as $B = (b_{ij})$ can be explicitely written as

$$b_{ij} = \begin{cases} \int_{x_j}^{1}\frac{ds}{p(s)} & \text{if } i \leq j \\ \\ \int_{x_i}^{1}\frac{ds}{p(s)} & \text{if } i \geq j \end{cases}. \tag{5.25}$$

Therefore,

$$\begin{aligned} |e_i| &\leq \sum_{j=1}^{n} b_{ij}|d_j| \\ &= \sum_{j=1}^{i}\int_{x_i}^{1}\frac{ds}{p(s)}|d_j| + \sum_{j=i+1}^{n}\int_{x_j}^{1}\frac{ds}{p(s)}|d_j| \\ &\leq \sum_{j=1}^{n}\int_{x_j}^{1}\frac{ds}{p(s)}|d_j|. \end{aligned} \tag{5.26}$$

We see that

$$\int_{x_1}^1 \frac{ds}{p(s)}|d_1| \leq \int_{x_1}^1 \frac{ds}{p(s)} \int_{x_0}^{x_1} |g(s)|w(s)ds + \int_{x_1}^1 \frac{ds}{p(s)} \frac{\int_{x_1}^{x_2} |g(s)|w(s) \int_s^{x_2} \frac{dt}{p(t)} ds}{\int_{x_1}^{x_2} \frac{dt}{p(t)}}$$

$$= \int_{x_1}^1 \frac{ds}{p(s)} \int_{x_0}^{x_1} |g(s)|w(s)ds + \int_{x_1}^{x_2} \frac{ds}{p(s)} \frac{\int_{x_1}^{x_2} |g(s)|w(s) \int_s^{x_2} \frac{dt}{p(t)} ds}{\int_{x_1}^{x_2} \frac{dt}{p(t)}}$$

$$+ \int_{x_2}^1 \frac{ds}{p(s)} \frac{\int_{x_1}^{x_2} |g(s)|w(s) \int_s^{x_2} \frac{dt}{p(t)} ds}{\int_{x_1}^{x_2} \frac{dt}{p(t)}}$$

$$\leq \int_{x_1}^1 \frac{ds}{p(s)} \int_{x_0}^{x_1} |g(s)|w(s)ds + \int_{x_1}^{x_2} |g(s)|w(s) \int_s^{x_2} \frac{dt}{p(t)} ds +$$

$$\int_{x_2}^1 \frac{ds}{p(s)} \int_{x_1}^{x_2} |g(s)|w(s)ds$$

$$= \int_{x_1}^1 \frac{ds}{p(s)} \int_{x_0}^{x_1} |g(s)|w(s)ds + \int_{x_1}^{x_2} |g(s)|w(s) \int_s^1 \frac{dt}{p(t)} ds$$

$$\leq \int_{x_0}^{x_1} |g(s)|w(s) \int_s^1 \frac{dt}{p(t)} ds + \int_{x_1}^{x_2} |g(s)|w(s) \int_s^1 \frac{dt}{p(t)} ds.$$

Also for $j = 2, \cdots, n$, by a similar approach, we have

$$\int_{x_j}^1 \frac{ds}{p(s)}|d_j| \leq \int_{x_j}^1 \frac{ds}{p(s)} \int_{x_{j-1}}^{x_j} |g(s)|w(s)ds +$$

$$\int_{x_j}^1 \frac{ds}{p(s)} \frac{\int_{x_i}^{x_{i+1}} |g(s)|w(s) \int_s^{x_{i+1}} \frac{dt}{p(t)} ds}{\int_{x_i}^{x_{i+1}} \frac{dt}{p(t)}}$$

$$\leq \int_{x_{j-1}}^{x_j} |g(s)|w(s) \int_s^1 \frac{dt}{p(t)} ds + \int_{x_j}^{x_{j+1}} |g(s)|w(s) \int_s^1 \frac{dt}{p(t)} ds.$$

Substituting these two inequalities in (5.26) we obtain

$$
\begin{aligned}
|e_i| &\leq \int_{x_0}^{x_n} |g(s)|w(s) \int_s^1 \frac{dt}{p(t)} ds + \int_{x_1}^{x_{n+1}} |g(s)|w(s) \int_s^1 \frac{dt}{p(t)} ds \\
&\leq 2\int_0^1 |g(s)|w(s) \int_s^1 \frac{dt}{p(t)} ds \\
&= 2\int_0^1 |q(s)(u(s) - u^G(s))|w(s) \int_s^1 \frac{dt}{p(t)} ds.
\end{aligned}
$$

This completes the proof.

Remark 5.3 *When $u \in AC[0,1]$ we have*

$$
\begin{aligned}
\max_{1\leq i\leq n} |\alpha_i - u_i| &\leq 2\|u - u^G\|_\infty \int_0^1 |q(s)|w(s) \int_s^1 \frac{dt}{p(t)} ds \\
&= 2C_q\|u - u^G\|_\infty.
\end{aligned}
$$

We now deduce an inequality which gives a relation with the V-norm of the Galerkin error.

Lemma 5.7

$$
\max_{1\leq i\leq n} |\alpha_i - u_i| \leq \left\{ \int_0^1 |q(s)| \left(\int_s^1 \frac{dt}{p(t)} \right)^{3/2} w(s)ds \right\} \|u - u^G\|_V. \qquad (5.27)
$$

Proof: Let $u - u^G = v$. Then by the lemma 5.6

$$
\max_{1\leq i\leq n} |\alpha_i - u_i| \leq 2\int_0^1 |q(s)| \left| \int_s^1 v'(t)dt \right| \int_s^1 \frac{dt}{p(t)} w(s)ds
$$

$$\begin{aligned}
&\le 2\int_0^1 |q(s)| \left(\int_s^1 p|v'|^2\right)^{1/2} \left(\int_s^1 \frac{dt}{p(t)}\right)^{1/2} \int_s^1 \frac{dt}{p(t)} w(s)ds \\
&\le 2\|v\|_V \int_0^1 |q(s)|w(s) \left(\int_s^1 \frac{dt}{p(t)}\right)^{3/2} ds \\
&= 2\|u-u^G\|_V \int_0^1 |q(s)| \left(\int_s^1 \frac{dt}{p(t)}\right)^{3/2} w(s)ds.
\end{aligned}$$

Remark 5.4 *For the general case (where q is not necessarily nonnegative), for the uniform norm estimation we will need*

$$C' = \int_0^1 |q(s)| \left(\int_s^1 \frac{dt}{p(t)}\right)^{3/2} w(s)ds < \infty. \tag{5.28}$$

We remark that for LC if $q \in L^4_w(0,1)$ is sufficient to satisfy this. For LP1 (with $q \in L^\infty_w(0,1)$), $\int_s^1 \frac{1}{p} \in L^{3/2}_w(0,1)$ is a sufficient condition to satisfy this.

Lemma 5.8

$$\|u^G - u^I\|_\infty \le 2 \max_{1\le i\le n} |\alpha_i - u_i|.$$

Proof: For $x \in [x_0, x_1]$,

$$\left|u^G - u^I\right| = |\alpha_1 - u_1|.$$

For $x \in [x_i, x_{i+1}]$, $i = 1, \cdots, n-1$, we have

$$\begin{aligned}
\left|u^G - u^I\right| &= \left|\alpha_i r_i^+ + \alpha_{i+1} r_{i+1}^- - u_i r_i^+ - u_{i+1} r_{i+1}^-\right| \\
&\le \left|(\alpha_i - u_i) r_i^+\right| + \left|(\alpha_{i+1} - u_{i+1}) r_{i+1}^-\right|
\end{aligned}$$

$$\leq \; |\alpha_i - u_i| + |\alpha_{i+1} - u_{i+1}|$$

$$\leq \; 2\max\{|\alpha_i - u_i|, |\alpha_{i+1} - u_{i+1}|\}.$$

For $x \in [x_n, x_{n+1}]$,

$$\left|u^G - u^I\right| = \left|\alpha_n r_n^+ - u_n r_n^+\right| \leq |\alpha_n - u_n|.$$

Therefore,

$$\|u^G - u^I\|_\infty \leq 2 \max_{1\leq i\leq n} |\alpha_i - u_i|.$$

We now state and prove the uniform convergence of the Galerkin approximation in the following theorem:

Theorem 5.7 *Let q satisfy (4.5),(4.6),(4.12) and (5.28) then for both LC and LP1 (for LP1 with $f, q \in L_w^\infty(0,1)$) the Galerkin approximation u^G converges uniformly to the solution u. If $f, q \in L_w^\infty(0,1)$ then the order of convergence is given by*

$$\max_{0\leq i\leq n} \left(\int_{x_i}^{x_{i+1}} \left(\int_s^{x_{i+1}} \frac{dt}{p(t)}\right) w(s)ds\right)^{1/2},$$

otherwise, (i.e.,for LC only) it is given by

$$\max\left\{\max_{0\leq i\leq n} \left(\int_{x_i}^{x_{i+1}} \left(\int_s^{x_{i+1}} \frac{dt}{p(t)}\right) w(s)ds\right)^{1/2}, \max_{0\leq i\leq n} \left(\int_{x_i}^{x_{i+1}} \left(\int_s^{x_{i+1}} \frac{dt}{p(t)}\right)^2 w(s)ds\right)^{1/2}\right\}.$$

Proof:

$$\|u - u^G\|_\infty \;\leq\; \|u - u^I\|_\infty + \|u^G - u^I\|_\infty$$

$$
\begin{aligned}
&\leq \|u - u^I\|_\infty + 2 \max_{1 \leq i \leq n} |\alpha_i - u_i| \quad [\textit{by lemma } 5.8] \\
&\leq \|u - u^I\|_\infty + 4C'\|u - u^G\|_V \quad [\textit{by lemma } 5.7] \\
&\leq \|u - u^I\|_\infty + 4CC' \max_{0 \leq i \leq n} \left(\int_{x_i}^{x_{i+1}} \left(\int_s^{x_{i+1}} \frac{dt}{p(t)} \right) w(s) ds \right)^{1/2} \\
&\qquad [\textit{by theorem } 5.6],
\end{aligned}
$$

where C and C' are given by (5.23) and (5.28) respectively. Therefore, the results follow from theorems 5.4, 5.5.

5.5 Higher order of convergence in the uniform norm for special data

In this section we show that if $C_q < \frac{1}{4}$ or $q \geq 0$ then for both LC and LP1 (with $q, f \in L_w^\infty(0,1)$) we have uniform convergence. In this case we do not need the restriction (5.28). Also we show that in particular cases we have higher order of convergence.

Lemma 5.9 *Suppose $C_q < \frac{1}{4}$ where $f \in L_w^2(0,1)$, q satisfies (4.12) for LC or $f, q \in L_w^\infty(0,1)$ for LP1. Then*

$$\|u - u^G\|_\infty \leq \frac{1}{1 - 4C_q} \|u - u^I\|_\infty.$$

Proof:

$$\|u - u^G\|_\infty \leq \|u - u^I\|_\infty + 2 \max_{1 \leq i \leq n} |u_i - \alpha_i|$$

$$\leq \; \|u - u^I\|_\infty + 4C_q\|u - u^G\|_\infty$$

by remark 5.3. The result thus follows.

Lemma 5.10 *Suppose $q \geq 0$. Suppose the solution $u \in AC[0,1]$ (i.e. q satisfies conditions (4.5); for LC q satisfies (4.12) and $f \in L^2_w(0,1)$; for LP1 $f, q \in L^\infty_w(0,1)$). Then*

$$\|u - u^G\|_\infty \leq (1 + 4C_q)\|u - u^I\|_\infty. \tag{5.29}$$

Proof: By lemma 5.2 we have for $i = 1, \cdots, n$

$$B(u^G - u^I, r_i) = \langle q(u - u^I), r_i\rangle_H$$

$$\Rightarrow \; \sum_{i=1}^{n} [\langle r_j, r_i\rangle_V + \langle q r_j, r_i\rangle_H](\alpha_j - u_j) = \langle q(u - u^I), r_i\rangle_H.$$

This gives the system

$$(A + Q)\mathbf{e} = \mathbf{d}$$

where A and Q are given by (5.8)-(5.10) and (5.11)-(5.13) respectively and $\mathbf{d} = (d_i) = (\langle q(u - u^I), r_i\rangle_H)$. Therefore,

$$\mathbf{e} = (A + Q)^{-1}\mathbf{d}.$$

Now $q_{ij} \geq 0$ and

$$q_{i,i+1} \;=\; \frac{\int_{x_i}^{x_{i+1}} q(s)\left(\int_{x_i}^{s} \frac{1}{p(t)}dt\right)\left(\int_{s}^{x_{i+1}} \frac{1}{p(t)}dt\right) w(s)ds}{\left(\int_{x_i}^{x_{i+1}} \frac{1}{p(s)}ds\right)^2}$$

$$\leq \frac{\int_{x_i}^{x_{i+1}} q(s)\left(\int_s^{x_{i+1}} \frac{1}{p(t)}dt\right) w(s)ds}{\int_{x_i}^{x_{i+1}} \frac{1}{p(s)}ds}$$

$$\leq \frac{1}{\int_{x_i}^{x_{i+1}} \frac{1}{p(s)}ds}$$

for sufficiently small interval $[x_i, x_{i+1}]$. Therefore, by theorem 2.12 $A+Q$ is an M-matrix and

$$(A+Q)^{-1} \leq A^{-1}.$$

Thus

$$\begin{aligned} |\mathbf{e}| &\leq (A+Q)^{-1}|\mathbf{d}| \\ &\leq A^{-1}|\mathbf{d}|. \end{aligned}$$

Here $|\mathbf{e}| = (|e_i|)$ and $|\mathbf{b}| = (|d_i|)$. By a similar approach to the proof of lemma 5.6 it can be shown that for any $i = 1, \cdots, n$,

$$|e_i| \leq 2\int_0^1 |q(s)(u(s) - u^I(s))| \int_s^1 \frac{dt}{p(t)} w(s)ds.$$

So,

$$\max_{1\leq i\leq n} |\alpha_i - u_i| \leq 2C_q \|u - u^I\|_\infty.$$

Therefore,

$$\begin{aligned} \|u - u^G\|_\infty &\leq \|u - u^I\|_\infty + \|u^G - u^I\|_\infty \\ &\leq \|u - u^I\|_\infty + 2\max_{1\leq i\leq n} |u_i - \alpha_i| [\textit{ by lemma } 5.8] \end{aligned}$$

$$
\begin{aligned}
&\leq \|u-u^I\|_\infty + 4C_q\|u-u^I\|_\infty \\
&= (1+4C_q)\|u-u^I\|_\infty .
\end{aligned}
$$

This completes the proof.

Now we state and prove the main theorems of this section.

Theorem 5.8 *For LC let $f \in L^2_w(0,1)$ and q satisfy (4.12). If either $C_q < \frac{1}{4}$ or $q \geq 0$ (satisfying (4.5)) then*

$$\|u-u^G\|_\infty \leq M \max_{0\leq i\leq n} \left(\int_{x_i}^{x_{i+1}} \left(\int_s^{x_{i+1}} \frac{dt}{p(t)} \right)^2 w(s)ds \right)^{1/2}$$

where

$$M = (1+4C_q)\|f-qu\|_{L^2_w(0,1)}.$$

Proof: The proof follows from theorem 5.4 and (5.29) of the above lemma.

Theorem 5.9 *If $f, q \in L^\infty_w(0,1)$ (for both LC and LP1) such that either $C_q < \frac{1}{4}$ or $q \geq 0$ then*

$$\|u-u^G\|_\infty \leq M' \max_{0\leq i\leq n} \int_{x_i}^{x_{i+1}} \left(\int_s^{x_{i+1}} \frac{dt}{p(t)} \right) w(s)ds$$

where

$$M' = (1+4C_q)\|f-qu\|_{L^\infty_w(0,1)}.$$

Proof: The proof follows from (5.29) of the above lemma and theorem 5.5.

We can apply this theorem to get higher order convergence for special cases. For example

Corollary 5.4 *Suppose in addition to the assumptions of the above theorem* $w = p$ *and* p *is monotone increasing function. Then* $O(h^2)$ *convergence is obtained.*

Proof:

$$
\begin{aligned}
& \max_{0 \le i \le n} \int_{x_i}^{x_{i+1}} \left(\int_s^{x_{i+1}} \frac{dt}{p(t)} \right) w(s) ds \\
\le \; & \max_{0 \le i \le n} \int_{x_i}^{x_{i+1}} \int_s^{x_{i+1}} dt ds \\
= \; & \max_{0 \le i \le n} \frac{(x_{i+1} - x_i)^2}{2} \\
= \; & \frac{h^2}{2}
\end{aligned}
$$

where $\max_{0 \le i \le n}(x_{i+1} - x_i) = h$.

Chapter 6

The Nonlinear Problem

Consider the nonlinear BVP, (1.11)-(1.13) written formally as:

$$Lu + f(x,u) = 0 \tag{6.1}$$

with both LC and LP1 cases. Here $f(x,u)$ is nonlinear, continuous in u such that for any real u, $f(.,u) \in H = L^2_w(0,1)$.

In section 6.1 sufficient conditions on $f(x,u)$ are made so that a variational boundary value problem (VBVP) on V can be defined. It is then shown that under these assumptions on $f(x,u)$ the VBVP has a unique solution in V.

In section 6.2 a regularity condition on $f(x,u)$ is made which insures that the weak solution is indeed the classical solution.

In section 6.3 some stronger assumptions on $f(x,u)$ are given which are useful for many problems in practice.

In section 6.4 a class of examples is treated. The reults are then applied to examples available in the literature.

Finally in section 6.5 the Galerkin method with the same basis of chapter 5 is applied for the approximation of the solution of the VBVP. The convergence results are found to be same as those of chapter 5.

6.1 The nonlinear variational boundary value problem

Let V be the Hilbert space defined in section 4.1. Parallel to the assumptions on q in section 4.2 we make the following assumptions on $f(x, u)$. We define for $x \in (0,1]$ and $u, v \in V,\ u \neq v$

$$q(u, v, x) := \frac{f(x, u(x)) - f(x, v(x))}{u(x) - v(x)}, \quad u(x) \neq v(x); (:= 0 \quad if\ u(x) = v(x)).$$

Assumptions on f(x,u):

Let $f(x, u)$ satisfy the following three assumptions:

(A) $\quad f(x, u_0(x)) \in H$ for some $u_0 \in V$.

(B) $\quad \int_0^1 |q(u, v, x)| \int_x^1 \frac{ds}{p(s)} w(x)dx \leq \hat{C} < \infty \quad \forall\, u, v \in V.$

(C) $\quad \frac{\int_0^1 q(u,v,x)\hat{u}^2(x)w(x)dx}{||\hat{u}||_V^2} \geq \gamma > -1 \quad \forall\, u, v, \hat{u} \in V,\ \hat{u} \neq 0.$

Lemma 6.1 *Under the assumption (A) and (B), for any $u, v \in V$ the integral $\int_0^1 f(x, u)v(x)w(x)dx$ exists and for any fixed $u \in V$ the mapping $v \mapsto \int_0^1 f(x, u)v(x)w(x)dx$ is continuous.*

Proof:

$$
\begin{aligned}
& \left| \int_0^1 f(x,u)v(x)w(x)dx \right| \\
\leq & \left| \int_0^1 \frac{f(x,u(x)) - f(x,u_0(x))}{u(x)-u_0(x)}(u(x)-u_0(x))v(x)w(x)dx \right| + \\
& \left| \int_0^1 f(x,u_0(x))v(x)w(x)dx \right| \\
\leq & \left(\int_0^1 |q(u,u_0,x)| \int_x^1 \frac{ds}{p(s)} w(x)dx \right) \|u-u_0\|_V \|v\|_V \\
& + \|f(x,u_0)\|_H \|v\|_H \quad \text{(by lemma 4.2 and assumption } (A)) \\
\leq & \hat{C}\|u-u_0\|_V\|v\|_V + \sqrt{C}\|f(x,u_0)\|_H\|v\|_V \\
& \qquad \text{(by assumption } (B) \text{ and theorem 4.1).}
\end{aligned}
$$

This completes the proof.

Now we define the variational boundary value problem (VBVP) in the following way:

Given f satisfying (A) and (B) find $u \in V$ such that

$$a(u,v) := \langle u,v\rangle_V + \int_0^1 f(x,u(x))v(x)w(x)dx = 0 \qquad (6.2)$$

for all $v \in V$.

We have the following existence and uniqueness theorem:

Theorem 6.1 *Suppose $f(x,u)$ satisfies assumptions $(A)-(C)$. Then the VBVP (6.2) has a unique solution in V.*

Proof: By lemma 6.1, for any fixed $u \in V$ the linear functional

$$v \mapsto \int_0^1 f(x,u)vw - \gamma\langle u,v\rangle_V$$

is continuous (in the topology of V). Therefore, we can define an operator

$$B : V \to V$$

by

$$\langle Bu, v\rangle_V = \int_0^1 f(x,u)vw - \gamma\langle u,v\rangle_V \quad \forall\, v \in V.$$

Let $\alpha = 1+\gamma$. Then

$$\alpha I + B : V \to V$$

is given by

$$\langle(\alpha I + B)u, v\rangle_V = \langle u,v\rangle_V + \int_0^1 f(x,u)vw = a(u,v) \quad \forall\, v \in V$$

and the VBVP (6.2) is equivalent to the equation

$$(\alpha I + B)u = 0. \tag{6.3}$$

Therefore it is enough to show that (6.3) has a unique solution in V. Clearly $D(B) = V$.

B is monotone:

For any $u, v \in V$, $u \neq v$ (for $u = v$ it is trivial)

$$
\begin{aligned}
& \langle Bu - Bv, u - v \rangle_V \\
= & \int_0^1 (f(x,u) - f(x,v))(u-v)w - \gamma \|u - v\|_V^2 \\
= & \int_0^1 q(u,v,x)(u-v)^2 w - \gamma \|u - v\|_V^2 \\
\geq & \; 0 \quad \text{(by assumption } (C)).
\end{aligned}
$$

B is hemicontinuous:

Let $u, \tilde{u}, v \in V$. Then

$$
\begin{aligned}
& \lim_{t \to 0} \langle B(u + t\tilde{u}) - Bu, v \rangle_V \\
= & \lim_{t \to 0} \int_0^1 (f(x, u + t\tilde{u}) - f(x,u))\, vw - \lim_{t \to 0} \gamma \langle t\tilde{u}, v \rangle_V \\
= & \lim_{t \to 0} \int_0^1 q(u + t\tilde{u}, u, x) t\tilde{u}(x) v(x) w(x) dx \\
= & \lim_{t \to 0} t \int_0^1 q(u + t\tilde{u}, u, x) \tilde{u}(x) v(x) w(x) dx \\
= & \; 0
\end{aligned}
$$

because

$$
\begin{aligned}
& \left| \int_0^1 q(u + t\tilde{u}, v, x) \tilde{u}(x) v(x) w(x) dx \right| \\
\leq & \left(\int_0^1 |q(u + t\tilde{u}, v, x)| \int_x^1 \frac{ds}{p(s)} w(x) dx \right) \|\tilde{u}\|_V \|v\|_V
\end{aligned}
$$

$$\leq \hat{C}\|\tilde{u}\|_V\|v\|_V.$$

Hence by corollary 2.2 the operator

$$\alpha I + B : V \to V$$

is bijective and therefore, (6.3) has a unique solution. This completes the proof.

6.2 Regularity of the classical solution

In this section we derive sufficient conditions on $f(x,u)$ so that the BVP (6.1) has a unique solution. We show this by proving that, under a stronger assumption than (A), the solution of the VBVP is also a solution of the BVP. In other words, the weak solution is same as the strong solution of the BVP. We replace (A) by a stronger one:

$(\hat{A})$ $\quad u \in V \Rightarrow f(x,u) \in H.$

The results of this section are parallel to those of section 4.3.

Let

$$X := \{\ u \in V\ :\ v \mapsto a(u,v) \text{ is continuous on } V \text{ in the topology of } H\ \}.$$

Since V is dense in H, for each $u \in X$ the linear correspondense $v \mapsto a(u,v)$ can be extended to a continuous linear functional $G(v)$ on H. Therefore, there exists a unique element, say, Su in H such that

$$G(v) = a(u,v) = \langle Su, v\rangle_H \quad \forall\, v \in V.$$

This gives rise to an operator (nonlinear)

$$S \;:\; V \subset H \to H$$

such that $D(S) = X$.

Lemma 6.2 *Let* $(\hat{A})$ *is satisfied. Then*

$$D(L) \subset D(S)$$

and for any $u \in D(L)$

$$Lu + f(x,u) = Su.$$

Proof: Fix $u \in D(L)$. Then for any $v \in V$ we have

$$\begin{aligned}
a(u,v) &= \int_0^1 p(x)u'(x)v'(x)dx + \int_0^1 f(x,u(x))v(x)w(x)dx \\
&= p(x)u'(x)v(x)|_0^1 - \int_0^1 (p(x)u'(x))'v(x)dx + \\
&\quad \int_0^1 f(x,u(x))v(x)w(x)dx \\
&= -\int_0^1 (p(x)u'(x))'v(x)dx + \int_0^1 f(x,u(x))v(x)w(x)dx \\
&\qquad\qquad \text{(by lemma 4.1)}
\end{aligned}$$

$$= \langle Lu + f(x,u), v\rangle_H$$

$$= \langle g, v\rangle_H$$

where $Lu + f(x,u) \doteq g \in H$.

Therefore,

$$|a(u,v)| \leq \|g\|_H \|v\|_H$$

and the mapping $v \mapsto a(u,v)$ is continuous in the topology of H. Thus $D(L) \subset D(S)$.

Also for any $u \in D(L)$

$$\langle Su, v\rangle_H = a(u,v) = \langle Lu + f(x,u), v\rangle_H \quad \forall\, v \in V.$$

Since V is dense in H, then

$$Su = Lu + f(x,u).$$

This completes the proof.

Lemma 6.3 *With the same assumption $(\hat{A})$ on $f(x,u)$ we have*

$$D(S) \subset D(L).$$

Proof: Let $u \in D(S)$. Then $v \mapsto a(u,v)$ is a linear continuous functional on V in the toplology of H. Also $v \mapsto \langle f(x,u), v\rangle_H$ is a continuous linear functional on V in the topology of H. Therefore, $v \mapsto a(u,v) - \langle f(x,u), v\rangle_H = \langle u, v\rangle_V$ is a continuous linear functional on V in the topology of H. In particular it is a linear continuous functional on $D(L)$ in the topology of H.

But for $v \in D(L)$,

$$\langle u, v\rangle_V = \langle u, Lv\rangle_H = \langle Lv, u\rangle_H.$$

Thus $v \mapsto \langle Lv, u\rangle_H$ is continuous on $D(L)$. This implies that $u \in D(L^*)$ (see definition 2.1). Since $L^* = L$, we have $u \in D(L)$. This completes the proof.

Corollary 6.1 *Under the assumption* $(\hat{A})$, *for all* $u \in D(S) = D(L)$

$$Su = Lu + f(x, u).$$

Theorem 6.2 *The following two statements are equivalent:*

(i) $\quad u \in D(L)$ *and* $Lu + f(x, u) = 0.$

(ii) $\quad u \in V$ *and* $a(u, v) = 0 \;\; \forall\, v \in V.$

Proof: $(i) \Rightarrow (ii)$ is clear.

$(ii) \Rightarrow (i)$:

Let $u \in V$ such that $a(u, v) = 0$. Then clearly $v \mapsto a(u, v) = 0$ is continuous in the topology of H. Hence $u \in D(S) = D(L)$. Thus

$$0 = a(u, v) = \langle Su, v\rangle_H = \langle Lu + f(x, u), v\rangle_H \;\; \forall\, v \in V.$$

This implies that $Lu + f(x, u) = 0$ (since V is dense in H). This completes the proof.

Corollary 6.2 *Suppose $f(x,u)$ satisfies $(\hat{A})$, (B) and (C). Then the BVP (6.1) has a unique solution. Furthermore, this solution is also the unique solution of the VBVP (6.2).*

6.3 About the assumptions on $f(x,u)$

In this section we give some assumption on $f(x,u)$ which are stronger than (B) and (C). These assumptions are sufficient for many problems in practice. Let

$(\hat{B})$ $\quad \left|\frac{f(x,u)-f(x,v)}{u-v}\right| \le \hat{q}(x)$

for $-\infty < u,v < \infty$, $u \neq v$, where $\hat{q}(x)$ (independent of u,v) satisfies $\int_0^1 \hat{q}(x) \int_x^1 \frac{ds}{p(s)} w(x)dx := \hat{C} < \infty$.

We notice that if $f(x,u)$ is Lipchitz continuous in u then $(\hat{B})$ is clearly satisfied. Also if $f(x,u)$ is differentiable w.r.t. u then $\left|\frac{\partial f}{\partial u}(x,u)\right| \le \hat{q}(x)$ for any $u \in \mathbf{R}$ implies $(\hat{B})$.

Remark 6.1

$$(\hat{B}) \Rightarrow (B).$$

Proof: For any $u,v \in V$, $u \neq v$

$$\int_0^1 |q(u,v,x)| \int_x^1 \frac{ds}{p(s)} w(x)dx$$

$$= \int_0^1 \left|\frac{f(x,u(x)) - f(x,v(x))}{u(x)-v(x)}\right| \int_x^1 \frac{ds}{p(s)} w(x)dx$$

$$\leq \int_0^1 |\hat{q}(x)| \int_x^1 \frac{ds}{p(s)} w(x)dx$$

$$= \hat{C} < \infty.$$

This completes the proof.

Let us recall the number Λ given by (4.2) as:

$$\Lambda := \inf_{u \in V} \frac{||u||_V^2}{||u||_H^2}$$

and λ_1 given by (4.3) as:

$$\lambda_1 := \inf_{u \in D(L)} \frac{\langle Lu, u \rangle_H}{||u||_H^2}$$

and the relation (4.4):

$$0 < \frac{1}{\int_0^1 \left(\int_s^1 \frac{dt}{p(t)} \right) w(s)ds} \leq \Lambda \leq \lambda_1.$$

Let

$$(\hat{C}) \qquad \frac{\int_0^1 q(u,v,x)\hat{u}^2(x)w(x)dx}{||\hat{u}||_H^2} \geq \hat{\gamma} > -\Lambda \quad \forall\, u, v, \hat{u} \in V,\ u \neq v,\ \hat{u} \neq 0.$$

Remark 6.2 $(\hat{C}) \Rightarrow (C)$.

Proof: The proof is similar to the proof of remark 4.8.

$$\frac{\int_0^1 q(u,v,x)\hat{u}^2(x)w(x)dx}{||\hat{u}||_V^2}$$

$$\geq \hat{\gamma}.\frac{||\hat{u}||_H^2}{||\hat{u}||_V^2}$$

$$
\begin{aligned}
&\geq \min(0,\hat{\gamma}).\frac{\|\hat{u}\|_H^2}{\|\hat{u}\|_V^2} \\
&\geq \min(0,\hat{\gamma}).\sup_{u\in V}\frac{\|\hat{u}\|_H^2}{\|\hat{u}\|_V^2} \\
&= \min(0,\hat{\gamma}).\frac{1}{\inf_{u\in V}\frac{\|\hat{u}\|_V^2}{\|\hat{u}\|_H^2}} \\
&= \min(0,\hat{\gamma}).\frac{1}{\Lambda} := \gamma \\
&> \min(0,-\Lambda).\frac{1}{\Lambda} \\
&= -1.
\end{aligned}
$$

This completes the proof.

Let

$(\hat{\hat{C}})$ $\quad \frac{f(x,u)-f(x,v)}{u-v} \geq \hat{\gamma} \geq -\Lambda$ for $-\infty < u, v < \infty,\ u \neq v$.

Remark 6.3 $(\hat{\hat{C}}) \Rightarrow (\hat{C})$.

Proof:

$$
\begin{aligned}
&\frac{\int_0^1 q(u,v,x)\hat{u}^2(x)w(x)dx}{\|\hat{u}\|_H^2} \\
=\ &\frac{\int_0^1 \frac{f(x,u(x))-f(x,v(x))}{u(x)-v(x)}\hat{u}^2(x)w(x)dx}{\|\hat{u}\|_H^2}
\end{aligned}
$$

$$\geq \frac{\int_0^1 \hat{\gamma}\hat{u}^2(x)w(x)dx}{\|\hat{u}\|_H^2}$$

$$= \hat{\gamma} > -\Lambda.$$

This completes the proof.

Remark 6.4 *If $D(L)$ is dense in V then $\Lambda = \lambda_1$. Therefore, in such a case Λ of $(\hat{C})$ and $(\hat{\hat{C}})$ can be replaced by λ_1.*

Remark 6.5 *Suppose u is a solution of the VBVP (6.2). Also suppose that $f(x,u)$ satisfies $(\hat{\hat{C}})$. Then u is the unique minimizer of the functional*

$$J(u) = \frac{1}{2}\langle u, u\rangle_V + \int_0^1 \int_0^{u(x)} f(x,s)dsw(x)dx.$$

(Such kind of functionals for nonlinear boundary value problems have been considered by Levinson [25], Mikhlin [28] and Ciarlet [9].)

Proof: Suppose u is a solution of the VBVP. Then for any $v \in V$

$$\begin{aligned} J(v) - J(u) &= \frac{1}{2}\langle v, v\rangle_V - \frac{1}{2}\langle u, u\rangle_V + \int_0^1 \int_{u(x)}^{v(x)} f(x,s)dsw(x)dx \\ &= \frac{1}{2}\langle v-u, v-u\rangle_V + \langle v-u, u\rangle_V + \int_0^1 \int_{u(x)}^{v(x)} f(x,s)dsw(x)dx \\ &= \frac{1}{2}\|u-v\|_V^2 + \left\{\langle u, v-u\rangle_V + \int_0^1 f(x,u(x))(v(x)-u(x))w(x)dx\right\} \\ &\quad + \int_0^1 \int_{u(x)}^{v(x)} f(x,s)dsw(x)dx - \int_0^1 f(x,u(x)) \int_{u(x)}^{v(x)} dsw(x)dx \end{aligned}$$

$$= \frac{1}{2}\|u-v\|_V^2 + \int_0^1 \int_{u(x)}^{v(x)} [f(x,s) - f(x,u(x))] ds w(x) dx$$

(since u is a solution of VBVP)

$$\geq \frac{1}{2}\|u-v\|_V^2 + \hat{\gamma} \int_0^1 \int_{u(x)}^{v(x)} (s - u(x)) ds w(x) dx$$

(by $(\hat{\hat{C}})$)

$$= \frac{1}{2}\|u-v\|_V^2 + \frac{\hat{\gamma}}{2} \int_0^1 (v(x) - u(x))^2 w(x) dx$$

$$\geq \frac{1}{2}\left[\Lambda\|u-v\|_H^2 + \hat{\gamma}\|u-v\|_H^2\right]$$

$$= \frac{1}{2}(\hat{\gamma} + \Lambda)\|u-v\|_H^2$$

$$> 0 \quad \text{unless } u = v.$$

This completes the proof.

6.4 A class of examples

Theorem 6.3 *Suppose*

$$f(x,u) = q(x)h(u) + g(x)$$

where $g \in L_w^2(0,1)$ *and* $q \in L_w^4(0,1)$ *for the LC case,* $q \in L_w^\infty(0,1)$ *for the LP1 case,* $h(u)$ *satisfies:*

$$0 \leq \frac{h(u) - h(v)}{u - v} \leq M, \quad -\infty < u, v < \infty, \quad u \neq v, \tag{6.4}$$

and

$$\text{ess inf } q(x) > -\frac{\Lambda}{M}, \tag{6.5}$$

then the BVP (6.1) has a unique solution in $D(L)$.

Proof: To show that $(\hat{A})$ is satisfied, let $u \in V$, then substituting $u(x)$ for u and 0 for v in (6.4) we obtain

$$\begin{aligned} |h(u(x))| &\leq |h(u(x) - h(0)| + |h(0)| \\ &\leq M|u(x)| + |h(0)|. \end{aligned}$$

Therefore,

$$|f(x,u)| \leq |q(x)|\,(M|u(x)| + |h(0)|) + |g(x)| \in H$$

by remark 4.4 and remark 4.10.

To show that $(\hat{B})$ is satisfied, note that for any $u, v \in \mathbf{R}$ $u \neq v$,

$$\begin{aligned} \left|\frac{f(x,u) - f(x,v)}{u - v}\right| &= \left|\frac{q(x)(h(u) - h(v))}{u - v}\right| \\ &\leq M|q(x)|, \end{aligned}$$

where $q(x)$ clearly (for both LC and LP1) satisfies:

$$\int_0^1 |q(x)| \int_x^1 \frac{ds}{p(s)} w(x)dx < \infty.$$

To show that $(\hat{C})$ is satisfied, let u, v $(u \neq v)$ be any real numbers. We have

$$\begin{aligned}
\frac{f(x,u)-f(x,v)}{u-v} &= q(x).\frac{h(u)-h(v)}{u-v} \\
&\geq \text{ess inf } q(x).\frac{h(u)-h(v)}{u-v} \\
&\geq \min(0, \text{ess inf } q(x)).\frac{h(u)-h(v)}{u-v} \\
&\geq \min(0, \text{ess inf } q(x)).M \\
&\geq \min(0, -\frac{\Lambda}{M}).M \\
&= -\Lambda.
\end{aligned}$$

Hence the BVP has a unique solution in V. But $f(x,u) \in H$, so $u \in D(L)$. This completes the proof.

Example 6.1 *Let $q \in L^{\infty}_{w}(0,1)$, $g \in L^{2}_{w}(0,1)$ so that both LC and LP1 cases can be applied. Let $h(u)$ be given by*

$$h(u) = \begin{cases} e^{u} & \text{if } u \leq 0 \\ 1 & \text{if } u > 0 \end{cases} . \tag{6.6}$$

Clearly (6.4) is satisfied with $M = 1$. Therefore, if q satisfies

$$\text{ess inf } q > -\Lambda$$

then there exists a unique solution in $D(L)$.

(a) In particular if we take $p(x) = w(x) = x$ (LC), $q(x) = 1$, and $g(x) = 0$ then the BVP (6.1) becomes

$$\begin{aligned} -\frac{1}{x}(xu')' + h(u) &= 0, \quad 0 < x < 1, \\ \lim_{x \to 0^+} xu'(x) &= 0, \\ u(1) &= 0 \end{aligned}$$

where $h(u)$ is given by (6.6). The solution u exists and is unique and $u \in D(L)$. In this case case (since $p = w = x$) the first boundary condition can be replaced by

$$\lim_{x \to 0^+} u'(x) = 0 \quad i.e., \quad u'(0) = 0$$

In this example it is seen that

$$Lu = -h(u) < 0$$

and so $u(x) < 0$ for all $x \in [0, 1)$. Thus in any iteration process $h(u)$ is given by the first part i.e., e^u only. So it is immaterial if the second part is replaced by any other continuous function e.g., by e^u itself. The problem then becomes:

$$\begin{aligned} -\frac{1}{x}(xu')' + e^u &= 0, \quad 0 < x < 1 \\ u'(0) = u(1) &= 0. \end{aligned}$$

This problem was numerically solved by Russel and Shampine [36] and Chawla et al. [6]. The exact solution is known:

$$u(x) = 2ln\left(\frac{1+\beta}{1+\beta x^2}\right)$$

where $\beta = -5 + 2\sqrt{6}$.

(b) Let $p(x) = w(x) = x$ (LC). But consider $q(x) = 3cos(\pi x)$ and $g(x)$ any continuous function. Then

$$\inf q = -3 > -\Lambda = -\lambda_1 = -5.781$$

($\lambda_1 = 5.781$, see section 3.3). Therefore, there exists a unique solution $u \in AC[0,1]$. The BVP becomes

$$\begin{aligned} -\frac{1}{x}(xu')' &= 3cos(\pi x)h(u) - g(x), \quad 0 < x < 1 \\ u'(0) = u(1) &= 0. \end{aligned}$$

The analytic solution is not known.

Example 6.2 *Consider the problem*

$$u''(x) + \frac{2}{x}u'(x) + [u(x)]^5 = 0, \quad 0 < x < 1,$$

$$u'(0) = 0, \quad u(1) = \frac{\sqrt{3}}{2}$$

This problem was numerically treated by Russel and Shampine [36], and Chawla et. al. [6]). The solution is known:

$$u(x) = \frac{1}{\sqrt{1 + x^2/3}}.$$

We want to justify our theorem with this example.

If we trasform this problem to our setting it becomes:

$$-\frac{1}{x^2}(x^2u')' - \left(u + \frac{\sqrt{3}}{2}\right)^5 = 0$$

$$u'(0) = u(1) = 0$$

with the analytical solution given by

$$u(x) = \frac{1}{\sqrt{1 + x^2/3}} - \frac{\sqrt{3}}{2}. \tag{6.7}$$

Here $q(x) = -1$ and $h(u) = (u + \sqrt{3}/2)^5$. Therefore, conditions (6.4) and (6.5) are not satisfied and the existence and uniqueness of the solution is not garunteed. In this case $\lambda_1 = 9.865$ (see section 3.3). So if we replace $h(u) = (u + \sqrt{3}/2)^5$ by

$$h(u) = \begin{cases} (a + \sqrt{3}/2)^5 & \text{if } u \leq a, \\ (u + \sqrt{3}/2)^5 & \text{if } a \leq u \leq b, \\ (b + \sqrt{3}/2)^5 & \text{if } u \geq b, \end{cases} \tag{6.8}$$

where a and b are such that (6.4) and (6.5) are satisfied i.e..

$$-1 > -\frac{9.865}{M} \quad \Rightarrow M < 9.865$$

and

$$h'(u) < M, \quad a < u < b.$$

Choose $M = 9.8$, say and solve :

$$5(u + \sqrt{3}/2)^4 < 9.8$$

which gives:

$$-2.049 < u < .317$$

Choose $a = -2$ and $b = .3$; then by theorem (6.3) this equation has a unique solution. It is observed that the analytical solution (6.7) satisfies:

$$0 \leq u(x) \leq 1 - \frac{\sqrt{3}}{2} < .134, \quad x \in [0,1].$$

But

$$[0, .134] \subset [a, b] = [-2, .3].$$

So it is also the unique solution of our problem with $h(u)$ given by (6.8).

Example 6.3 *Consider the example*

$$u'' + \frac{n-1}{x}u' + q(x)u^{-\alpha} = 0$$

$$u'(0) = u(1) = 0$$

where $\alpha > 0$. This example was treated by Fink et. al. [16] where he proved the existence and uniqueness of the positive solution for $q : [0,1] \rightarrow [0,\infty)$, continuous and $q(x) > 0, \quad x \in [0,1)$. They first considered the problem

$$u'' + \frac{n-1}{x}u' + q(x)(u+\epsilon)^{-\alpha} = 0 \tag{6.9}$$

$$u'(0) = u(1) = 0 \tag{6.10}$$

and proved the existence and uniqeness of the positive solution of this problem. The existence and uniqueness of the positive solution of the original *problem was shown as the uniform limit of the solutions u_ϵ of this problem as $\epsilon \to 0^+$.*

We want to use our results to prove the existence and uniqueness of the regularized problem (6.9). In our setting this problem becomes

$$Lu + q(x)h_\epsilon(u) = 0$$

$$u'(0) = u(1) = 0$$

where $h_\epsilon(u)$ is given by

$$h_\epsilon(u) = \begin{cases} -\frac{1}{(u+\epsilon)^\alpha} & \text{if } u \geq 0, \\ -\frac{1}{(\epsilon)^\alpha} & \text{otherwise.} \end{cases} \tag{6.11}$$

Here, since $q(x) > 0$, condition (6.5) is clearly satisfied. Also for any real u, v

$$0 \leq \frac{h(u) - h(v)}{u - v} \leq \frac{\alpha}{\epsilon^{\alpha+1}}$$

i.e., condition (6.4) is also satisfied. Hence by theorem (6.3) there exists a unique solution. Since the solution is nonngative the second part of $h_\epsilon(u)$ in (6.4) is immaterial (if the initial guess of the solution in the iteration process is taken to be nonnegative).

6.5 Galerkin approximation

Consider the VBVP (6.2)

$$a(u, v) = 0, \quad \forall\, v \in V \tag{6.12}$$

with $f(x,u)$ satisfying conditions $(\hat{A})$, (B) and (C). Let u be the unique solution of (6.12). We consider the same approximation subspace V_n of chapter 5. Since $V_n \subset V$ all the assumptions are also sayisfied in V_n and the finite dimensional problem (Galerkin):

$$a(u_n, v_n) = 0 \quad \forall\, v_n \in V_n \tag{6.13}$$

has also a unique solution $u_n = u^G \in V_n$. This unique solution u^G is the Galerkin approximation of the solution u in the subspace V_n.

Since $V_n \subset V$ equation (6.12) is also satisfied for all $v_n \in V_n$. Thus we have

$$a(u, v_n) = 0 \quad \forall\, v_n \in V_n$$

and we also have from (6.13)

$$a(u^G, v_n) = 0 \quad \forall\, v_n \in V_n. \tag{6.14}$$

Subtracting these two equations we obtain, for all $v_n \in V_n$

$$a(u - u^G, v_n) = 0$$

$$\Rightarrow \quad \langle u - u^G, v_n\rangle_V + \int_0^1 \frac{f(x,u) - f(x,u^G)}{u - u^G}(u - u^G)v_n w = 0. \tag{6.15}$$

Let $\tilde{q}(x)$ be the unique function (because u and u^G are unique) defined by:

$$\tilde{q}(x) := \frac{f(x,u(x)) - f(x,u^G(x))}{u(x) - u^G(x)},$$

(and $\tilde{q}(x) := 0$ when $u(x) = u^G(x)$).

Then

$$\langle u, v\rangle_{\tilde{q}} = \langle u, v\rangle_V + \int_0^1 \tilde{q}(x)u(x)v(x)w(x)dx$$

defines an inner product in V and by virtue of (6.15) the Galerkin approximation u^G can be thought of as the orthogonal projection of u on V_n with respect to this inner product.

Using

$$u^G = \sum_{j=1}^{n} \alpha_j r_j$$

and $v_n = r_i,\ i = 1, \cdots, n$ in (6.14) we obtain the system

$$\sum_{j=1}^{n} \alpha_j \langle r_j, r_i\rangle_V + \int_0^1 f(x, \sum_{j=1}^{n} \alpha_j r_j(x)) r_i(x) w(x) dx.$$

This gives

$$A\alpha + G\alpha = 0 \tag{6.16}$$

where the matrix A is given by (5.8)-(5.10) and $G\alpha$ is a nonlinear system given by

$$g_1(\alpha) = \int_{x_0}^{x_1} f(x, \alpha_1 r_1^-) r_1^- w + \int_{x_1}^{x_2} f(x, \alpha_1 r_1^+ + \alpha_2 r_2^-) r_1^+ w, \tag{6.17}$$

$$\begin{aligned} g_i(\alpha) &= \int_{x_{i-1}}^{x_i} f(x, \alpha_{i-1} r_{i-1}^+ + \alpha_i r_i^-) r_i^- w + \\ &\quad \int_{x_i}^{x_{i+1}} f(x, \alpha_i r_i^+ + \alpha_{i+1} r_{i+1}^-) r_i^+ w, \quad i = 2, \cdots, n-1, \end{aligned} \tag{6.18}$$

$$g_n(\alpha) = \int_{x_{n-1}}^{x_n} f(x, \alpha_{n-1} r_{n-1}^+ + \alpha_n r_n^-) r_n^- w + \int_{x_n}^{x_{n+1}} f(x, \alpha_n r_n^+) r_n^+ w \tag{6.19}$$

Since (6.14) has a unique solution u^G (determined uniquely by α_j's), then, (6.16) has also a unique solution $\alpha \in R^n$. This can also be shown independently by showing that $A + G$ is monotone and hemicontinuous on R^n. Then by remark 2.8 and by uniform monotonicity theorem 2.13 it follows that equation (6.16) has a unique solution in R^n.

Convergence:

All the convergence results of chapter 5 hold true for this nonlinear case also. In the proofs we only need to replace q by $\tilde{q}$, C_q by $\hat{C}$, $B(u,v)$ by $\langle u, v\rangle_{\tilde{q}}$ and $qu - f$ by $f(x, u)$.

Remark 6.6 *Note that the class of nonlinear problems and the nonlinear analysis methods chosen in this chapter directly extends our work on the linear case. More general classes of nonlinear singular problems is open for investigation.*

Chapter 7

Numerical Examples

In this chapter we give numerical examples for both linear and nonlinear cases. The programs were written in FORTRAN 77. For solving the nonlinear system (6.14) Picard's iteration

$$(A + \gamma I)\alpha^{k+1} = \gamma\alpha^k - G\alpha^k$$

and the secant method

$$\alpha^{k+1} = \alpha^k - (A + J_k)^{-1}(A\alpha^k + G\alpha^k)$$

were used where $\gamma = \hat{C}/2$ ($\hat{C}$ is the constant of assumption (B) of chapter 6) and J_k is given by the following. Let $\delta = .00001$. Define

$$\hat{f}_{k,i}(x) = \begin{cases} \frac{f(x,u_i^k) - f(x,u_{i-1}^k)}{u_i^k - u_{i-1}^k} & \text{if } |u_i^k - u_{i-1}^k| \geq \delta, \\ \\ \frac{f(x,u_i^k) - f(x,u_i^k - \delta)}{\delta} & \text{if } |u_i^k - u_{i-1}^k| < \delta. \end{cases} \qquad (7.1)$$

(If $\hat{f}_{k,i} = 0.0$ then it is replaced by γ.) Then the entries of the symmetric tridiagonal matrix J_k are given by

$$
\begin{aligned}
[J_k]_{ii} &= \int_{x_{i-1}}^{x_i} \hat{f}_{k,i}(x)(r_i^-)^2 w + \int_{x_i}^{x_{i+1}} \hat{f}_{k,i+1}(x)(r_i^+)^2 w \\
[J_k]_{i,i+1} &= \int_{x_i}^{x_{i+1}} \hat{f}_{k,i+1}(x) r_{i+1}^- r_i^+ w.
\end{aligned}
$$

The results are compared with the analytical solution and the error is shown in the uniform norm. We note that in the following examples, the data $f(x)$ was not always continuous or even bounded. Also the condition $u'(0) = 0$ was not always satisfied. Such cases were not considered in the literature. Some of the examples are limit point and other are limit circle cases as can be checked. Example 5 below considers a case where $w(x) = 0$ on an interval of positive measure.

Example 1

$$p(x) = 1 - e^{-x}$$

$$w(x) = 1.0$$

$$q(x) = x$$

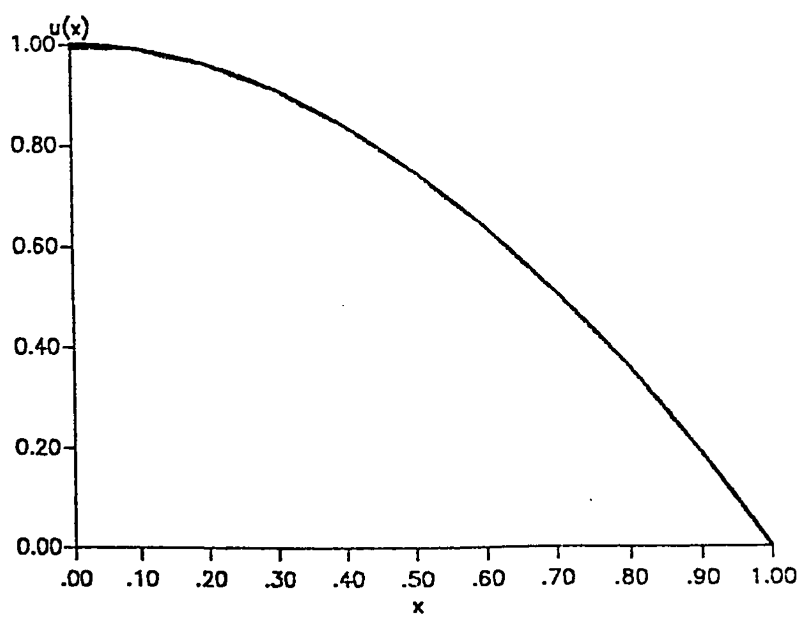
$$f(x) = -x^3 + x + 2xe^{-x} - 2e^{-x} + 2$$

$$u(x) = 1 - x^2$$

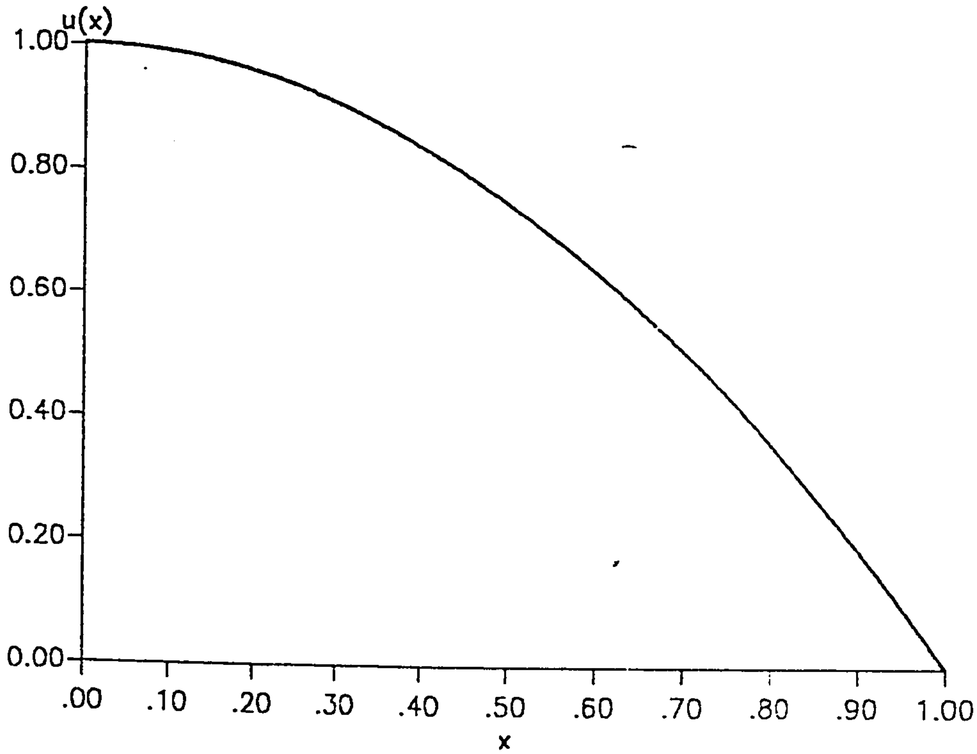
$$\|e\|_\infty = .91671 \times 10^{-2} \quad \text{when} \quad h = .1$$

$$\|e\|_\infty = .91863 \times 10^{-4} \quad \text{when} \quad h = .01$$

Graph of the solution (exact and approximate with $h = .1$):



Graph of the solution (exact and approximate with $h = .01$):



Example 2

$$p(x) = x^2$$

$$w(x) = x$$

$$q(x) = -x$$

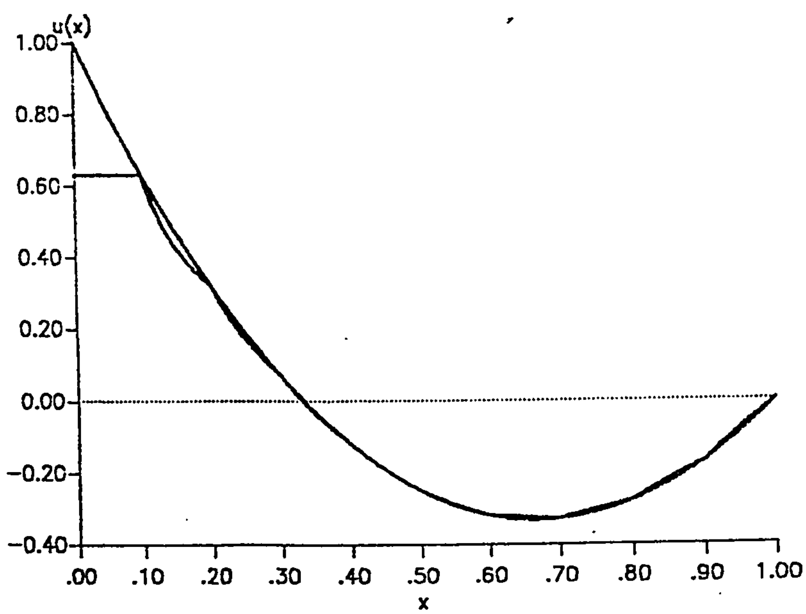
$$f(x) = -3x^3 + 4x^2 - 19x + 8$$

$$u(x) = 3x^2 - 4x + 1$$

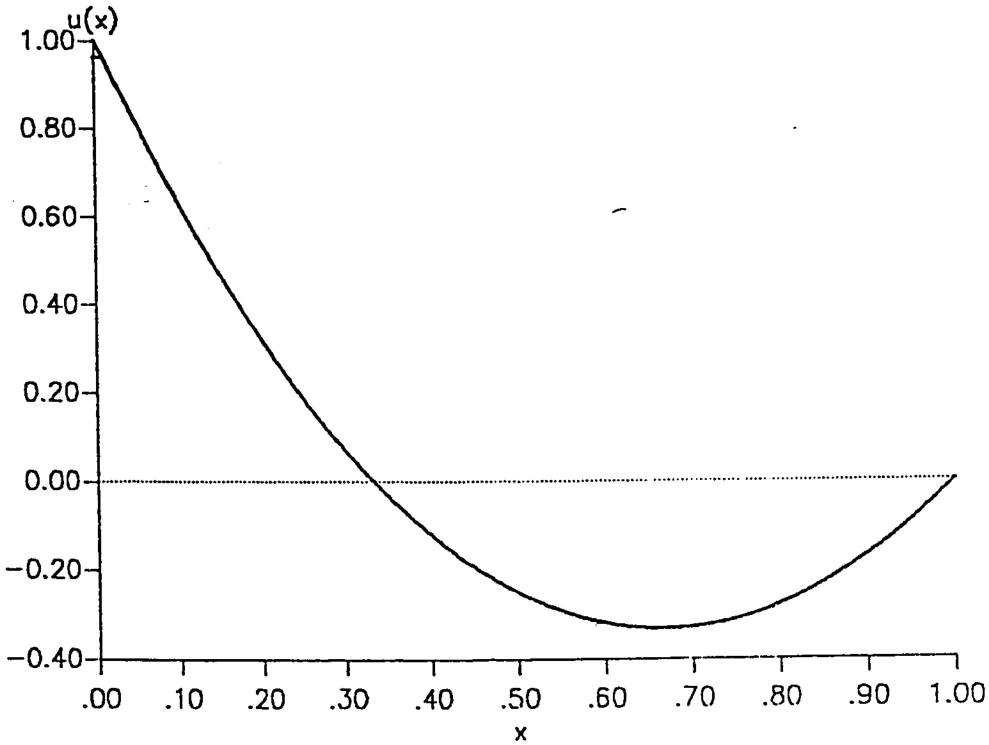
$$\|e\|_\infty = .37051 \quad \text{when} \quad h = .1$$

$$\|e\|_\infty = .039708 \quad \text{when} \quad h = .01$$

Graph of the solution (exact and approximate with $h = .1$):



Graph of the solution (exact and approximate with $h = .01$):



Example 3

$$p(x) = x$$

$$w(x) = 1$$

$$q(x) = x$$

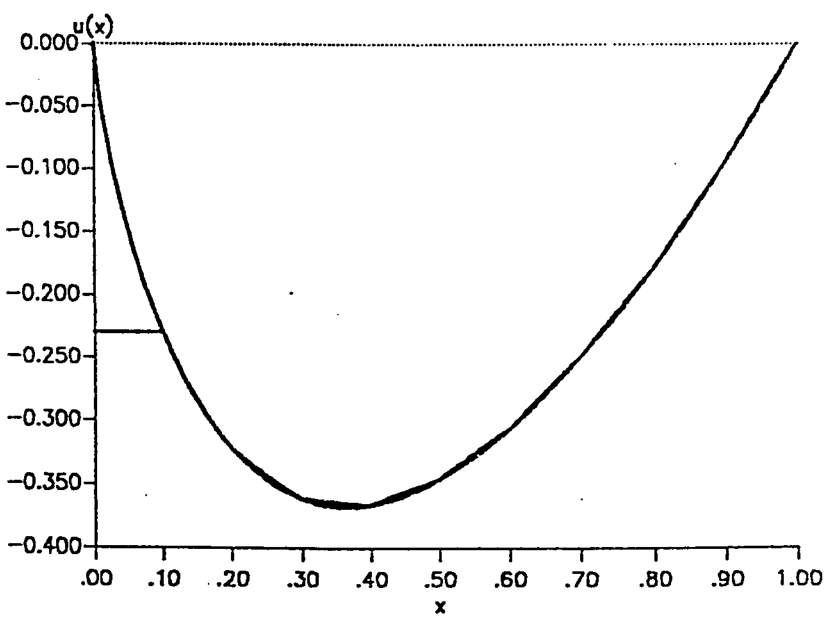
$$f(x) = x^2 lnx - lnx - 2$$

$$u(x) = xlnx$$

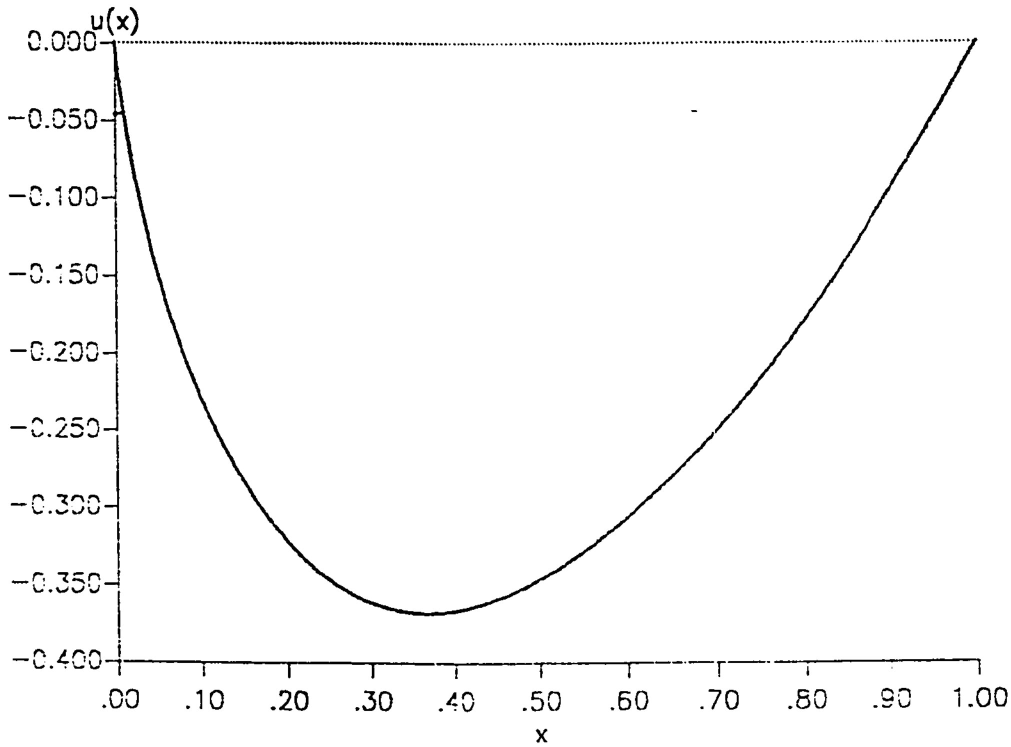
$$\|e\|_\infty = .23006 \quad \text{when} \quad h = .1$$

$$\|e\|_\infty = .04605 \quad \text{when} \quad h = .01$$

Graph of the solution (exact and approximate with $h = .1$):



Graph of the solution (exact and approximate with $h = .01$):



Example 4

$$p(x) = x\left[1 + x(1-x)^{\frac{1}{7}}\right]$$

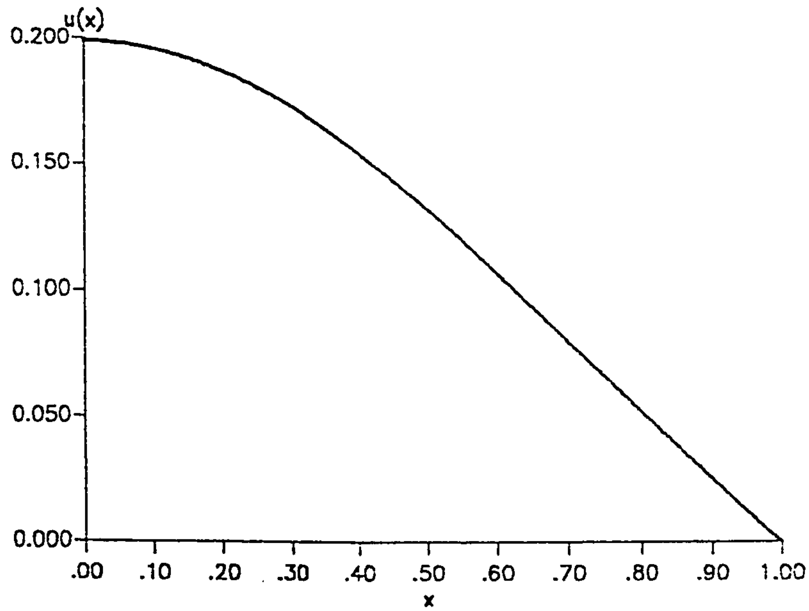
$$w(x) = x(1-x)^{\frac{1}{7}}$$

$$q(x) = -1$$

$$f(x) = cos\left(\frac{\pi}{2}x\right)$$

$$u(x) = \text{unknown}$$

Graph of the solution (approximate with $h = .02$):



Example 5

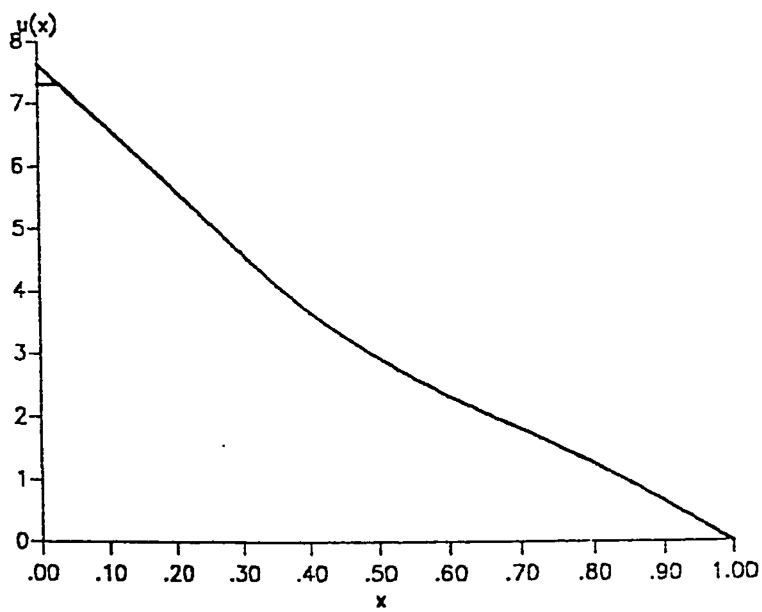
$$p(x) = x, \quad q(x) = 0.0, \quad f(x) = 10.0$$

$$w(x) = \begin{cases} 1 & \text{if } 0 \le x \le \frac{1}{3}, \\ 0 & \text{if } \frac{1}{3} < x < \frac{2}{3}, \\ 1 & \text{if } \frac{2}{3} \le x \le 1. \end{cases} \tag{7.2}$$

$$u(x) = \begin{cases} -10x + \frac{20}{3} + \frac{10}{3} ln\left(\frac{4}{3}\right) & \text{if } 0 \le x \le \frac{1}{3}, \\ -\frac{10}{3} ln x + \frac{10}{3} + \frac{20}{3} ln\left(\frac{2}{3}\right) & \text{if } \frac{1}{3} \le x \le \frac{2}{3}, \\ \frac{10}{3} ln x + 10(1-x) & \text{if } \frac{2}{3} \le x \le 1. \end{cases} \tag{7.3}$$

$$\|\epsilon\|_\infty = .3333 \quad \text{when} \quad h = .0333$$

Graph of the solution (exact and approximate with $h = .0333$):



Example 6

$$-\frac{1}{x}(xu')' + e^u = 0, \quad 0 < x < 1$$

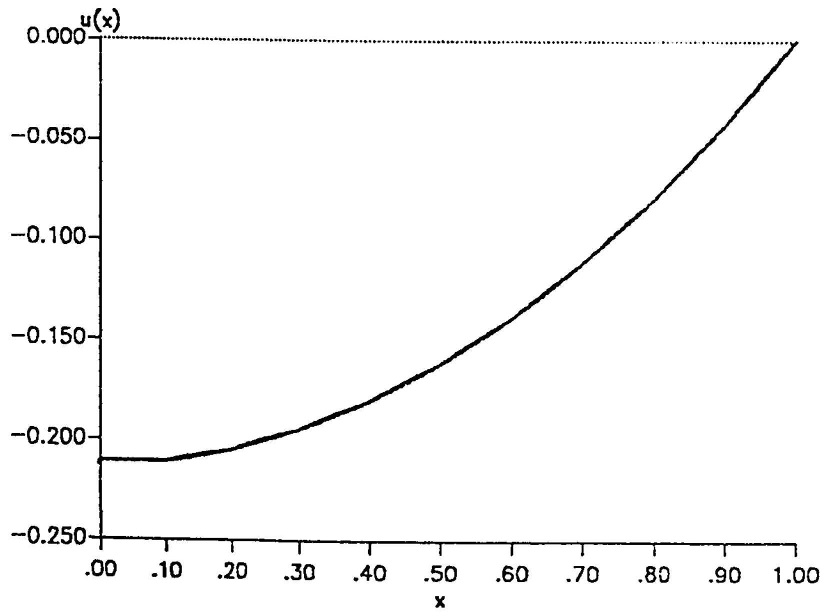
$$u'(0) = u(1) = 0.$$

$$u(x) = 2ln\left(\frac{1+\beta}{1+\beta x^2}\right), \quad \beta = -5 + 2\sqrt{6}.$$

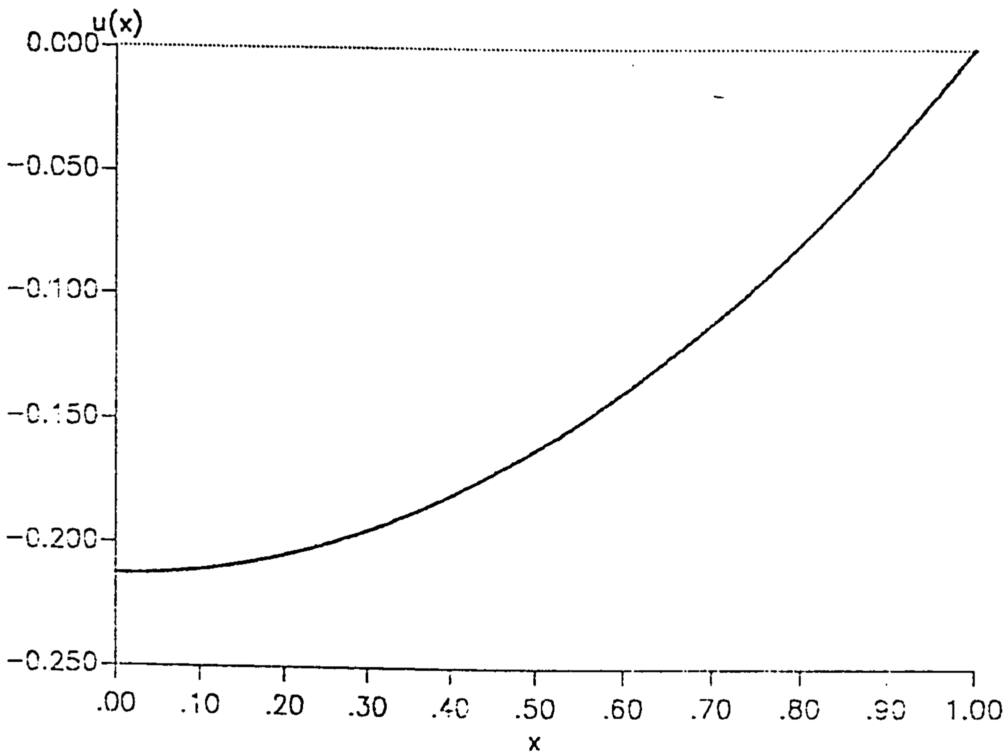
$$\|e\|_\infty = .188845 \times 10^{-2} \quad \text{when} \quad h = .1$$

$$\|e\|_\infty = .189 \times 10^{-4} \quad \text{when} \quad h = .01$$

Graph of the solution (exact and approximate with $h = .1$):



Graph of the solution (exact and approximate with $h = .01$):



Example 7

$$-\frac{1}{x^2}(x^2u')' - \left(u + \frac{\sqrt{3}}{2}\right)^5 = 0$$

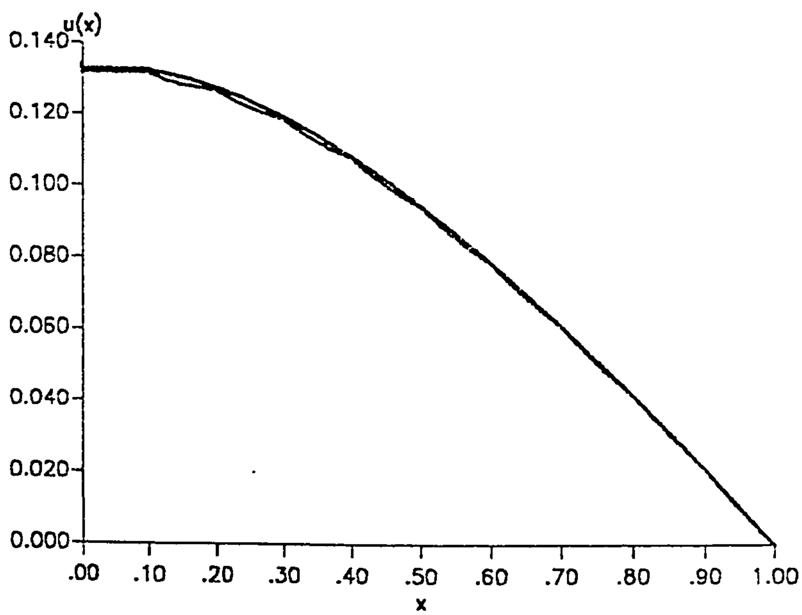
$$u'(0) = u(1) = 0$$

$$u(x) = \frac{1}{\sqrt{1 + x^2/3}} - \frac{\sqrt{3}}{2}.$$

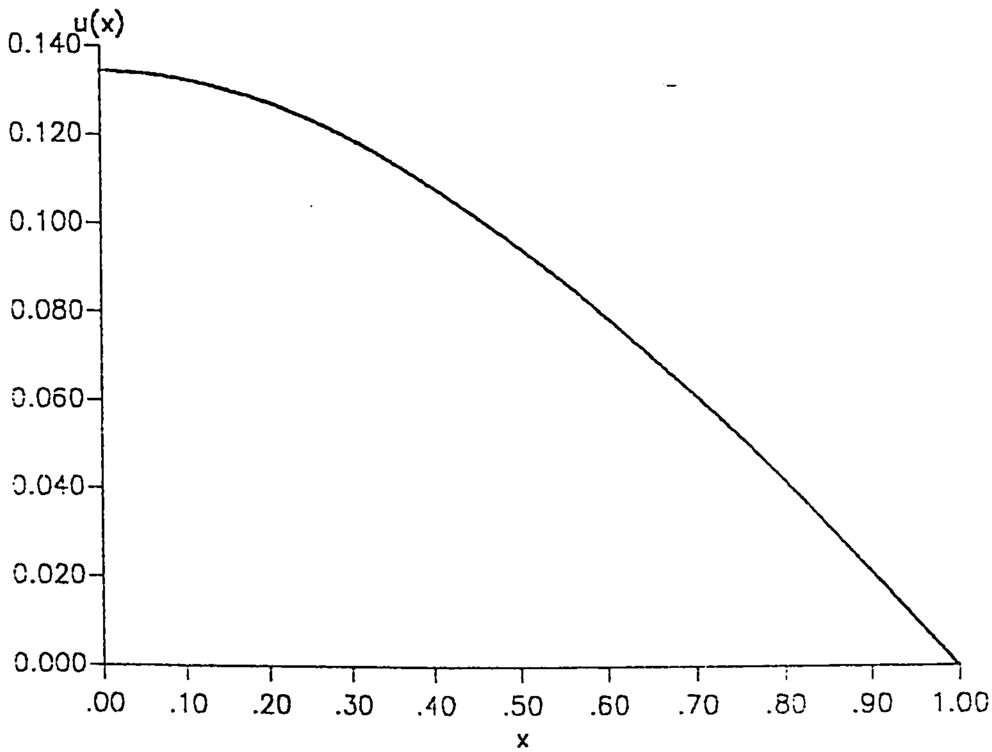
$$\|e\|_\infty = .25109 \times 10^{-2} \quad \text{when} \quad h = .1$$

$$\|e\|_\infty = .254 \times 10^{-4} \quad \text{when} \quad h = .01$$

Graph of the solution (exact and approximate with $h = .1$):



Graph of the solution (exact and approximate with $h = .01$):



Example 8

$$
\begin{aligned}
-\frac{1}{x}(xu')' - \frac{1}{(u+\epsilon)^{\frac{1}{3}}} &= 0 \\
u'(0) = u(1) &= 0
\end{aligned}
$$

The analytical solution is not known. Fink et. al. numerically solved it for $\epsilon = 10^{-3}$ and $\epsilon = 10^{-4}$. We also solved with these ϵ. The approximate solution $u^G(x)$ is shown in the following tables where $u(x)$ denotes the Fink's result.

Table of the solution when $h = .02$:

$\varepsilon = 10^{-3}$

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
u(x)	0.3820	0.3716	0.3543	0.3298	0.2979	0.2581	0.2100	0.1525	0.0840	0.0000
uG(x)	0.3820	0.3717	0.3543	0.3298	0.2979	0.2582	0.2100	0.1525	0.0840	0.0000

$\varepsilon = 10^{-4}$

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
u(x)	0.3823	0.3720	0.3547	0.3302	0.2982	0.2584	0.2102	0.1527	0.0842	0.0000
uG(x)	0.3824	0.3720	0.3547	0.3302	0.2982	0.2585	0.2103	0.1527	0.0842	0.0000

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APPENDIX

Computer Codes

Sample programe for linear problem (Example 1)

```
      IMPLICIT REAL*8(A-H,Q-Z)
      DIMENSION A(100),B(100),D(100)
      EXTERNAL F1,F2,F3,F4,FF1,FF2,FF0
      NZERO=0
      NONE=1
      NTWO=2
      N=9
      MM=20
      NP1=N+1
      NP=NP1*MM+1
      C1=1.
      H=C1/NP1
      M=10
      X0=0.0
      X1=H
      X2=2*H
      CALL DSIM(F4,X0,X1,M,Y1)
      CALL DSIM(F3,X1,X2,M,Y2)
      Y3=XP(X2)-XP(X1)
      D(1)=1/Y3 + Y1 + Y2/(Y3)**2
      DO 10 I=2,N
      XI=I*H
      XII=XI-H
      XIJ=XI+H
      CALL DSIM(F2,XII,XI,M,Y1)
      CALL DSIM(F3,XI,XIJ,M,Y2)
      Y1=Y1/(XP(XI)-XP(XII))**2
      Y2=Y2/(XP(XIJ)-XP(XI))**2
10    D(I)=1/(XP(XI)-XP(XI-H)) + 1/(XP(XI+H)-XP(XI)) + Y1 + Y2
      DO 20 I=1,N-1
      XA=I*H
      XB=XA+H
      CALL DSIM(F1,XA,XB,M,Y)
20    A(I)= - 1/(XP(XB)-XP(XA))+ Y/(XP(XB)-XP(XA))**2
      X0=0.0
      CALL DSIM(FF0,X0,H,M,Y1)
      X1=H
      X2=X1+H
      CALL DSIM(FF2,X1,X2,M,Y2)
      B(1)=Y1 + Y2/(XP(X2)-XP(X1))
      DO 30 I=2,N
      X2=I*H
      X1=X2-H
      X3=X2+H
      CALL DSIM(FF1,X1,X2,M,Y1)
      CALL DSIM(FF2,X2,X3,M,Y2)
30    B(I)= Y1/(XP(X2)-XP(X1)) + Y2/(XP(X3)-XP(X2))
      CALL DTRI(N,A,D,A,B)
      ERR0=0.0
      X0=0.0
      U0=B(1)
      B(N+1)=0.0
        HH=H/MM
        Y1=U0
      WRITE(7, *)NZERO,NP,NTWO
        DO 60 I=1,MM
        X1=X0+(I-1)*HH
      Y=SOL(X1)
      ERR=DABS(Y-Y1)
      IF(ERR.GE.ERR0)ERR0=ERR
      WRITE(8,2)X1,Y,Y1,ERR
60    WRITE(7,1)X1,Y,Y1
      DO 15 I=1,N
      A1=I*H
```

```
      U1=B(I)
      U2=B(I+1)
      B1=A1+H
        HH=(B1-A1)/MM
        DO 5 J=1,MM
        X1=A1+HH*(J-1)
        IF(I.EQ.N)THEN
        Y1=(XP(B1)-XP(X1))/(XP(B1)-XP(A1))*U1
        ELSE
        Y1=(XP(B1)-XP(X1))/(XP(B1)-XP(A1))*U1 +
     +      (XP(X1)-XP(A1))/(XP(B1)-XP(A1))*U2
        ENDIF
      Y=SOL(X1)
      ERR=DABS(Y-Y1)
      IF(ERR.GE.ERRO)ERRO=ERR
      WRITE(8,2)X1,Y,Y1,ERR
5     WRITE(7,1)X1,Y ,Y1
15    CONTINUE
      WRITE(8,4)ERRO
      X1=1.
      U1=0.
      WRITE(7,2)X1,U1,U1
1     FORMAT(3F15.8)
2     FORMAT(4G15.5)
      DO 50 I=1,N
      X=I*H
      Y=SOL(X)
      ERR=DABS(Y-B(I))
50    WRITE(9,3)X,Y,B(I),ERR
3     FORMAT(F15.10,3G18.10)
4     FORMAT(///' ERROR = ',G15.5////)
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C     INTEGRATION  BY   SIMPSON'S  RULE        C
C     F : SUPPLIED FUNCTION                    C
C     A : LOWER LIMIT                          C
C     B : UPPER LIMIT                          C
C     M : HALF OF THE NUMBER OF DIVISIONS      C
C     Y : APPROXIMATE VALUE OF INTEGRATION     C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      SUBROUTINE DSIM(F,A,B,M,Y)
      IMPLICIT REAL*8(A-H,Q-Z)
      EXTERNAL F
      H=(B-A)/2/M
      XI0=F(A,B,A)+F(A,B,B)
      XI1=0.0
      XI2=0.0
      DO 10 I=1,2*M-1
         X=A+I*H
         N=I/2
         N=I-2*N
         IF(N.EQ.0)THEN
            XI2=XI2+F(A,B,X)
         ELSE
            XI1=XI1+F(A,B,X)
         ENDIF
10    CONTINUE
      Y=H*(XI0+2*XI2+4*XI1)/3
      RETURN
      END
C---------------------------------------------
      SUBROUTINE DTRI(N,A,D,C,B)
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                          C
```

```
C   INPUT :                                                       C
C                                                                 C
C          -                                              -       C
C         ] D(1)    C(1)                        '   B(1)    ]     C
C         ]                         O           '           ]     C
C         ] A(1)    D(2)    C(2)                '   B(2)    ]     C
C         ]                                     '           ]     C
C         ]         A(2)    D(3)    C(3)        '   B(3)    ]     C
C         ]                                     '           ]     C
C         ]     O           A(3)    D(4)        '   B(4)    ]     C
C         ]                                     '          _]     C
C          -                                                      C
C                                                                 C
C   OUTPUT:                                                       C
C                                                                 C
C      RETURNS THE SOLUTION IN  THE  VECTOR  B                    C
C                                                                 C
C                                                                 C
C   METHOD:                                                       C
C                                                                 C
C      NAIVE  GAUSSIAN ELIMINATION (WITHOUT PIVOTING)             C
C                                                                 C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      IMPLICIT REAL*8(A-H,Q-Z)
      DIMENSION A(N),D(N),C(N),B(N)
      DO 2 I=2,N
         XMULT=A(I-1)/D(I-1)
         D(I)=D(I)-XMULT*C(I-1)
2        B(I)=B(I)-XMULT*B(I-1)
      B(N)=B(N)/D(N)
      DO 3 I=1,N-1
3     B(N-I)=(B(N-I)-C(N-I)*B(N-I+1))/D(N-I)
      RETURN
      END
      FUNCTION F1(A,B,X)
      IMPLICIT REAL*8(A-H,Q-Z)
      F1=QW(X)*(XP(B)-XP(X))*(XP(X)-XP(A))
      RETURN
      END
C---------------------------------------------
      FUNCTION F2(A,B,X)
      IMPLICIT REAL*8(A-H,Q-Z)
      F2=QW(X)*(XP(X)-XP(A))**2
      RETURN
      END
C---------------------------------------------
      FUNCTION F3(A,B,X)
      IMPLICIT REAL*8(A-H,Q-Z)
      F3=QW(X)*(XP(B)-XP(X))**2
      RETURN
      END
C---------------------------------------------
      FUNCTION F4(A,B,X)
      IMPLICIT REAL*8(A-H,Q-Z)
      F4=QW(X)
      RETURN
      END
C---------------------------------------------
      FUNCTION FF1(A,B,X)
      IMPLICIT REAL*8(A-H,Q-Z)
      FF1=FFW(X)*(XP(X)-XP(A))
      RETURN
      END
C---------------------------------------------
```

```
      FUNCTION FF2(A,B,X)
      IMPLICIT REAL*8(A-H,Q-Z)
      FF2=FFW(X)*(XP(B)-XP(X))
      RETURN
      END
C----------------------------------------------
      FUNCTION FF0(A,B,X)
      IMPLICIT REAL*8(A-H,Q-Z)
      FF0=FFW(X)
      RETURN
      END
C-----------------------------------------------------
C*****************************************************
C-----------------------------------------------------
      FUNCTION XP(X)
      IMPLICIT REAL*8(A-H,Q-Z)
      XP=DLOG(DEXP(X)-1)
      RETURN
      END
C-----------------------------------------------------
      FUNCTION W(X)
      IMPLICIT REAL*8(A-H,Q-Z)
      IF ( X .LT. 1.D-10)THEN
          X=0.
      ENDIF
       W=1.0
      RETURN
      END
C-----------------------------------------------------
      FUNCTION QW(X)
      IMPLICIT REAL*8(A-H,Q-Z)
      QW=X*W(X)
      RETURN
      END
C-----------------------------------------------------
      FUNCTION FFW(X)
      IMPLICIT REAL*8 (A-H,Q-Z)
      IF(DABS(X).LT. 1.D-20) THEN
         X=0.
      ENDIF
      FFW=-X*X*X +X+2*X*DEXP(-X)-2*DEXP(-X)+2
      FFW=FFW*W(X)
      RETURN
      END
C-----------------------------------------------------
      FUNCTION SOL(X)
      IMPLICIT REAL*8 (A-H,Q-Z)
      IF(DABS(X).LT. 1.D-20) THEN
         X=0.
      ENDIF
      SOL=1-X*X
      RETURN
      END
```

Sample programe for nonlinear problem (Example 6)

```
      IMPLICIT REAL*8(A-H,Q-Z)
      DIMENSION U(100)
      N=99
      M=4
      NM1=(N+1)*M+1
      N1=30
      N2=50
      CALL GAMA(GAMMA)
      CALL PICARD(N,N1,GAMMA,100,U,NSIG)
      IF (NSIG .EQ. 1) STOP
      CALL SECANT(N,N2,100,U,ERROR)
      PRINT*,ERROR
      C1=1.
      H=C1/(N+1)
      WRITE(7,*)'0',NM1,'2'
      ERRO=0.0
      X0=0.0
      U0=U(1)
      U(N+1)=0.0
        HH=H/M
        Y1=U0
        DO 60 I=1,M
        X1=X0+(I-1)*HH
      Y=SOL(X1)
      ERR=DABS(Y-Y1)
      IF(ERR.GE.ERRO)ERRO=ERR
      WRITE(7,1)X0,Y1,Y
      WRITE(8,2)X1,Y1,Y,ERR
60    CONTINUE
      DO 10 I=1,N
      A=I*H
      U1=U(I)
      U2=U(I+1)
      B=A+H
        HH=(B-A)/M
        DO 5 J=1,M
        X1=A+HH*(J-1)
        IF(I.EQ.N)THEN
        Y1=(XP(B)-XP(X1))/(XP(B)-XP(A))*U1
        ELSE
        Y1=(XP(B)-XP(X1))/(XP(B)-XP(A))*U1 +
     +      (XP(X1)-XP(A))/(XP(B)-XP(A))*U2
        ENDIF
      Y=SOL(X1)
      ERR=DABS(Y-Y1)
      IF(ERR.GE.ERRO)ERRO=ERR
      WRITE(8,2)X1,Y1,Y,ERR
5     WRITE(7,1)X1,Y1,Y
10    CONTINUE
      PRINT*,ERRO
      X=1.
      Y=0.
      WRITE(7,1)X,Y,Y
1     FORMAT(3F10.5)
2     FORMAT(4G15.5)
      DO 40 I=1,N
      X=I*H
      Y=SOL(X)
      ERR=DABS(Y-U(I))
40    WRITE(9,2)X,U(I),Y,ERR
      END
C
      SUBROUTINE PICARD(N,NI,GAMMA,M,U,NSIG)
      IMPLICIT REAL*8(A-H,Q-Z)
```

```
      DIMENSION A(100),B(100),D(100),U(100),A1(100),D1(100)
      EXTERNAL FF0,FF1,FF2
      H=1./(N+1)
      M=4
      X0=0.0
      X1=H
      X2=2*H
      D(1)=1/(XP(X2)-XP(X1))
      DO 10 I=2,N
      XI=I*H
10    D(I)=1/(XP(XI)-XP(XI-H)) + 1/(XP(XI+H)-XP(XI))
      DO 20 I=1,N-1
      XA=I*H
      XB=XA+H
20    A(I)= - 1/(XP(XB)-XP(XA))
C - - - - - - - - - - - - - - - - - - - - - - - - - - -
C INITIAL GUESS FOR U(I)
      DO 25 I=1,N
25    U(I)=0.0
C -----------------------------------------------------
      DO 45 J=1,NI
         DO 41 I=1,N
         A1(I)=A(I)
41       D1(I)=D(I)
      X0=0.0
      U0=U(1)
      X1=H
      U1=U(1)
      CALL DSIM(FF0,X0,X1,M,Y1,U0,U1)
      X2=X1+H
      U2=U(2)
      CALL DSIM(FF2,X1,X2,M,Y2,U1,U2)
      B(1)=Y1 + Y2/(XP(X2)-XP(X1)) + GAMMA*U(1)
      D1(1)=D1(1)+GAMMA
      DO 30 I=2,N
      X2=I*H
      U2=U(I)
      X1=X2-H
      U1=U(I-1)
      X3=X2+H
      U3=U(I+1)
      CALL DSIM(FF1,X1,X2,M,Y1,U1,U2)
      CALL DSIM(FF2,X2,X3,M,Y2,U2,U3)
      B(I)= Y1/(XP(X2)-XP(X1)) + Y2/(XP(X3)-XP(X2)) + GAMMA*U(I)
30    D1(I)=D1(I)+GAMMA
      CALL DTRI(N,A1,D1,A1,B)
C  ESTIMATE THE NORM  OF THE DIFFERENCE
      ERR=0.0
      DO 35 I=1,N
      DIFF=DABS(U(I)-B(I))
      IF( DIFF .GT. ERR) ERR=DIFF
 35   CONTINUE
      IF (ERR .GT. 1.D5) THEN
         PRINT*,'PICARD ITERATINS DIVERGES'
         NSIG=1
         RETURN
      ENDIF
      IF (ERR .LT. 1.D-5) THEN
         PRINT*,'PICARD ITERATINS CONVERGES'
         NSIG=0
         RETURN
      ENDIF
      DO 40 I=1,N
40    U(I)=B(I)
```

```
45    CONTINUE
      PRINT*,ERR
      NSIG=0
      RETURN
      END
C
      SUBROUTINE SECANT(N,NI,M,U,ERROR)
      IMPLICIT REAL*8(A-H,Q-Z)
      DIMENSION A(100),B(100),D(100),U(100),A1(100),D1(100)
      EXTERNAL F1,F2,F3,FF0,FF1,FF2,FFN
      C1=1.
      H=C1/(N+1)
      M=10
C -----------------------------------------------------
      X1=H
      X2=X1+H
      D(1)=1/(XP(X2)-XP(X1))
      DO 70 I=2,N
      XI1=H*I
      XI0=XI1-H
      XI2=XI1+H
      D(I)=1/(XP(XI1)-XP(XI0)) + 1/(XP(XI2)-XP(XI1))
70    A(I-1)= - 1/(XP(XI1)-XP(XI0))
C-----------------------------------------------------
      DO 45 J=1,NI
      X0=0.0
      X1=H
      X2=2*H
      U1=U(1)
      U2=U(2)
      CALL DSIM(F3,X1,X2,M,Y2,U1,U2)
      D1(1)= Y2/(XP(X2)-XP(X1))**2
C
      CALL DSIM(FF0,X0,X1,M,Y1,U0,U1)
      CALL DSIM(FF2,X1,X2,M,Y2,U1,U2)
      B(1)=Y1 + Y2/(XP(X2)-XP(X1))
C
      DO 10 I=2,N-1
      X2=I*H
      U2=U(I)
      X1=X2-H
      U1=U(I-1)
      X3=X2+H
      U3=U(I+1)
      CALL DSIM(FF1,X1,X2,M,Y1,U1,U2)
      CALL DSIM(FF2,X2,X3,M,Y2,U2,U3)
      B(I)= Y1/(XP(X2)-XP(X1)) + Y2/(XP(X3)-XP(X2))
      CALL DSIM(F2,X1,X2,M,Y1,U1,U2)
      CALL DSIM(F3,X2,X3,M,Y2,U2,U3)
      Y1=Y1/(XP(X2)-XP(X1))**2
      Y2=Y2/(XP(X3)-XP(X2))**2
10    D1(I)= Y1 + Y2
C
      X1=(N-1)*H
      X2=X1+H
      X3=X2+H
      U1=U(N-1)
      U2=U(N)
      U3=0.0
      CALL DSIM(FF1,X1,X2,M,Y1,U1,U2)
      CALL DSIM(FFN,X2,X3,M,Y2,U2,U3)
      B(N)= Y1/(XP(X2)-XP(X1)) + Y2/(XP(X3)-XP(X2))
      CALL DSIM(F2,X1,X2,M,Y1,U1,U2)
      CALL DSIM(F3,X2,X3,M,Y2,U2,U3)
```

```
      Y1=Y1/(XP(X2)-XP(X1))**2
      Y2=Y2/(XP(X3)-XP(X2))**2
      D1(N)= Y1 + Y2
C
      DO 20 I=1,N-1
      XA=I*H
      UA=U(I)
      XB=XA+H
      UB=U(I+1)
      CALL DSIM(F1,XA,XB,M,Y,UA,UB)
20    A1(I)=  Y/(XP(XB)-XP(XA))**2
C
      DO 41 I=1,N
      A1(I)=A(I)-A1(I)
41    D1(I)=D(I)-D1(I)
C
      B(1) =D(1)*U(1) + A(1)*U(2) - B(1)
      DO 30 I=2,N-1
30    B(I)= A(I-1)*U(I-1) + D(I)*U(I) + A(I)*U(I+1) - B(I)
      B(N)= A(N-1)*U(N-1) + D(N)*U(N) -B(N)
C
      CALL DTRI(N,A1,D1,A1,B)
C  UPDATE
      ERR=0.0
      DO 35 I=1,N
      DIFF=DABS(B(I))
      IF( DIFF .GT. ERR) ERR=DIFF
 35   CONTINUE
      IF( ERR .GT. 1:D5) THEN
         PRINT*,' SECANT DIVERGES'
         RETURN
      ENDIF
      IF( ERR .LT. 1.D-15) THEN
         PRINT*,' SECANT CONVERGES WITH NO. OF ITERATIONS:', J
         ERROR=ERR
         RETURN
      ENDIF
      U0=U(1)
      DO 40 I=1,N
40    U(I) =U(I)-B(I)
45    CONTINUE
      RETURN
      END
C-----------------------------------------------------------------------
C    INTEGRATION  BY   SIMPSON'S  RULE              C
      SUBROUTINE DSIM(F,A,B,M,Y,UA,UB)
      IMPLICIT REAL*8(A-H,Q-Z)
      DIMENSION U(100)
      EXTERNAL F
      H=(B-A)/2/M
      XI0=F(A,B,A,UA,UB)+F(A,B,B,UA,UB)
      XI1=0.0
      XI2=0.0
      DO 10 I=1,2*M-1
         X=A+I*H
         N=I/2
         N=I-2*N
         IF(N.EQ.0)THEN
            XI2=XI2+F(A,B,X,UA,UB)
         ELSE
            XI1=XI1+F(A,B,X,UA,UB)
         ENDIF
10    CONTINUE
      Y=H*(XI0+2*XI2+4*XI1)/3
```

```
      RETURN
      END
C-------------------------------------------
      SUBROUTINE DTRI(N,A,D,C,B)
      IMPLICIT REAL*8(A-H,Q-Z)
      DIMENSION A(N),D(N),C(N),B(N)
      DO 2 I=2,N
         XMULT=A(I-1)/D(I-1)
         D(I)=D(I)-XMULT*C(I-1)
2        B(I)=B(I)-XMULT*B(I-1)
      B(N)=B(N)/D(N)
      DO 3 I=1,N-1
3     B(N-I)=(B(N-I)-C(N-I)*B(N-I+1))/D(N-I)
      RETURN
      END
C-------------------------------------------
      FUNCTION FF0(A,B,X,UA,UB)
      IMPLICIT REAL*8(A-H,Q-Z)
      FF0=FFW0(A,B,X,UA,UB)
      RETURN
      END
C-------------------------------------------
      FUNCTION FF1(A,B,X,UA,UB)
      IMPLICIT REAL*8(A-H,Q-Z)
      FF1=FFW(A,B,X,UA,UB)*(XP(X)-XP(A))
      RETURN
      END
C-------------------------------------------
      FUNCTION FF2(A,B,X,UA,UB)
      IMPLICIT REAL*8(A-H,Q-Z)
      FF2=FFW(A,B,X,UA,UB)*(XP(B)-XP(X))
      RETURN
      END
C--------------------------------------------------
      FUNCTION FFN(A,B,X,UA,UB)
      IMPLICIT REAL*8(A-H,Q-Z)
      FFN=FFWN(A,B,X,UA,UB)*(XP(B)-XP(X))
      RETURN
      END
C-------------------------------------------
      FUNCTION F1(A,B,X,UA,UB)
      IMPLICIT REAL*8(A-H,Q-Z)
      F1=DFW(A,B,X,UA,UB)*(XP(B)-XP(X))*(XP(X)-XP(A))
      RETURN
      END
C-------------------------------------------
      FUNCTION F2(A,B,X,UA,UB)
      IMPLICIT REAL*8(A-H,Q-Z)
      F2=DFW(A,B,X,UA,UB)*(XP(X)-XP(A))**2
      RETURN
      END
C-------------------------------------------
      FUNCTION F3(A,B,X,UA,UB)
      IMPLICIT REAL*8(A-H,Q-Z)
      F3=DFW(A,B,X,UA,UB)*(XP(B)-XP(X))**2
      RETURN
      END
C-------------------------------------------
      FUNCTION FFW0(A,B,X,UA,UB)
      IMPLICIT REAL*8(A-H,Q-Z)
      FFW0= FW(X,UB)
      RETURN
      END
C-------------------------------------------
```

```
      FUNCTION FFW(A,B,X,UA,UB)
      IMPLICIT REAL*8(A-H,Q-Z)
        C=XP(B)-XP(A)
               U1=UA*(XP(B)-XP(X))
               U2=UB*(XP(X)-XP(A))
        U=( U1 + U2 ) / C
      FFW = FW(X,U)
      RETURN
      END
C-----------------------------------------
      FUNCTION FFWN(A,B,X,UA,UB)
      IMPLICIT REAL*8(A-H,Q-Z)
        C=XP(B)-XP(A)
               U1=UA*(XP(B)-XP(X))
               U2=0.0
        U=( U1 + U2 ) / C
      FFWN= FW(X,U)
      RETURN
      END
C---------------------------------------------------
      FUNCTION FW(X,U)
      IMPLICIT REAL*8(A-H,Q-Z)
      FW= DEXP(U)
      FW=-FW*W(X)
      RETURN
      END
C ---------------------------------------------
      SUBROUTINE GAMA(GAMMA)
      IMPLICIT REAL*8(A-H,Q-Z)
      GAMMA= 0.5
      RETURN
      END
C ---------------------------------------------
      FUNCTION DFW(A,B,X,UA,UB)
      IMPLICIT REAL*8(A-H,Q-Z)
      IF(DABS(UB) .LT. 1.D-5 .AND. DABS(UA) .LT. 1.D-5  THEN
          DFW=-GAMMA
          RETURN
      ENDIF
      DELTA = .0001
      H=UB-UA
      IF(DABS(H) .LT. DELTA) THEN
         H=HB-DELTA
      ENDIF
      CALL GAMA(GAMMA)
      DFW = (FW(X,UB)-FW(X,UA))/H
        IF(DABS(DFW) .GT.1000.) THEN
           PRINT*,'|DFW| IS GREATER THAN 1000'
           DFW=-GAMMA
        ENDIF
        IF(DABS(DFW).LT. 1.D-5) THEN
           DFW=-GAMMA
        ENDIF
      RETURN
      END
C---------------------------------------------------
      FUNCTION XP(X)
      REAL*8 XP,X
       IF ( X .LT.1.D-10)THEN
              PRINT*,'ARGUMENT OF XP=',X
              XP=DLOG(DABS(X)+.0001)
       ELSE
       XP=DLOG(X)
       ENDIF
```

```
      RETURN
      END
C-----------------------------------------------------------
      FUNCTION W(X)
      REAL*8 W,X
      IF ( X .LT. 1.D-10)THEN
          W=0.0
      ELSE
          W=X
      ENDIF
      RETURN
      END
C-----------------------------------------------------------
      FUNCTION SOL(X)
      IMPLICIT REAL*8(A-H,Q-Z)
      IF ( X .LT. 1.D-10)THEN
          X=0.0
      ENDIF
      C6=6.
      BETA=-5.+2*DSQRT(C6)
      SOL= 2*DLOG( (1+BETA)/(1+BETA*X*X))
      RETURN
      END
```

THE END