Perfect Numbers

What is a perfect number?

A perfect number is a number which equals the sum of its positive proper divisors. An example of a perfect number is 6. The positive divisors of 6 are 1, 2, 3, and 6. The proper divisors of 6 are 1, 2, and 3. Thus the number 6 is considered an improper divisor of itself. We have

$$6 = 1+ 2+ 3.$$

Another example of a perfect number is 28; and we have

$$28 = 1+ 2+ 4+ 7+ 14.$$

Perfect numbers were probably first introduced by the Pythagoreans (for mystical reasons).

How to find perfect numbers?

The first to give an answer about this question was Euclid. Euclid showed that if p is a prime number such that $2^p - 1$ is also a prime number, then the number $2^{p-1}(2^p - 1)$ is a perfect number. This was stated as Proposition 36 of Book IX of Euclid's book "The Elements", written about 300 BC. Note that the perfect numbers given by Euclid are even numbers. About two thousands years later, Euler showed that any even perfect number has to have the form given by Euclid. In other words, the even perfect numbers are described as follows:

Euclid-Euler Theorem

Any even perfect number N is of the form

$$N = 2^{p-1}(2^p - 1),$$

where p and $2^p - 1$ are prime numbers.

With this formula, we can produce even perfect numbers by trying prime numbers p that make $2^p - 1$ prime.

p	$2^p - 1$	$2^{p-1}(2^p - 1)$
2	3, prime	6
3	7, prime	28
5	31, prime	496
7	127, prime	8128

11	2047, not prime	-
13	8191, prime	33, 550, 336
17	131071, prime	8, 589, 869, 056

One can see that finding an even perfect number depends on whether the number $2^p - 1$ is prime or not. Prime numbers of the form $2^p - 1$ are called **Mersenne primes** and are denoted by M_p:

$$M_p = 2^p - 1.$$

Marin Mersenne (1588-1648), a French monk and an amateur mathematician, claimed, without proof, that M_p is prime when

$$p = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257$$

and composite for all primes $p < 257$. Later on, it was shown that his claim is false for $p = 67, 257$ and that he missed the primes $p = 61, 89, 107$ from his list. In addition to producing even perfect numbers, Mersenne primes are now considered a main supply of large primes. If the prime p is very large, M_p becomes very large too. How can one check if M_p is prime. The common test used to check primality of M_p is due to Edouard Lucas (1842-1891). An improved version of the test was given later by Derrick Lehmer (1905-1991). This test can be stated as follows:

Lucas-Lehmer Test: Let $\{L_k\}$ be the sequence defined recursively as follows: $L_1 = 4,\ L_{k+1} = L_k^2 - 2$ for $k \geq 1$. Then, for $p > 2$,

M_p is prime if and only if M_p divides L_{p-1}.

Let us illustrate Lucas-Lehmer test by two examples. First we have $\{L_k\} = \{4, 14, 194, 37634, \cdots\}$. As $M_3 = 7$ divides $L_2 = 14$ and $M_5 = 31$ divides $L_4 = 37634$, then both M_3 and M_5 are primes. Nowadays, Luca-Lehmer Test is used by supercomputers to discover larger and larger Mersenne primes. Up to the time of writing this article, the largest known Mersenne prime is

$$2^{32,582,657} - 1.$$

It was discovered in September of 2006. It has 9,808,358 decimal digits.

We state now some properties of even perfect numbers. The units digit of an even perfect number is either 6 or 8. In case the units digit is 8, then in fact it must ends with the digits 28. Every even perfect number, except the first, leaves a remainder of 1 when divided by 9 and leaves a remainder of 4 when divided by 6. Every even perfect number, except the first, can be expressed as a sum of consecutive odd cubes; for example,

$$28 = 1^3 + 3^3,$$

$$496 = 1^3 + 3^3 + 5^3 + 7^3.$$

How many even perfect numbers are there?

According to the special form of an even perfect number, the question is equivalent to the following question: How many Mersenne primes are there? The exact answer is not known; although evidence suggests that maybe there are infinitely many Mersenne primes. Up to the writing of this article, there are 44 known Mersenne primes and hence there are 44 known even perfect numbers. The latest on Mersenne primes can be found in the web site: http://www.mersenne.org.

Odd Perfect Numbers

Our discussion above was concentrated on even perfect numbers. What about odd perfect numbers? Is there any example of an odd perfect number? This is maybe the oldest unsolved problem in Mathematics. Up to now, no single example has been found for an odd perfect number. What we know is some properties that an odd perfect number, if it exists, must satisfy. We list some of these properties:

- An odd perfect number has to be of the form $p^a M^2$, where p is prime, p and a leave a remainder of 1 when divided by 4, and M is a positive integer.(This result is due to Euler)
- An odd perfect number must have at least eight distinct prime factors.
- An odd perfect number must be larger than 10^{300}.

Amicable Numbers

Related to perfect numbers are amicable numbers. Two positive integers are called amicable numbers or form an amicable pair if each is equal to the sum of the proper divisors of the other. For example the two numbers 220 and 284 are amicable numbers since

$$220 = 1 + 2 + 4 + 71 + 142$$

and

$$284 = 1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110.$$

The Arab mathematician Thabit ben Korrah (ninth century AD) formulated a rule to find amicable numbers. This rule can be stated as follows:

Thabit's Rule for amicable pairs

For a natural number n, if

$$\begin{cases} p = 3 \cdot 2^{n-1} - 1 \\ q = 3 \cdot 2^{n} - 1 \\ r = 9 \cdot 2^{2n-1} - 1 \end{cases}$$

are all primes, then the pair

$$(2^n \cdot p \cdot q, 2^n \cdot r)$$

is an amicable pair.

For example if we take $n = 2$, then $p = 5, q = 11$, and $r = 71$ are all prime numbers and hence we get the amicable numbers $M = 2^4 p_1 p_2 = 220$ and $N = 2^2 q_2 = 284$ indicated above. If we take $n = 4$, we get the amicable numbers $M = 2^4 \times 23 \times 47 = 17296$ and $N = 2^4 \times 1151 = 18416$. Euler (1707-1783) gave a generalization of Thabit's Rule which can be stated as follows:

Euler's Rule for amicable pairs

Let n and m be positive integers such that $1 \leq m \leq n-1$. If

$$\begin{cases} p = 2^m (2^{n-m} + 1) - 1 \\ q = 2^n (2^{n-m} + 1) - 1 \\ r = 2^{n+m} (2^{n-m} + 1)^2 - 1 \end{cases}$$

are all primes, then the pair

$$(2^n \cdot p \cdot q, 2^n \cdot r)$$

is an amicable pair.

Note that if $n - m = 1$ in Euler's Rule, we get Thabit's Rule. Even though there are rules to generate amicable numbers, it is not known whether or not there are infinitely many amicable pairs.

References:

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