A GALERKIN METHOD FOR NONLINEAR SINGULAR TWO POINT BOUNDARY VALUE PROBLEMS

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الخلاصــة :

اعتبرنا نوعاً من المعادلات الحدية ذات النقطتيْن الشاذة ، وحلّلنا موضوع الوجود واللأحادية لها ، ومن ثم استعملنا طريقة جالركن مع دوال قاعدية خاصة لتفتيت مسألة التغاير المرتبطة بها. كما أوجدنا معدل التقارب في كلٍّ من القياتن الطاقي والمنتظم .

ABSTRACT

A class of nonlinear singular two point boundary value problems is considered and the existence and uniqueness of the solution is addressed. A Galerkin method with special basis functions is used to discretize the variational problem and the order of convergence in energy and uniform norms are shown.

Subject Classification: 65L15, 34E05.

Keywords: Singular Differential Equations, Galerkin Method, Limit Point, Limit Circle, Order of Convergence.

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1. INTRODUCTION

In this work we consider a class of two-point boundary value problems:

$$-\frac{1}{w}(pu')' + f(x,u) = 0, \qquad 0 < x < 1$$
$$(pu')(0^+) = 0, \qquad u(1) = 0. \tag{1}$$

Problems of the form (1) appear in many applications in physics and engineering, for example, see [1–4].

There is a considerable literature on numerical methods for singular boundary value problems. We refer to our earlier paper [5] for the solution of the two point boundary value problem of the linear type by the Galerkin method and the references therein. In this paper we extend the results about the convergence and rate of convergence of the numerical solutions of the linear singular differential equation treated in [5] to a class of nonlinear problems represented by Equation (1).

We require that the coefficient functions satisfy the following assumptions:

A1. $p, w > 0$ almost everywhere, $\frac{1}{p} \in L^1_{loc}(0,1]$;

A2. $r(x) \equiv \int_x^1 \frac{1}{p} \in L^1_w(0,1)$ (the class of absolutely integrable functions on $(0,1)$ relative to the weight w);

the problem becomes singular when

$$\frac{1}{p} \notin L^1_{loc}([0,\alpha)) \text{ for any } \alpha > 0. \tag{2}$$

This is the type of singularity to be considered in this paper.

In operator form, we rewrite (1) as

$$Lu + Fu = 0,$$

where:

$$Lu \equiv -\frac{1}{w}(pu')',$$

with domain $D(L) \subset L^2_w(0,1)$ given by:

$$D(L) = \left\{ u \in L^2_w(0,1) : u,\ pu' \in AC_{loc}(0,1],\ \text{and}\ \frac{1}{w}(pu')' \in L^2_w(0,1) \right\},$$

and F is the nonlinear operator defined on $L^2_w(0,1)$ by:

$$Fu(x) = f(x, u(x)) \text{ for almost all } x \in (0,1).$$

Here $AC_{loc}(0,1]$ denotes the space of functions which are absolutely continuous on any compact subinterval of $(0,1]$. It is known that the operator L defined above is self adjoint on $L^2_w(0,1)$ (see [6].)

This paper consists of three sections besides the introduction. In Section 2 the variational formulation of problem (1) is undertaken and the assumptions about the class of nonlinear functions to be considered are stated. In Section 3, a Galerkin method is given for the numerical approximation of the variational problem. A discussion of the rate of convergence of the method is then presented. Two illustrative examples are given in Section 4.

2. VARIATIONAL FORMULATION

In what follows, $\langle\cdot,\cdot\rangle_w, \langle\cdot,\cdot\rangle_p, \|\cdot\|_w, \|\cdot\|_p$ will denote the inner product and the norm in the spaces $L^2_w(0,1)$ and $L^2_p(0,1)$, respectively. By V_p we denote the space:

$$V_p = \left\{u \in L^2_w(0,1) : u \in AC_{loc}(0,1],\ u' \in L^2_p(0,1),\ u(1) = 0\right\}.$$

The inner product $\langle\cdot,\cdot\rangle_{V_p}$ in V_p is defined by:

$$\langle u, v\rangle_{V_p} = \int_0^1 u'(t)\, v'(t)\, p(t)\, dt\,,$$

and the norm induced by this inner product will be denoted by $\|\cdot\|_{V_p}$. It can be easily shown that, if the measure generated by p is absolutely continuous with respect to the measure generated by w, then V_p is a complete Hilbert space in its norm and that V_p is dense in $L^2_w(0,1)$. Furthermore, we have the following lemma.

Lemma 1. *V_p is continuously embedded in $L^2_w(0,1)$.*

Proof. Let $u \in V_p$, then:

$$\begin{aligned} \|u\|_w^2 &= \int_0^1 u^2 w = \int_0^1 \left(\int_s^1 \frac{\sqrt{p}}{\sqrt{p}} u'\right)^2 w \\ &\le \int_0^1 \int_s^1 \frac{1}{p} \int_s^1 p\,(u')^2\, w \\ &\le \left(\int_0^1 rw\right) \|u\|_p^2 = C\,\|u\|_p^2\,. \end{aligned}$$

□

The following assumptions are related to the class of nonlinear functions f we allow in this work:

B1. $Fu_0 \in L^2_w(0,1)$ for some $u_0 \in V_p$;

B2. $\langle \frac{|Fu - Fv|}{|u-v|}, r\rangle_w \le \widehat{C} < \infty\ \forall\, u, v \in V_p$;

B3. $\langle Fu - Fv, u - v\rangle_w \ge \gamma\,\|u - v\|^2_{V_p} > -\,\|u - v\|^2_{V_p}\ \forall u, v \in V_p,\ u \ne v$.

The following lemma is a result of assumption B2.

Lemma 2. *Suppose $u, v \in V_p$ such that $Fu - Fv \in L^2_w(0,1)$, and assumption B2) holds, then:*

$$|\langle Fu - Fv, v\rangle_w| \leq \langle \frac{|Fu - Fv|}{|u-v|}, r\rangle_w \, \|u - v\|_p \, \|v\|_p \, .$$

Proof.

$$\begin{aligned}
\left|\int_0^1 (Fu - Fv)\, vw\right| &\leq \int_0^1 \frac{|Fu - Fv|}{|u-v|} \, |(u-v)\, v|\, w \\
&\leq \int_0^1 \frac{|Fu - Fv|}{|u-v|} \left|\int_s^1 \frac{\sqrt{p}}{\sqrt{p}} (u-v)' \int_s^1 \frac{\sqrt{p}}{\sqrt{p}} v'\right| w \\
&\leq \int_0^1 \frac{|Fu - Fv|}{|u-v|} \left(\int_s^1 p\, |(u-v)'|^2\right)^{1/2} \left(\int_s^1 p|v'|^2\right)^{1/2} \int_s^1 \frac{1}{p}\, w \\
&\leq \langle \frac{|Fu - Fv|}{|u-v|}, r\rangle_w \, \|u - v\|_p \, \|v\|_p \, .
\end{aligned}$$

□

Lemma 3. *If assumptions B1, B2 are satisfied then*

(1) $\langle Fu, v\rangle_w$ *exists* $\forall\, u, v \in V_p$,

(2) *for any fixed* $u \in V_p$, *the mapping* $v \longmapsto \langle Fu, v\rangle_w$ *is continuous.*

Proof. For $u, v \in V_p$,

$$\begin{aligned}
|\langle Fu, v\rangle_w| &\leq |\langle Fu - Fu_0, v\rangle_w| + |\langle Fu_0, v\rangle_w| \\
&\leq \langle \frac{|Fu - Fv|}{|u-v|}, r\rangle_w \, \|u - u_0\|_{V_p} \, \|v\|_{V_p} + \|Fu_0\|_w \, \|v\|_w \\
&\leq \widehat{C}\, \|u - u_0\|_{V_p} \, \|v\|_{V_p} + \sqrt{C}\, \|Fu_0\|_w \, \|v\|_{V_p} \, .
\end{aligned}$$

□

As a result of Lemma 3, the nonlinear functional:

$$a(u,v) \equiv \langle u, v\rangle_{V_p} + \langle Fu, v\rangle_w \, , \tag{3}$$

is well defined on V_p. We now define the variational boundary value problem (VBVP) corresponding to (1) as: given F satisfying B1–B3, find $u \in V_p$ such that

$$a(u,v) = 0 \quad \forall\, v \in V_p \, . \tag{4}$$

It follows, from Lemma 3 and the Riesz Representation Theorem , that there exists an operator $B : V_p \to V_p$ defined by:

$$\langle Bu, v\rangle_{V_p} = \langle Fu, v\rangle_w - \gamma \langle u, v\rangle_{V_p} \qquad \forall\, v \in V_p,$$

which has the following properties.

Lemma 4. *B is monotone and hemicontinuous on V_p.*

Proof. Let $u, v \in V_p$. Then:

$$\langle Bu - Bv, u - v\rangle_{V_p} = \langle Fu - Fv, u - v\rangle_w - \gamma \|u - v\|^2_{V_p} \geq 0\,,$$

by assumption B3.

The hemicontinuity of B can be shown as follows. Let $u, \tilde{u}, v \in V_p$, $t \in \mathbb{R}$, then:

$$\begin{aligned}\langle B(u + t\tilde{u}) - Bu, v\rangle_{V_p} &= \langle F(u + t\tilde{u}) - Fu, v\rangle_w - \gamma\langle t\tilde{u}, v\rangle_{V_p} \\ &\leq |t| \left(\langle \frac{|F(u + t\tilde{u}) - Fu|}{|t\tilde{u}|}, r\rangle_w + |\gamma| \right) \|\tilde{u}\|_{V_p} \|v\|_{V_p}\,, \text{ (by Lemma 2)}, \\ &\leq |t| \left(\widehat{C} + |\gamma| \right) \|\tilde{u}\|_{V_p} \|v\|_{V_p}\,, \text{ (by assumption B2)},\end{aligned}$$

which goes to zero as $t \to 0$ □

It follows, since $1 + \gamma > 0$, that the operator

$$(1 + \gamma)I + B : V_p \to V_p$$

is bijective (see [7]) and therefore the equation

$$((1 + \gamma)I + B)u = 0 \tag{5}$$

has a unique solution in V_p. Note that:

$$\langle((1 + \gamma)I + B)u, v\rangle_{V_p} = \langle u, v\rangle_{V_p} + \langle Fu, v\rangle_w = a(u, v)\,,$$

and therefore (5) is equivalent to (4).

Next we proceed to show that, under an assumption somewhat stronger than B1, the solution of the VBVP (4) is equivalent to the solution of the BVP (1). We replace B1 by the assumption

$\widetilde{B}1$ $\;F : V_p \to H.$

The procedure is standard in the case of linear problems. We are going to show that it can be extended to our class of nonlinear problems. Let

$$X := \left\{u \in V_p : v \mapsto a(u, v) \text{ is continuous on } V_p \text{ in the topology of } L^2_w(0, 1)\right\}\,.$$

Since V_p is dense in $L^2_w(0, 1)$, then, for each $u \in X$, the linear mapping $v \mapsto a(u, v)$ can be extended to a continuous linear functional $G(v)$ on $L^2_w(0, 1)$. Thus there exists a unique element, say, $Su \in L^2_w(0, 1)$ such that:

$$G(v) = a(u, v) = \langle Su, v\rangle_w \qquad \forall\, v \in V_p. \tag{6}$$

This gives rise to the (nonlinear) operator:

$$S : V_p \subset L^2_w(0,1) \to L^2_w(0,1)\,,$$

defined by (6) on $D(S) = X$.

Lemma 5. *If* $\widetilde{B}1$ *holds, then* $D(L) \subset D(S)$ *and for any* $u \in D(L)$

$$Lu + Fu = Su.$$

Proof. Fix $u \in D(L)$, then for any $v \in V_p$

$$\begin{aligned} a(u,v) &= \langle u,v\rangle_{V_p} + \langle Fu,v\rangle_w \\ &= -\int_0^1 (pu')'v\,dx + \int_0^1 Fuvw\,dx \text{ (by Lemma 5.2(b) of [5])} \\ &= \langle Lu + Fu, v\rangle_w = \langle g, v\rangle_w, \end{aligned}$$

where $g = Lu + Fu \in L^2_w(0,1)$. Thus $|a(u,v)| \le \|g\|_w \|v\|_w$ and the mapping $v \mapsto a(u,v)$ is continuous in the topology of $L^2_w(0,1)$. So $D(L) \subset D(S)$.

Also for any $u \in D(L)$

$$\langle Su, v\rangle_w = a(u,v) = \langle Lu + Fu, v\rangle_w \quad \forall\, v \in V_p.$$

Since V_p is dense in $L^2_w(0,1)$, then $Su = Lu + Fu$. This completes the proof. □

Lemma 6. *If* $\widetilde{B}1$ *holds, then* $D(S) \subset D(L)$.

Proof. Let $u \in D(S)$. Then $v \mapsto a(u,v)$ is a linear continuous functional on V_p in the topology of $L^2_w(0,1)$. Also $v \mapsto \langle Fu, v\rangle_w$ is a continuous linear functional on V_p in the topology of $L^2_w(0,1)$. Therefore, $v \mapsto a(u,v) - \langle Fu, v\rangle_w$ is a linear continuous functional on $D(L) \subset V_p$ in the topology of $L^2_w(0,1)$. But for $v \in D(L)$,

$$\langle u, v\rangle_{V_p} = \langle u, Lv\rangle_w = \langle Lv, u\rangle_w.$$

Thus $v \mapsto \langle Lv, u\rangle_w$ is continuous on $D(L)$. Therefore $u \in D(L^*) = D(L)$. This completes the proof. □

Hence, if $\widetilde{B}1$ holds then:

$$Su = Lu + Fu\,,$$

$\forall\, u \in D(S) = D(L)$.

Theorem 7. *The following two statements are equivalent:*

(*i*) $u \in D(L)$ and $Lu + Fu = 0$

(*ii*) $u \in V_p$ and $a(u,v) = 0 \quad \forall\, v \in V_p$.

Proof. $(i) \Rightarrow (ii)$ is clear. To show that $(ii) \Rightarrow (i)$, let $u \in V_p$ such that $a(u,v) = 0$. Then $v \mapsto a(u,v) = 0$ is continuous in the topology of $L^2_w(0,1)$. Hence $u \in D(S) = D(L)$. Thus:

$$0 = a(u,v) = \langle Su, v\rangle_w = \langle Lu + Fu, v\rangle_w \quad \forall v \in V_p.$$

This implies that $Lu + Fu = 0$ (since V_p is dense in $L^2_w(0,1)$). □

Corollary 8. *Suppose f_u satisfies $\widetilde{B}1$, B2, and B3. Then the BVP (1.1) has a unique solution. Furthermore, this solution is also the unique solution of the VBVP (2.2).*

Assumption A2 requires that:

$$r \in L^1_w(0,1).$$

More generally, if $r \notin L^2_w(0,1)$ the problem is said to be in the *Limit Point* case. Thus A2 includes a special class of problems in the limit point case. We refer to this class as *Limit Point one (LP1)*. If, on the other hand,

$$r \in L^2_w(0,1),$$

the problem is said to be in the *Limit Circle (LC)* case. The following theorem illustrates the trade off that occurs between weakening the assumptions on F and strengthening the assumptions on r. The proof of the theorem is similar to that of [5].

Theorem 9. *Under the assumptions $\widetilde{B}1$, B2, and B3 we have:*

(1) (LC) *If $Fu \in L^2_w(0,1)$ then the solution u of (1.1) is absolutely continuous on* $[0,1]$,

(2) (LP1) *If $Fu \in L^\infty_w(0,1)$ then the solution u is absolutely continuous on* $[0,1]$.

3. THE GALERKIN APPROXIMATION AND CONVERGENCE RESULTS

Let $\pi : 0 = x_0 < x_1 < \cdots < x_{N+1} = 1$ be a mesh on the interval $[0,1]$ and for $i = 1, 2, \cdots N$ define the patch functions:

$$r_i(x) = \begin{cases} r_i^-(x) & \text{if } x_{i-1} \le x \le x_i, \\ r_i^+(x) & \text{if } x_i \le x \le x_{i+1}, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

where:

$$r_1^-(x) = 1,$$

$$r_i^-(x) = \frac{\int_{x_{i-1}}^{x} \frac{1}{p(s)}\,ds}{\int_{x_{i-1}}^{x_i} \frac{1}{p(s)}\,ds}, \quad i = 2, 3, \cdots, N$$

and:

$$r_i^+(x) = \frac{\int_x^{x_{i+1}} \frac{1}{p(s)}\,ds}{\int_{x_i}^{x_{i+1}} \frac{1}{p(s)}\,ds}, \quad i = 1, 2, \cdots, N.$$

Clearly, $r_i \in V_p$ for $i = 1, 2, \cdots, N$. Define the finite dimensional subspace V_N of V_p by

$$V_N = \text{span}\,\{r_i\}_{i=1}^N.$$

The discrete version of the weak problem (4) reads:

Find $u^G \in V_N$ such that

$$a(u^G, v) = 0 \text{ for all } v \in V_N. \tag{8}$$

Note that (8) has a unique solution $u^G \in AC[0,1]$. To see this we note that the operator $B_N : V_N \to V_N$ defined by

$$\langle B_N u, v\rangle_{V_p} = \langle Fu, v\rangle_w - \gamma\langle u, v\rangle_{V_p} \qquad \forall v \in V_N$$

inherits the monotonicity and hemicontinuity from its continuous counterpart B. If u is the solution of (4) and u^G is the solution of (8), then:

$$\langle u - u^G, v\rangle_{V_p} + \langle Fu - Fu^G, v\rangle_w = 0 \text{ for all } v \in V_N. \tag{9}$$

We can now state our results on the convergence of the Galerkin solution u^G to the weak solution u of 4.

Theorem 10. (LC, LP1) *If the function F satisfies B1 and $Fu \in L_w^2(0,1)$, then*

$$\|u^G - u\|_{V_p} \le C\sqrt{\ell(\pi_N)}\,\|Fu\|_w$$

where C depends only on the data and $\ell(\pi_N)$ is given by

$$\ell(\pi_N) = \max_{0\le i\le N} \int_{x_i}^{x_{i+1}} \left(\int_s^{x_{i+1}} \frac{1}{p(t)}dt\right) w(s)ds\,. \tag{10}$$

Theorem 11. (LC, LP1) *If $Fu \in L_w^\infty(0,1)$, then*

$$\|u^G - u\|_\infty \le C\sqrt{\ell(\pi_N)}\,\|Fu\|_\infty\,.$$

Theorem 12. (LC) *If $Fu \in L_w^2(0,1)$ and $\frac{Fu - Fu^G}{u - u^G} \in L_w^4(0,1)$, then*

$$\|u^G - u\|_\infty \le C\sqrt{\ell_1(\pi_N)}\,\|Fu\|_w\,,$$

where

$$\ell_1(\pi_N) = \max\left\{\ell(\pi_N), \max_{0\le i\le N} \int_{x_i}^{x_{i+1}} \left(\int_s^{x_{i+1}} \frac{1}{p(t)}dt\right)^2 w(s)ds\right\}. \tag{11}$$

Proofs of the above theorems are straightforward extensions of their counterparts in [5]. We only need Lemma 13 below. The following notation is needed to state the Lemma.

Let

$$u^G(x) = \sum_{i=1}^{N} \alpha_i r_i(x),$$

and u^I be the V_N-interpolant of the solution u given by:

$$u^I(x) = \sum_{i=1}^{N} u_i r_i(x),$$

where $u_i = u(x_i)$ and r_i is given by (3.1), $i = 1, \ldots, N$. We note here that u^I is the orthogonal projection of u with respect to the inner product $\langle \cdot, \cdot \rangle_{V_p}$:

$$\langle u - u^I, v \rangle_{V_p} = 0 \ \forall \ v \in V_N. \tag{12}$$

The following relation is easily checked (using (9) and (12)):

$$\langle u^G - u^I, v \rangle_{V_p} = \langle Fu - Fu^G, v \rangle_w \ \forall \ v \in V_N. \tag{13}$$

Lemma 13.

$$\left\| u - u^G \right\|_{V_p} \le \left(1 + \frac{\widehat{C}}{1+\gamma}\right) \left\| u - u^I \right\|_{V_p} \tag{14}$$

where $\widehat{C}$ and γ are given by assumptions B2 and B3.

Proof. We note that:

$$\langle u^G - u^I, v \rangle_{V_p} + \langle Fu - Fu^G, v \rangle_w = \langle Fu - Fu^G, v \rangle_w. \tag{15}$$

Now, putting $v = u^G - u^I$ in 15 we get, using assumptions B2 and B3:

$$(1+\gamma) \left\| u^G - u^I \right\|_{V_p}^2 \le \widehat{C} \left\| u - u^I \right\|_{V_p} \left\| u^G - u^I \right\|_{V_p}.$$

Thus:

$$\begin{aligned} \left\| u - u^G \right\|_{V_p} &\le \left\| u - u^I \right\|_{V_p} + \left\| u^G - u^I \right\|_{V_p} \\ &\le \left(1 + \frac{\widehat{C}}{1+\gamma}\right) \left\| u - u^I \right\|_{V_p}. \end{aligned}$$

□

4. EXAMPLES

In this Section we give two examples which are solved by the Galerkin method just described above with equal mesh size h.

Example 1. *Consider the boundary value problem:*

$$-\frac{1}{x^2}(x^2u')' - \left(u + \frac{\sqrt{3}}{2}\right)^5 = 0, \quad 0 < x < 1$$

$$u'(0) = u(1) = 0.$$

The exact solution is known:

$$u(x) = \frac{1}{\sqrt{1 + x^2/3}} - \frac{\sqrt{3}}{2}.$$

It is seen that $\|u^G - u\|_\infty = 0.25109 \times 10^{-2}$ *for* $h = 0.1$ *and* $\|u^G - u\|_\infty = 0.254 \times 10^{-4}$ *for* $h = 0.01$.

Example 2. *Consider the boundary value problem:*

$$-\frac{1}{w}(pu')' + f(x,u) = 0, \quad 0 < x < 1$$

$$(pu')(0^+) = u(1) = 0,$$

where

$$w = 1.0$$

$$p = 1 - e^{-x}$$

$$f(x,u) = (\frac{1}{2} - x)e^u.$$

All the assumptions B1–B3 are satisfied and therefore by Corollary 6 and Theorem 7 there exits a unique solution to this problem which is absolutely continuous. We take $h = 0.01$ *and solve by the Galerkin method and the graph of the solution is shown in Figure 1. The theoretical order of convergence for this example is* $h \log h$.

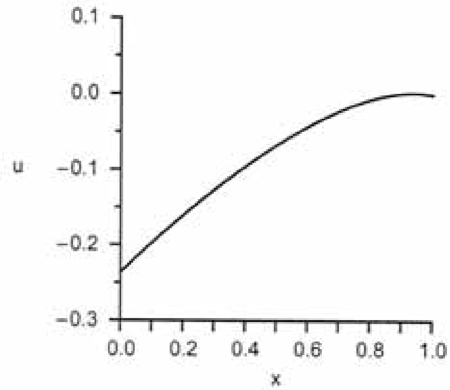

Figure 1. Solution of Example 2.

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