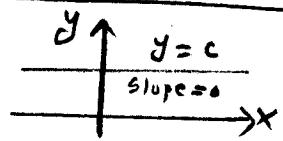


* Derivatives of Polynomials and Exponential Functions *

1

Objectives: 1. To introduce the rules of differentiation2. Consider \Rightarrow derivative of e^x .Rule 1: If $y = f(x) = c$ then $y' = f'(x) = \frac{d}{dx}(c) = 0$.

$$\text{Proof}, \quad f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} 0 = 0$$

Rule 2: $f(x) = x \rightarrow f'(x) = 1$ or $\frac{d}{dx}(x) = 1$.Rule 3 Power Rule: If n is a positive integer then $\frac{d}{dx}(x^n) = nx^{n-1}$ Ex. Find the derivative:

a) $f(x) = x^5 \Rightarrow f'(x) = 5x^4$

b) $y = x^{100}, \quad y' = 100x^{99}$

c) $\frac{d}{dt}(t^3) = 3t^2$

d) $\frac{d}{dr}(r^3) = 3r^2$

Rule 4 Power rule (General): If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Ex. Differentiate each function.

Q.4, $f(x) = \sqrt{30} \Rightarrow f'(x) = 0$ Q.11, $y = x^{-\frac{2}{5}} \Rightarrow \frac{dy}{dx} = -\frac{2}{5}x^{-\frac{7}{5}}$

Q.18, $y = \sqrt[3]{x} \Rightarrow y = x^{\frac{1}{3}} \Rightarrow y' = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}} = \frac{1}{3\sqrt[3]{x^2}}$

Q.25, $y = 4\pi^2 \Rightarrow y' = 0$ because $4\pi^2$ is a constant

Q.42, Find the tangent eqn. to the curve of $y = x\sqrt{x}$, (4, 8)

191 $y = x \cdot x^{\frac{1}{2}} = x^{\frac{3}{2}}$

Rule 5

If C is a constant, and f is differentiable, then

$$\frac{d}{dx}[Cf(x)] = C \frac{d}{dx}(f(x))$$

Rule 6: The sum-difference rules: if f and g are differentiable, then:

$$1. \frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

and in general: $(f+g+h)' = f' + g' + h'$.

$$2. \frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

and also: $(f+g-h)' = f' + g' - h'$

Ex: Differentiate:

$$\begin{aligned} \text{Q.20: } f(t) &= \sqrt{t} - \frac{1}{\sqrt{t}} \Rightarrow f(t) = t^{\frac{1}{2}} - t^{-\frac{1}{2}} \Rightarrow f'(t) = \frac{1}{2}t^{-\frac{1}{2}} + \frac{1}{2}t^{-\frac{3}{2}} \\ &= \frac{1}{2\sqrt{t}} + \frac{1}{2\sqrt{t^3}}. \end{aligned}$$

$$\begin{aligned} \text{Q.22: } y &= \sqrt{x}(x-1) = x^{\frac{3}{2}} - x^{\frac{1}{2}} \\ \frac{dy}{dx}(y) &= \frac{3}{2}x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}} = \frac{3}{2}\sqrt{x} - \frac{1}{2\sqrt{x}}. \end{aligned}$$

$$\begin{aligned} \text{Q.24: } y &= \frac{x^2 - 2\sqrt{x}}{x} \Rightarrow y = (x^2 - 2x^{\frac{1}{2}})\bar{x}' = x - 2x^{-\frac{1}{2}} \\ \frac{dy}{dx} &= 1 - 2(-\frac{1}{2})x^{-\frac{3}{2}} = 1 + \frac{1}{\sqrt{x^3}}. \end{aligned}$$

$$\text{Q.26: } g(u) = \sqrt{2}u + \sqrt{3}u \Rightarrow g(u) = \sqrt{2}u + \sqrt{3}\sqrt{u}$$

$$\therefore g(u) = \sqrt{2}u + \sqrt{3}u^{\frac{1}{2}} \Rightarrow g'(u) = \sqrt{3} + \frac{\sqrt{2}}{2}u^{-\frac{1}{2}} = \sqrt{3} + \frac{\sqrt{3}}{2\sqrt{u}}$$

Ex.6: Find the points on $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

$$\text{Slope} = \frac{dy}{dx} = 4x^3 - 12x$$

$$\begin{aligned} \text{The tangent horizontal} \Rightarrow \frac{dy}{dx} &= 0 \Rightarrow 4x^3 - 12x^2 = 0 \\ &\quad 4x(x^2 - 3) = 0 \\ 4x = 0 &\Rightarrow x = 0, x^2 - 3 = 0 \\ &\Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3} \end{aligned}$$

\therefore The points are $(0, 4)$, $(-\sqrt{3}, -5)$, $(\sqrt{3}, 5)$.

Def: e is the number such that: $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$

Def: The derivative of e^x is e^x : $\frac{d}{dx}(e^x) = e^x$.

Ex Differentiation

Q.28, $y = a e^v + \frac{b}{v} + \frac{c}{v^2}$.
 $y = a e^v + b v^{-1} + c v^{-2} \Rightarrow \frac{dy}{dv} = a e^v - b v^{-2} - 2 c v^{-3}$
 $\frac{dy}{dv} = a e^v - \frac{b}{v^2} - \frac{2c}{v^3}$.

Q.32, $y = e^{x+1} + 1 \Rightarrow y = e \cdot e^x + 1 \Rightarrow \frac{dy}{dx} = e \cdot e^x = e^{x+1}$

Q.48, At what point on $y = 1 + 2e^x - 3x$ the tangent line parallel to $3x - y = 5$.

191 $\frac{dy}{dx} = 2e^x - 3$, $y = 3x - 5 \Rightarrow \text{slope} = 3$

$$2e^x - 3 = 3 \Rightarrow 2e^x = 6 \Rightarrow e^x = 3 \Rightarrow x = \ln 3, f(\ln 3) = 1 + 2e^{\ln 3} - 3\ln 3 \\ = 1 + 2(3) - 3\ln 3 \\ = 7 - 3\ln 3$$

∴ The point is $(\ln 3, f(\ln 3)) = (\ln 3, 7 - 3\ln 3)$.

Q.39, Find the tangent eqn. at the given point.

191 $y = x^6 + 2e^x, (0, 2)$

$$\frac{dy}{dx} = 6x^5 + 2e^x \Rightarrow \text{slope} = m = \frac{dy}{dx} \Big|_{x=0} = 6(0) + 2e^0 = 2.$$

∴ The tangent eqn.: $y - y_1 = m(x - x_1)$

$$y - 2 = 2(x - 0) \Rightarrow y = 2x + 2.$$

Q.57, Find f' , $f(x) = |x^2 - 9|$ and for which f is not differentiable.

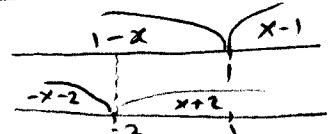
192 $f(x) = \begin{cases} x^2 - 9 & , x \leq -3 \\ 9 - x^2 & , -3 < x < 3 \\ x^2 - 9 & , x \geq 3. \end{cases}$

$$f'(x) = \begin{cases} 2x & , x < -3 \\ -2x & , -3 < x < 3 \\ 2x & , x > 3 \end{cases} \quad f'_-(-3) = -6, \quad f'_+(-3) = 6 \Rightarrow f'(-3) \text{ does not exist.} \\ f'_-(3) = -6, \quad f'_+(3) = 6 \Rightarrow f'(3) \text{ does not exist.}$$

∴ f is not differentiable at $x = -3, x = 3$.

Q.58, Where is $h(x) = |x-1| + |x+2|$ differentiable? Find $h'(x)$.

$$|x-1| = \begin{cases} x-1 & , x \geq 1 \\ 1-x & , x < 1 \end{cases}, \quad |x+2| = \begin{cases} x+2 & , x \geq -2 \\ -x-2 & , x < -2 \end{cases}$$



$$h(x) = \begin{cases} 1-x + (-x-2) & \text{if } x \leq -2 \\ 1-x+x+2 & \text{if } -2 < x < 1 \\ x-1+x+2 & \text{if } x \geq 1 \end{cases} \Rightarrow h(x) = \begin{cases} -2x-1 & \text{if } x \leq -2 \\ 3 & \text{if } -2 < x < 1 \\ 2x+1 & \text{if } x \geq 1 \end{cases}$$

$$h'(-2) = -2 \neq h'_+(-2) = 0 \Rightarrow h'(-2) \text{ does not exist.} \\ h'-(1) = 0 \neq h'_+(1) = 2 \Rightarrow h'(1) \text{ does not exist.} \\ \therefore h(x) \text{ not differentiable at } x = -2, 1.$$

Q.60, Let $f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ mx+b & \text{if } x > 2 \end{cases}$ Find m, b such that f is diff. everywhere

1. f is diff. $\Rightarrow f$ is conts. $\Rightarrow \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) \Rightarrow 2m+b = 4 \quad \dots (1)$

2. $f'(x) = \begin{cases} 2x & \text{if } x \leq 2 \\ m & \text{if } x > 2 \end{cases} \Rightarrow f'_-(2) = f'_+(2) \Rightarrow m = 4$
 $\therefore 2+b=4 \Rightarrow b=2$

The Product and Quotient Rules

Objectives: To introduce the rules of differentiation for the product and quotient

The Product Rule: If f and g are both differentiable; then:

$$\frac{d}{dx} [f(x)g(x)] = f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x)$$

$$\text{or } (fg)' = f'g + fg'$$

Ex, Differentiate:

Q.3, $f(x) = x^2 e^x \Rightarrow f'(x) = x^2 \cdot e^x + e^x \cdot (2x) = x^2 e^x + 2x e^x = x e^x (x+2)$

Q.4, $g(x) = \sqrt{x} e^x \Rightarrow g(x) = x^{\frac{1}{2}} e^x$
 $\frac{d}{dx} g(x) = x^{\frac{1}{2}} \cdot e^x + e^x \cdot \frac{1}{2} x^{-\frac{1}{2}} = x^{\frac{1}{2}} e^x + \frac{e^x}{2x^{\frac{1}{2}}} = \sqrt{x} e^x + \frac{e^x}{2\sqrt{x}}$

Q.33, If $f(x) = e^x g(x)$ when $g(0) = 2$, $g'(0) = 5$, find $f'(0)$

$$f'(x) = e^x \cdot g'(x) + g(x) \cdot e^x$$

$$f'(0) = e^0 \cdot g'(0) + g(0) \cdot e^0 = (1)(5) + (2)(1) = 7$$

Q.12, $R(t) = (t + e^t)(3 - \sqrt{t}) = (t + e^t)(3 - t^{\frac{1}{2}})$
 $R'(t) = (t + e^t)(0 - \frac{1}{2}t^{-\frac{1}{2}}) + (3 - t^{\frac{1}{2}})(1 + e^t)$
 $= -\frac{1}{2}t^{-\frac{1}{2}}(t + e^t) + (3 - t^{\frac{1}{2}})(1 + e^t)$
 $= \frac{-1}{2\sqrt{t}}(t + e^t) + (3 - \sqrt{t})(1 + e^t)$

The Quotient Rule, If f and g are both differentiable, then:

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx} f(x) - f(x) \frac{d}{dx} g(x)}{[g(x)]^2} \text{ or } \left(\frac{f}{g} \right)' = \frac{gf' - fg'}{g^2}.$$

Ex, Differentiate:

Q.8, $f(t) = \frac{2t}{4+t^2} \Rightarrow f'(t) = \frac{(4+t^2)(2) - 2t(2t)}{(4+t^2)^2} = \frac{8+2t^2-4t^2}{(4+t^2)^2} = \frac{8-2t^2}{(4+t^2)^2}$

Q.16, $y = \frac{1}{s+ke^s} \Rightarrow \frac{dy}{ds} = \frac{(s+ke^s)(0) - 1(1+ke^s)}{(s+ke^s)^2} = -\frac{1+ke^s}{(s+ke^s)^2}$

Q-34: If $h(2) = 4$, $h'(2) = -3$, find $\frac{d}{dx} \left(\frac{h(x)}{x} \right) \Big|_{x=2}$ See 3.2

$$\frac{d}{dx} \left(\frac{h(x)}{x} \right) = \frac{x \cdot h'(x) - h(x) \cdot 1}{x^2} = \frac{xh'(x) - h(x)}{x^2}$$

$$\frac{d}{dx} \left(\frac{h(x)}{x} \right) \Big|_{x=2} = \frac{(2) h'(2) - h(2)}{(2)^2} = \frac{2(-3) - 4}{4} = -\frac{10}{4} = -\frac{5}{2}$$

$$\underline{\text{Q.35}}: \quad u(x) = f(x)g(x), \quad v(x) = \frac{f(x)}{g(x)}$$

a. Find $u'(1)$

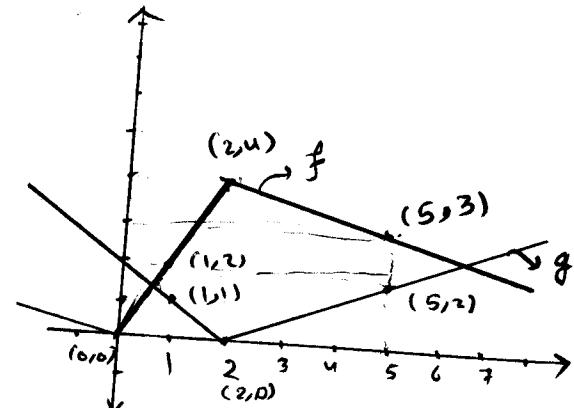
$$u'(x) = f(x)g'(x) + g(x)f'(x)$$

$$u'(1) = f(1) g'(1) + g(1) f'(1)$$

But $f(1) = 2$, $g(1) = 1$. From the graph

$$f'(1) = \frac{2-0}{1-0} = 2 , g'(1) = 0-1 = -1$$

$$u^1(1) = (2)(-1) + (1)(2) = -2 + 2 = 0$$



b. Find $v'(5)$

$$v'(x) = \frac{g(x) f'(x) - f(x) g'(x)}{\left[g(x)\right]^2}$$

$$g(5) = 2, \quad g'(5) = \text{slope} = \frac{2-0}{5-2} = \frac{2}{3}.$$

$$f(5) = 3, \quad f'(5) = \text{slope} = \frac{3-4}{5-2} = -\frac{1}{3}.$$

$$\therefore \sqrt[3]{5} = \frac{(2)(-\frac{1}{3}) - (3)(\frac{2}{3})}{2^2} = \frac{-\frac{2}{3} - 2}{4} = \frac{-2 - 6}{3} \cdot \frac{1}{4} = \frac{-8}{3} \cdot \frac{1}{4} = -\frac{2}{3}.$$

NOTE: If $f(x)$ is differentiable, then

$$\frac{d}{dx} \left[\frac{1}{f(x)} \right] = -\frac{f'(x)}{(f(x))^2}.$$

$$\text{Proof: } \frac{d}{dx} \left[\frac{1}{f(x)} \right] = \frac{f(x)(0) - 1 \cdot f'(x)}{(f(x))^2} = -\frac{f'(x)}{(f(x))^2}$$

$$y = \frac{1}{x^4 + x^2 + 1} \Rightarrow \frac{dy}{dx} = \frac{-(4x^3 + 2x)}{(x^4 + x^2 + 1)^2}$$

The End

Sec. 3.3* Rate of change in Physics *

objective: To solve velocity problems using differentiation rules

Ex. 1 The position of a particle is given by: $S = f(t) = t^3 - 6t^2 + 9t$
 199 t : in seconds, S : in meters.

a. Find v at t (velocity)

$$v(t) = \frac{ds}{dt} = f'(t) = 3t^2 - 12t + 9$$

b. What is the velocity after 2 s? 4 s?

$$\text{i)} v(2) = \frac{ds}{dt} \Big|_{t=2} = 3(2)^2 - 12(2) + 9 = -3 \text{ m/s}$$

$$\text{ii)} v(4) = \frac{ds}{dt} \Big|_{t=4} = 3(4)^2 - 12(4) + 9 = 9 \text{ m/s}$$

c. When is the particle at rest?

The particle is at rest when $v(t) = 0$

$$\Rightarrow 3t^2 - 12t + 9 = 0 \quad (\div 3) \Rightarrow t^2 - 4t + 3 = 0$$

$$(t-1)(t-3) = 0 \Rightarrow t=1, t=3$$

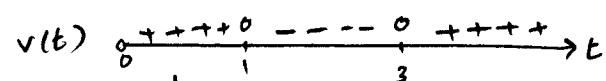
It is at rest after 1 s and after 3 s.

d. When the particle is moving forward (positive direction)

It moves forward when $v(t) > 0$

$$3t^2 - 12t + 9 > 0 \Rightarrow t^2 - 4t + 3 > 0$$

$$\Rightarrow (t-1)(t-3) > 0$$

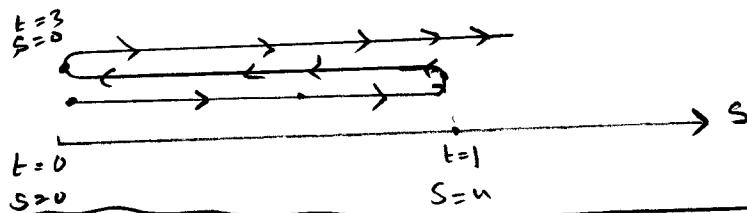


\Rightarrow It moves forward in the time intervals

$t < 1$ and $t > 3$ (or $(-\infty, 1) \cup (3, \infty)$)

\Rightarrow It moves backward on the time interval $1 < t < 3$ or $(1, 3)$

e. Draw a diagram to represent the motion of a particle.



f. Find the total distance traveled by the particle during the first 5 seconds.
 We need to compute the distance on $[0, 1]$, $[1, 3]$, $[3, 5]$ (note $v(t) \rightarrow$ sign above).

$$|f(1) - f(0)| = |4 - 0| = 4 \text{ m}$$

$$|f(3) - f(1)| = |0 - 4| = 4 \text{ m}$$

$$|f(5) - f(3)| = |20 - 0| = 20 \text{ m} \Rightarrow \text{The total distance} = 4 + 4 + 20$$

$$= 28 \text{ m}$$

Q.10, $v_0 = 80 \text{ ft/s}$, $s = 80t - \frac{16}{2} t^2$

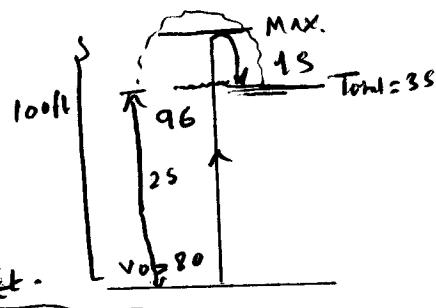
a. What is the maximum height reached by a ball?

At max. height $v(t) = 0$

$$v(t) = \frac{ds}{dt} = 80 - 32t \stackrel{\text{set } 0}{=} 0$$

$$80 = 32t \Rightarrow t = \frac{80}{32} = \frac{5}{2} \text{ second.}$$

$$\therefore \text{The max. height} = s\left(\frac{5}{2}\right) = 80\left(\frac{5}{2}\right) - 16\left(\frac{5}{2}\right)^2 \\ = 200 - 100 = 100 \text{ feet.}$$



b. What is the velocity when it is 96 ft above the ground when its way up? Way down.

i) When its way up.

$$s(t) = 96 \Rightarrow 80t - 16t^2 = 96$$

$$16t^2 - 80t + 96 = 0 \quad (\div 16) \Rightarrow t^2 - 5t + 6 = 0$$

$$(t-3)(t-2) = 0 \Rightarrow t=2 \text{ or } t=3.$$

$$\Rightarrow \text{Way up} \Rightarrow t=2$$

$$\therefore \text{velocity} = v(2) = \frac{ds}{dt} \Big|_{t=2} = 80 - 32(2) = 16 \text{ ft/s}$$

ii) A way down $\Rightarrow t=3 \text{ s}$.

$$v(3) = \frac{ds}{dt} \Big|_{t=3} = 80 - 32(3) = -16 \text{ ft/s}$$

The End

Derivatives Of Trigonometric Functions

See 3.4

Objectives

1. To introduce the derivatives of trigonometric functions.
2. Consider $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

Rules :

$$\begin{aligned} 1. \frac{d}{dx}(\sin x) &= \cos x & 2. \frac{d}{dx}(\cos x) &= -\sin x & 3. \frac{d}{dx}(\tan x) &= \sec^2 x \\ 4. \frac{d}{dx}(\sec x) &= \sec x \tan x & 5. \frac{d}{dx}(\csc x) &= -\csc x \cot x & 6. \frac{d}{dx}(\cot x) &= -\csc^2 x \end{aligned}$$

Note : $\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = 0$

Ex, Differentiate:

Q. 9 1. $y = 2 \csc x + 5 \cos x \Rightarrow y' = -2 \csc x \cot x - 5 \sin x$

Q. 8 2. $y = e^u (\cos u + \sin u)$

$$\frac{dy}{du} = e^u (-\sin u + c) + (\cos u + u) e^u = e^u [-\sin u + c + \cos u + u]$$

Q. 12 2. $y = \frac{\tan x - 1}{\sec x}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\sec x (\sec^2 x) - (\tan x - 1)(\sec x \tan x)}{\sec^2 x} = \frac{\sec^3 x - \sec x \tan^2 x + \sec x \tan x}{\sec^2 x} \\ &= \frac{\sec x (\sec^2 x - \tan^2 x + \tan x)}{\sec^2 x} \quad \text{But } 1 + \tan^2 x = \sec^2 x \\ &\Rightarrow \sec^2 x - \tan^2 x = 1 \\ &= \frac{1 + \tan x}{\sec x} \end{aligned}$$

Ex, Find the tangent eqn. to the curve on the given point.

Q. 21 1. $y = \tan x, (\frac{\pi}{4}, 1)$

$$\frac{dy}{dx} = \sec^2 x \Rightarrow m = \frac{dy}{dx} \Big|_{x=\frac{\pi}{4}} = \sec^2\left(\frac{\pi}{4}\right) = 2$$

$$\therefore \text{The tangent eqn. } y - y_1 = m(x - x_1) \Rightarrow y - 1 = 2(x - \frac{\pi}{4})$$

$$y = 2x - \frac{\pi}{2} + 1.$$

Q. 29 : For what values $f(x) = x + 2 \sin x$ does f have a horizontal tangent?

Horizontal tangent $\Rightarrow m = 0 \Rightarrow f'(x) = 1 + 2 \cos x \stackrel{\text{set}}{=} 0 \Rightarrow \cos x = -\frac{1}{2}$

or $x_1 = \frac{2\pi}{3} + 2n\pi \quad n \text{ is integer}$

$x_2 = \frac{4\pi}{3} + 2n\pi \quad n \text{ is integer}$

But both are at $\pm \frac{\pi}{3}$ of $\pi, 3\pi, 5\pi, \dots$ etc $\Rightarrow x = (2n+1)\pi \pm \frac{\pi}{3}$

n is an integer



See 3.4

(2)

Fact, $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ And $\lim_{x \rightarrow 0} \frac{x}{\sin x} = 1$.

Ex: Find the limit

Q.35 : $\lim_{x \rightarrow 0} \frac{\sin 3x}{x} = \lim_{x \rightarrow 0} \frac{\sin 3x}{x} \cdot \frac{3}{3} = 3 \lim_{x \rightarrow 0} \frac{\sin 3x}{3x}$ put $\theta = 3x$
As $x \rightarrow 0$, $\theta \rightarrow 0$
 $= 3 \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 3(1) = 3$

Q.36 : $\lim_{x \rightarrow 0} \frac{\sin 4x}{\sin 6x} = \lim_{x \rightarrow 0} \frac{\frac{1}{x} \sin 4x}{\frac{1}{x} \sin 6x} \Rightarrow \lim_{x \rightarrow 0} \frac{\sin 4x}{x} \cdot \frac{4}{4}$
 $= \frac{4}{6} \lim_{x \rightarrow 0} \frac{\sin 4x}{4x} \cdot \frac{\sin 6x}{6x} = \frac{(4)(1)}{(6)(1)} = \frac{4}{6} = \frac{2}{3}$.

Note: In general $\lim_{x \rightarrow 0} \frac{\sin ax}{bx} = \frac{a}{b}$ and $\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx} = \frac{a}{b}$

Q.38 : $\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\sin \theta} = \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\sin \theta} \cdot \frac{\cos \theta + 1}{\cos \theta + 1} = \lim_{\theta \rightarrow 0} \frac{\cos^2 \theta - 1}{\sin \theta (\cos \theta + 1)}$
 $= - \lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \theta}{\sin \theta (\cos \theta + 1)} = - \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\sin \theta (\cos \theta + 1)}$
 $= - \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\cos \theta + 1} = - \frac{0}{1+1} = \frac{0}{2} = 0$

Ex: Evaluate the following limits.

a. $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin(x + \frac{\pi}{4}) - 1}{x - \frac{\pi}{4}}$. let $\theta = x - \frac{\pi}{4} \Rightarrow x = \theta + \frac{\pi}{4}$.

$$\begin{aligned} &= \lim_{\theta \rightarrow 0} \frac{\sin(\theta + \frac{\pi}{4} + \frac{\pi}{4}) - 1}{\theta} \quad \text{as } x \rightarrow \frac{\pi}{4}, \theta \rightarrow 0 \\ &= \lim_{\theta \rightarrow 0} \frac{\sin(\theta + \frac{\pi}{2}) - 1}{\theta} \\ &= \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = 0 \end{aligned}$$

But $\sin(\frac{\pi}{2} - x) = \cos x$
 $\sin(\frac{\pi}{2} - (-\theta)) = \cos(-\theta) \underset{\text{even}}{\sim} \cos \theta$

b. $\lim_{x \rightarrow \pi} \frac{\sin 2x}{\sin x} \cdot \frac{0}{0} = \lim_{x \rightarrow \pi} \frac{2 \sin x \cos x}{\sin x} = 2 \lim_{x \rightarrow \pi} \cos x = 2(-1) = -2$

The End.

The Chain Rule

Objectives, 1. To introduce the chain rule

2. $s \rightarrow s^p \rightarrow \text{power} \rightarrow \text{combined with the chain rule}$.

The Chain Rule, If f and g are both differentiable functions and $F(x) = f(g(x))$
 $= f(g(x))$, then $F(x)$ is diff. and:
 $F'(x) = f'(g(x)) \cdot g'(x)$.

In Leibniz notation: If $y = f(u)$, $u = g(x)$ are diff. then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Ex: Write the composite function and find $\frac{dy}{dx}$.

Q.1 224, $y = \sin 4x$

$$y = \sin u, u = 4x \Rightarrow u = g(x) = 4x, y = f(u) = \sin u.$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u \cdot (4) = 4 \cos 4x.$$

Q.4 224, $y = \tan(\sin x)$

$$\text{Let } y = f(u) = \tan u, u = g(x) = \sin x$$

$$\frac{dy}{du} = \sec^2 u \quad \frac{du}{dx} = \cos x$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \sec^2 u \cdot \cos x = \sec^2(\sin x) \cdot \cos x.$$

Ex: Find the derivative:

Q.9 224, $F(x) = (x^2 - x + 1)^3$. let $f(x) = x^3$, $g(x) = x^2 - x + 1$
 $f'(x) = 3x^2$, $g'(x) = 2x - 1$

$$\text{then } F(x) = f(g(x))$$

$$F'(x) = f'(g(x)) \cdot g'(x)$$

$$\begin{aligned} F'(x) &= f'(x^2 - x + 1) \cdot (2x - 1) \\ &= 3(x^2 - x + 1)^2 (2x - 1) \end{aligned}$$

Q.12 224, $f(t) = \sqrt[3]{1 + \tan t} = (1 + \tan t)^{\frac{1}{3}}$

$$\text{let } h(t) = t^{\frac{1}{3}}, g(t) = 1 + \tan t \text{ then: } h'(t) = \frac{1}{3} t^{-\frac{2}{3}}$$

$$f(t) = (h \circ g)(t) = h(g(t))$$

$$g'(t) = \sec^2 t.$$

$$\begin{aligned} \therefore f'(t) &= h'(g(t)) g'(t) \\ &= h'(1 + \tan t) \cdot \sec^2 t \\ &= \frac{1}{3} (1 + \tan t)^{-\frac{2}{3}} \cdot \sec^2 t \end{aligned}$$

$$= \frac{\sec^2 t}{3(1 + \tan t)^{\frac{2}{3}}} = \frac{\sec^2 t}{3 \sqrt[3]{(1 + \tan t)^2}}.$$

Q.53, $F(x) = f(g(x))$ and $g(3) = 6, g'(3) = 4, f'(3) = 2, f'(6) = 7$
225 Find $F'(3)$.

$$F'(x) = f'(g(x)) \cdot g'(x)$$

$$F'(3) = f'(g(3)) \cdot g'(3) = f'(6) \cdot (4) = (7)(4) = 28$$

The Power Rule combined with the chain Rule:

If n is any real number and $u = g(x)$ is diff., then

$$\frac{d}{dx}(u^n) = n u^{n-1} \cdot \frac{du}{dx}$$

$$\text{or}, \quad \frac{d}{dx}[g(x)]^n = n [g(x)]^{n-1} \cdot g'(x)$$

Ex. Differentiate:

Q.17, $g(x) = (1+4x)^5 (3-x+x^2)^8$

$$\begin{aligned} g'(x) &= (1+4x)^5 \cdot 8(3-x+x^2)^7(-1-2x) + (3-x+x^2)^8 \cdot 5(1+4x)^4 \cdot (4) \\ &= 4(1+4x)^4(3-x+x^2)^7 [2(1+4x)(1-2x) + 5(3-x+x^2)]. \end{aligned}$$

Q.10, $f(x) = (1+x^4)^{\frac{2}{3}}$

$$f'(x) = \frac{2}{3}(1+x^4)^{-\frac{1}{3}}(4x^3) = \frac{8x^3}{3\sqrt[3]{1+x^4}}.$$

NOTE, 1. For any trigonometric function say $y = \sin(g(x))$, $\frac{dy}{dx} = \cos g(x) \cdot g'(x)$
 2. \Rightarrow exponential func of the form $y = e^{g(x)}$ then $\frac{dy}{dx} = g'(x) e^{g(x)}$
 where $g(x)$ is differentiable.

* Ex. Differentiate:

$$-5x \cos 3x$$

Q.22, $y = e^{-5x \cos 3x}$

$$\begin{aligned} \frac{dy}{dx} &= (-5x \cos 3x)' \cdot e^{-5x \cos 3x} \\ &= (-5x \cdot (-3 \sin 3x) + \cos 3x \cdot (-5)) e^{-5x \cos 3x} \\ &= (15x \sin 3x - 5 \cos 3x) e^{-5x \cos 3x}. \end{aligned}$$

Q.32, $y = \tan^2(3\theta) = (\tan 3\theta)^2$

$$\frac{dy}{d\theta} = 2(\tan 3\theta)^1 \cdot \sec^2 3\theta \cdot 3 = 6 \sec^2 3\theta \tan 3\theta.$$

Rule, General rule of expon. function derivation.

$$\frac{d}{dx}[a^x] = a^x \ln a \quad a \text{ is constant.}$$

Ex. Find $\frac{dy}{dx}$: $y = 10^x \Rightarrow y' = 10^x \ln 10$

Q.24, $y = 10^{1-x^2}$

Let $y = 10^u$, $u = 1-x^2$

$$\frac{dy}{du} = 10^u \cdot \ln(10) \quad \frac{du}{dx} = -2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 10^u \cdot \ln(10) \cdot (-2x) = -2x \ln(10) 10^{1-x^2}.$$

NOTE: Suppose that $y = f(u)$, $u = g(x)$, $x = h(t)$ where f, g and h are differentiable then

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dx} \cdot \frac{dx}{dt}.$$

Ex, Differentiate:

Q.38, $y = \sin(\sin(\sin x))$

$$\frac{dy}{dx} = \cos(\sin(\sin x)) \cdot \frac{d}{dx}(\sin(\sin x))$$

$$= \cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot \frac{d}{dx}(\sin x)$$

$$= \cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot \cos x.$$

Ex, Find the equ. of the tangent line at the given point-

Q.44, $y = \sin x + \sin^2 x$, $(0, 0)$

$$y' = \cos x + 2 \sin x \cos x = \cos x + \sin 2x$$

$$m = y'(0) = \cos(0) + \sin(2 \cdot 0) = 1 + 0 = 1$$

The tangent-eqn. $y - y_1 = m(x - x_1)$
 $y - 0 = 1(x - 0) \Rightarrow y = x$

Q.62: Suppose f is diff. on \mathbb{R} and α is a real number. Let
 $F(x) = f(x^\alpha)$, $G(x) = [f(x)]^\alpha$. Finds a. $F'(x)$ b. $G'(x)$

a. $F'(x) = f'(x^\alpha) \cdot \alpha x^{\alpha-1} = \alpha x^{\alpha-1} f'(x^\alpha).$

b. $G'(x) = \alpha [f(x)]^{\alpha-1} \cdot f'(x)$

Ex.10, Differentiate: $y = e^{\sec 3\theta}$

$$\frac{dy}{d\theta} = e^{\sec 3\theta} \cdot \frac{d}{d\theta}(\sec 3\theta)$$

$$= e^{\sec 3\theta} \cdot \sec 3\theta \cdot \tan 3\theta \cdot \frac{d}{d\theta}(3\theta)$$

$$= 3 e^{\sec 3\theta} \cdot \sec 3\theta \tan 3\theta.$$

The End

Implicit DifferentiationObjectives

1. To explain the method of implicit differentiation
2. = differentiate the inverse trigonometric functions
3. = consider the orthogonal trajectories.

If $y = f(x)$, then y is described explicitly in terms of x as $y = x^3 - 2x + 1$

But for a. $x^2 + y^2 = 14$ b. $x^3 + 2x^2y + y^2 = 3xy$

y is not described explicitly in terms of x , but y is defined implicitly by a relation between x and y .

Implicit differentiation: It is a way to find $\frac{dy}{dx}$ when y is defined implicitly by a relation between x and y .

Method to find $\frac{dy}{dx}$: 1. Differentiate both sides of the equ. where for the terms that contain x differentiate as usual, and for the terms containing y multiply the derivative by $\frac{dy}{dx}$.
2. Solve the equ. for $\frac{dy}{dx}$.

Ex. Find $\frac{dy}{dx}$:

Q.6 $x^2 - y^2 = 1 \Rightarrow 2x - 2y \frac{dy}{dx} = 0 \Rightarrow -2y \frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{x}{y}$.
233

Q.10, $y^5 + x^2y^3 = 1 + ye^{x^2}$.

233 $5y^4 \cdot \frac{dy}{dx} + (x^2 \cdot 3y^2 \frac{dy}{dx} + y^3 \cdot 2x) = 0 + y \cdot 2xe^{x^2} + e^{x^2} \cdot \frac{dy}{dx}$
 $5y^4 \frac{dy}{dx} + 3x^2y^2 \frac{dy}{dx} + 2xy^3 = 2xye^{x^2} + e^{x^2} \frac{dy}{dx}$

$5y^4 \frac{dy}{dx} + 3x^2y^2 \frac{dy}{dx} - e^{x^2} \frac{dy}{dx} = 2xye^{x^2} - 2xy^3$

$\frac{dy}{dx} (5y^4 + 3x^2y^2 - e^{x^2}) = 2xy(e^{x^2} - y^2) \Rightarrow \frac{dy}{dx} = \frac{2xy(e^{x^2} - y^2)}{5y^4 + 3x^2y^2 - e^{x^2}}$

Q.32 a) For the equ.: $y^2 = x^3 + 3x^2$ Find the tangent equ. at $(1, -2)$

234 $2yy' = 3x^2 + 6x \Rightarrow y' = \frac{3x^2 + 6x}{2y}$

$m = y'_{(1,-2)} = \frac{3(1)^2 + 6(1)}{2(-2)} = -\frac{9}{4}$.

\Rightarrow The tangent equ. is: $y - y_1 = m(x - x_1)$

$$y + 2 = -\frac{9}{4}(x - 1)$$

$$y = -\frac{9}{4}x + \frac{1}{4}$$

\rightarrow Q. 32 b) At what points does this curve have a horizontal asymptote?

Set $y' = 0 \Rightarrow \frac{3x^2 + 6x}{2y} = 0, y \neq 0$

$$\Rightarrow 3x^2 + 6x = 0 \Rightarrow 3x(x+2) = 0 \Rightarrow x=0, x=-2$$

So, there are two cases:

i) $x=0 \Rightarrow y^2 = 0 + 0 = 0 \Rightarrow y=0 \approx (0,0)$ But $y \neq 0$ is rejected.

ii) $x=-2 \Rightarrow y^2 = (-2)^3 + 3(-2)^2 = -8 + 12 = 4 \Rightarrow y = \pm 2$.

∴ There are two points $(-2, -2)$ and $(-2, 2)$.

Q. 12, Find y' if: $1+x = \sin(xy^2)$

$$0+1 = \cos(xy^2)(xy^2)'$$

$$1 = \cos(xy^2)[x \cdot 2yy' + y^2 \cdot 1]$$

$$\Rightarrow 1 = 2xyy'\cos(xy^2) + y^2\cos(xy^2) \Rightarrow 2xy\cos(xy^2)y' = 1 - y^2\cos(xy^2)$$

$$\therefore y' = \frac{1 - y^2\cos(xy^2)}{2xy\cos(xy^2)}$$

*Orthogonal Trajectories

Def: Two curves are called orthogonal if at each point of intersection their tangent lines are perpendicular.

Ex: Show that the given families of curves are orthogonal trajectories of each other.

Q. 62; $y = ax^3 \Rightarrow x^2 + 3y^2 = b$

$\leftarrow \frac{dy}{dx} = 3ax^2 ; 2x + 6y y' = 0 \Rightarrow y' = -\frac{2x}{6y} = -\frac{x}{3y}$

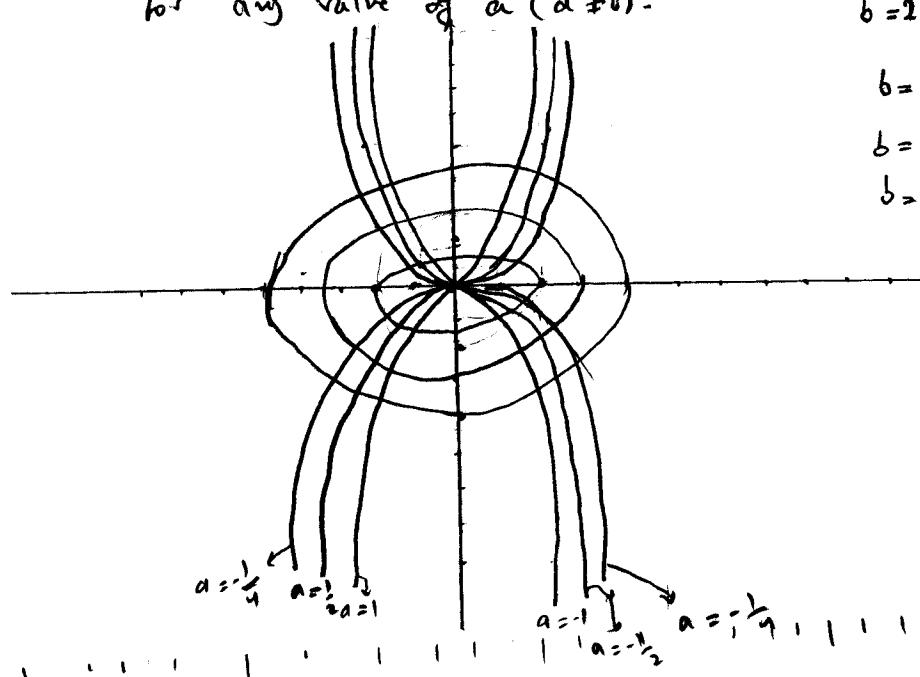
But at the point of intersection $y = ax^3$

$$\Rightarrow y' = \frac{-x}{3(ax^3)} = \frac{-x}{3ax^3} = \frac{1}{3ax^2}$$

NOTE that $m_1 \cdot m_2 = 3ax^2 \cdot \frac{-1}{3ax^2} = -1 \Rightarrow$ tangents are perpendicular.

for any value of a ($a \neq 0$).

$$\begin{aligned} a=1 &\rightarrow y=x^3 \\ a=\frac{1}{2} &\rightarrow y=\frac{1}{2}x^3 \\ a=\frac{1}{4} &\rightarrow y=\frac{1}{4}x^3 \\ a=-1 &\rightarrow y=-x^3 \\ a=-\frac{1}{2} &\rightarrow y=-\frac{1}{2}x^3 \\ a=-\frac{1}{4} &\rightarrow y=-\frac{1}{4}x^3 \end{aligned}$$



$$\begin{aligned} b=2 &\Rightarrow x^2 + 3y^2 = 2 \\ &\quad \frac{x^2}{1} + \frac{y^2}{\frac{2}{3}} = 2 \\ b=4 &\Rightarrow x^2 + 3y^2 = 4 \\ b=6 &\Rightarrow x^2 + 3y^2 = 6 \\ b=10 &\Rightarrow x^2 + 3y^2 = 10 \end{aligned}$$

Ex. Show that the two curves are orthogonal

$$\text{Q.56} \quad \frac{235}{x^2 - y^2 = 5}, \quad 4x^2 + 9y^2 = 72$$

$$2x - 2yy' = 0$$

$$2yy' = 2x$$

$$y' = \frac{2x}{2y} = \frac{x}{y}$$

$$8x + 18y y' = 0 \Rightarrow 18y y' = -8x$$

$$y' = \frac{-8x}{18y} = \frac{-4x}{9y}$$

$$\text{At the intersection point: } x^2 - y^2 = 5 \Rightarrow y^2 = x^2 - 5.$$

$$4x^2 + 9y^2 = 72 \Rightarrow y^2 = \frac{72 - 4x^2}{9}.$$

$$\therefore x^2 - 5 = \frac{72 - 4x^2}{9}$$

$$\Rightarrow 9x^2 - 45 = 72 - 4x^2$$

$$13x^2 = 72 + 45 = 117 \Rightarrow x^2 = \frac{117}{13} = 9 \Rightarrow x = \pm 3$$

$$\text{when } x = -3 \Rightarrow y^2 = 9 - 5 = 4 \Rightarrow y = \pm 2 \Rightarrow (-3, -2), (-3, 2)$$

$$x = 3 \Rightarrow y^2 = 9 - 5 = 4 \Rightarrow y = \pm 2 \Rightarrow (3, -2), (3, 2)$$

$$\text{i) At } (-3, -2) \Rightarrow m_1 = \frac{-3}{-2} = \frac{3}{2}, \quad m_2 = \frac{-4(-3)}{9(-2)} = -\frac{2}{3} \Rightarrow m_1 \cdot m_2 = -1$$

$$\text{ii) At } (-3, 2) \Rightarrow m_1 = \frac{-3}{2}, \quad m_2 = \frac{-4(-3)}{9(2)} = \frac{2}{3} \Rightarrow m_1 \cdot m_2 = -1$$

$$\text{iii) At } (3, -2) \Rightarrow m_1 = \frac{3}{-2} = -\frac{3}{2}, \quad m_2 = \frac{-4(3)}{9(-2)} = \frac{2}{3} \Rightarrow m_1 \cdot m_2 = -1$$

$$\text{iv) At } (3, 2) \Rightarrow m_1 = \frac{3}{2}, \quad m_2 = \frac{-4(3)}{9(2)} = -\frac{2}{3} \Rightarrow m_1 \cdot m_2 = -1$$

The two curves are orthogonal.

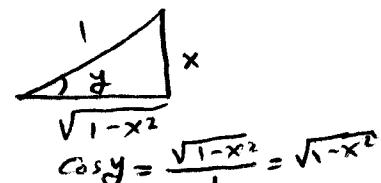
*Derivatives of Inverse Trigonometric Functions

Consider $y = \sin^{-1}(x) \Rightarrow \sin y = x \Rightarrow -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$$\cos y y' = 1 \Rightarrow y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}.$$

$$\therefore \frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}.$$

and similarly for others.



$$\rightarrow \frac{d}{dx}(\cos^{-1}(x)) = \frac{-1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}, \quad \frac{d}{dx}(\csc^{-1}(x)) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{x\sqrt{x^2-1}}, \quad \frac{d}{dx}(\cot^{-1}(x)) = -\frac{1}{1+x^2}.$$

Ex. Find the derivative of:

$$\text{Q.42} \quad \frac{234}{y = \sqrt{\tan^{-1}(x)}} \Rightarrow \frac{dy}{dx} = \frac{1}{2} (\tan^{-1}(x))^{-\frac{1}{2}} \cdot \frac{1}{1+x^2}$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{\tan^{-1}(x)}} \cdot \frac{1}{1+x^2}$$

$$= \frac{1}{2(1+x^2)\sqrt{\tan^{-1}(x)}}.$$

Q.44, 234, $h(x) = \sqrt{1-x^2} \arcsin(x)$

$$h'(x) = \sqrt{1-x^2} \cdot \frac{1}{\sqrt{1-x^2}} + \arcsin(x) \cdot \frac{-2x}{2\sqrt{1-x^2}}$$

$$= 1 + \frac{-x \arcsin(x)}{\sqrt{1-x^2}} = 1 - \frac{x \arcsin(x)}{\sqrt{1-x^2}}$$

Q.67, a) Suppose that f is 1-1 diff. show that $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

Proof: Let $y = f^{-1}(x) \Rightarrow f(y) = f(f^{-1}(x))$

$$\Rightarrow f(y) = x \\ f'(y) \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

$$\therefore (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

b) If $f(u)=5$, $f'(u)=\frac{2}{3}$, find $(f')'(5)$, $f(u)=5 \Rightarrow u=f^{-1}(5)$

$$(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(u)} = \frac{1}{\frac{2}{3}} = \frac{3}{2}.$$

The End.

Objectives: 1. To define the higher derivatives

2. To find a formula for the n -th derivative.

Def. If f is differentiable then f' is also a func., if f' is diff. then

$y'' = (f')' = f''$ is called the second derivative.

$y''' = (f'')' = f'''$... third =

$y^{(4)} = (f''')' = f^{(4)}$... forth =

\vdots ... \vdots ... \vdots ... \vdots

$y^{(n)} = (f^{(n-1)})' = f^{(n)}$... n -th =

These are called
higher derivatives

In Leibniz notation: $f' = \frac{dy}{dx}$

$$f'' = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2y}{dx^2}$$

$$f''' = \frac{d^3y}{dx^3}, \dots, f^{(n)} = \frac{d^n y}{dx^n}.$$

Another notation: $f'(x) = Df(x)$

$$f''(x) = D^2f(x), f'''(x) = D^3f(x), \dots, f^{(n)}(x) = D^n f(x)$$

Ex. Find the first and second derivatives

Q.5: $f(x) = x^5 + 6x^2 - 7x$

$$f'(x) = 5x^4 + 12x - 7, f''(x) = 20x^3 + 12.$$

Q.8: $y = \theta \sin \theta$

$$\frac{dy}{d\theta} = \theta \cos \theta + \sin \theta, \frac{d^2y}{d\theta^2} = \theta(-\sin \theta) + \cos \theta + \cos \theta \\ = -\theta \sin \theta + 2 \cos \theta.$$

Q.20: $h(x) = \tan^{-1}(x^2)$

$$h'(x) = \frac{1}{1+(x^2)^2} \cdot 2x = \frac{2x}{1+x^4}.$$

$$h''(x) = \frac{(1+x^4)(2) - 2x(4x^3)}{(1+x^4)^2} = \frac{2+2x^4-8x^4}{(1+x^4)^2} = \frac{2-6x^4}{(1+x^4)^2}.$$

Application in physics: If $S = S(t)$ is the position function of an object

moves in a straight line then $v(t) = S'(t)$ is the velocity at time t .

The instantaneous rate of change of velocity with respect to t is called the acceleration denoted by $a(t) = v'(t) = S''(t)$

$$\text{or } a = \frac{dv}{dt} = \frac{d^2S}{dt^2}.$$

And the derivative of $a(t) = a'(t) = S'''(t) = \frac{d^3S}{dt^3}$ is called jerk

Q.46, 241, $S = 2t^3 - 7t^2 + 4t + 1$, $t \geq 0$

a) Find v, a at t

$$v(t) = \frac{ds}{dt} = 6t^2 - 14t + 4 ; a(t) = \frac{d^2s}{dt^2} = v'(t) = 12t - 14$$

b) a(t) after 1s? $a(1) = 12(1) - 14 = -2 \text{ m/s}^2$.

c) The acceleration when velocity is 0

$$\text{Set } v(t) = 0 \Rightarrow 6t^2 - 14t + 4 = 0 (\div 2) \Rightarrow 3t^2 - 7t + 2 = 0$$

$$(3t - 1)(t - 2) = 0 \Rightarrow t = \frac{1}{3}, t = 2.$$

- There are two cases i) when $t = \frac{1}{3} \Rightarrow a(\frac{1}{3}) = 12(\frac{1}{3}) - 14 = -10 \text{ m/s}^2$
- ii) when $t = 2 \Rightarrow a(2) = 12(2) - 14 = 10 \text{ m/s}^2$.

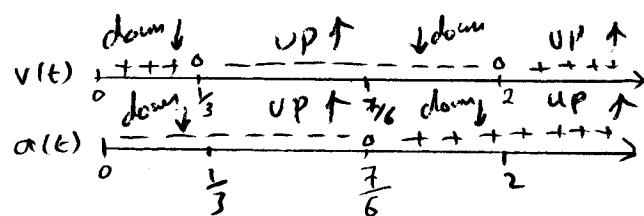
NOTE: The particle is speeding up when $v(t)$, $a(t)$ have the same sign, and slowing down when $v(t)$ and $a(t)$ have different signs.

Ex: Back to Q.46, When is the particle speeding up? When it is slowing down?

$$v(t) = 6t^2 - 14t + 4 = 0$$

$$\Rightarrow t = \frac{1}{3}, t = 2$$

$$a(t) = 12t - 14 \Rightarrow t = \frac{14}{12} = \frac{7}{6}.$$



The particle is speeding up when: $\frac{1}{3} < t < \frac{7}{6}$ and $t > 2$
or $(\frac{1}{3}, \frac{7}{6}) \cup (2, \infty)$

And it is slowing down when: $0 \leq t < \frac{1}{3}$, $(\frac{7}{6}, 2)$
or $[0, \frac{1}{3}) \cup (\frac{7}{6}, 2]$.

Ex: Find y''' at the given point

Q.28, If $g(x) = \sec x$, find $g'''(\frac{\pi}{4})$

$$g'(x) = \sec x \tan x$$

$$g''(x) = \sec \cdot \sec^2 x + \tan x \cdot \sec \tan x$$

$$= \sec^3 x + \sec x \tan^2 x.$$

$$g'''(x) = 3 \sec^2 x \cdot \sec x \tan x + \sec x \cdot 2 \tan x \cdot \sec^2 x + \tan^2 x \cdot \sec x \tan x$$

$$= 3 \sec^3 x \tan x + 2 \sec^3 x \tan^2 x + \sec x \tan^3 x.$$

$$= 5 \sec^3 x \tan x + \sec x \tan^3 x$$

$$g'''(\frac{\pi}{4}) = 5 \sec^3(\frac{\pi}{4}) \tan(\frac{\pi}{4}) + \sec(\frac{\pi}{4}) \cdot \tan^3(\frac{\pi}{4})$$

$$= 5(\sqrt{2})^3(1) + \sqrt{2}(1)^3 = 10\sqrt{2} + \sqrt{2}$$

$$= 11\sqrt{2}.$$

Ex-5, Find y'' if $x^4 + y^4 = 16$.

Sec. 3.7

(3)

$$\begin{aligned}
 & 4x^3 + 4y^3 \cdot y' = 0 \Rightarrow y' = -\frac{4x^3}{4y^3} = -\frac{x^3}{y^3}. \\
 & y' = -\frac{x^3}{y^3} \\
 & y'' = -\left(\frac{y^3 \cdot 3x^2 - x^3 \cdot 3y^2 y'}{(y^3)^2} \right) = -\frac{3x^2y^3 - 3x^3y^2 \cdot \left(-\frac{x^3}{y^3}\right)}{y^6} \\
 & = -\frac{3x^2y^3 + \frac{3x^6}{y}}{y^6} = -\frac{3x^2y^4 + 3x^6}{y^6} \cdot \frac{1}{y} \\
 & = -\frac{3x^2(y^4 + x^4)}{y^7} \quad \text{but } x^4 + y^4 = 16 \\
 & \Rightarrow y'' = -\frac{3x^2(16)}{y^7} = -48 \frac{x^2}{y^7}.
 \end{aligned}$$

Ex. Find a formula for $f^{(n)}(x)$

Q-35, $f(x) = e^{2x}$

$$f'(x) = 2e^{2x}$$

$$f''(x) = 2 \cdot 2 e^{2x} = 2^2 e^{2x}$$

$$f'''(x) = 2 \cdot 2 \cdot 2 e^{2x} = 2^3 e^{2x}$$

$$f^{(4)}(x) = 2 \cdot 2 \cdot 2 \cdot 2 e^{2x} = 2^4 e^{2x}$$

$$f^{(n)}(x) = \underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdots 2 e^{2x}}_{n-terms} = 2^n e^{2x}. \Rightarrow f^{(n)}(x) = 2^n e^{2x}.$$

Q-36, $f(x) = \sqrt{x} = x^{1/2}$.

$$n=1 \rightarrow f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2} x^{-1/2}$$

$$n=2 \rightarrow f''(x) = \frac{1}{2} \cdot \left(-\frac{1}{2}\right) x^{-3/2} = \frac{1 \cdot (-1)x^{-3/2}}{2 \cdot 2} = -\frac{1 \cdot 1}{2^2} x^{-3/2}$$

$$n=3 \rightarrow f'''(x) = \frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} x^{-5/2} = \frac{1 \cdot (-1) \cdot 3}{2^3} x^{-5/2} = +\frac{1 \cdot 1 \cdot 3}{2^3} x^{-5/2}$$

$$n=4 \rightarrow f^{(4)}(x) = \frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} \cdot \frac{-5}{2} x^{-7/2} = \frac{1 \cdot (-1) \cdot (-3) \cdot (-5)}{2^4} x^{-7/2} = -\frac{1 \cdot 3 \cdot 5}{2^4} x^{-7/2}$$

$$n=5 \rightarrow f^{(5)}(x) = \frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} \cdot \frac{-5}{2} \cdot \frac{-7}{2} x^{-9/2} = \frac{1 \cdot (-1) \cdot (-3) \cdot (-5) \cdot (-7)}{2^5} x^{-9/2} = -\frac{1 \cdot 3 \cdot 5 \cdot 7}{2^5} x^{-9/2}$$

$$\Rightarrow f^{(n)}(x) = \frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} \cdot \frac{-5}{2} \cdot \frac{-7}{2} \cdot \left(\frac{1}{2} - n+1\right) x^{-\frac{(2n-1)}{2}}$$

$$f^{(n)}(x) = (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n} x^{-\frac{(2n-1)}{2}}.$$

NOTE: $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n$

D $\overset{\hat{j}-1}{2x} \cos x$

Example 6

Ex. 6, Find $D^{27} \cos x$.

$$\text{Q. 39: } D \cos x = -\sin x$$

It occurs in a cycle of length 4

$$D^2 \cos x = -\cos x$$

and in general $D^{4n} \cos x = \cos x$.

$$D^3 \cos x = \sin x$$

$$D^4 \cos x = \cos x = f(x)$$

$$D^5 \cos x = -\sin x = D \cos x$$

$$\therefore D^{27} \cos x = D^3 \cos x = \sin x.$$

Q. 39; $D^{103} \cos 2x$

241

$$D \cos 2x = -2 \sin 2x$$

$$D^2 \cos 2x = -2 \cdot 2 \cos 2x = -2^2 \cos 2x$$

$$D^3 \cos 2x = -2^2 \cdot -2 \sin 2x = 2^3 \sin 2x$$

$$D^4 \cos 2x = 2^3 \cdot 2 \cos 2x = 2^4 \cos 2x$$

The successive derivatives occur in a cycle of length 4

$$\Rightarrow D^{103} \cos 2x = 2^3 \cdot D^3 \cos 2x = 2^3 \cdot \sin 2x$$

The End

Derivatives Of Logarithmic FunctionsObjectives:

1. To define the derivative of logarithmic functions.
2. \Rightarrow use the logarithmic differentiation method
3. \Rightarrow define e as a limit

Fact: $\frac{d}{dx} [\log_a x] = \frac{1}{x \ln a}$

Proof: Let $y = \log_a x \Leftrightarrow a^y = x$
 $a^y \ln a \cdot y' = 1 \Rightarrow y' = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}$

Result: $\frac{d}{dx} [\ln x] = \frac{1}{x}$. because $\ln e = 1$

In general: a) $\frac{d}{dx} [\log(g(x))] = \frac{g'(x)}{g(x) \ln a}$ if $g(x)$ is diff.

b) $\frac{d}{dx} [\ln g(x)] = \frac{g'(x)}{g(x)}$

Ex: Differentiate the function.

Q.2, 249, $f(x) = \ln(x^2 + 10) \Rightarrow f'(x) = \frac{2x}{x^2 + 10}$

Q.5, 249, $f(x) = \log_2(1-3x) \Rightarrow f'(x) = \frac{-3}{1-3x} \cdot \frac{1}{\ln 2} = \frac{-3}{(1-3x)\ln 2}$.

Q.14, 249, $F(y) = y \ln(1+e^y)$
 $F'(y) = y \cdot \frac{e^y}{1+e^y} + \ln(1+e^y) \cdot 1 = \frac{ye^y}{1+e^y} + \ln(1+e^y)$

Q.18, 249, $G(u) = \ln \sqrt{\frac{3u+2}{3u-2}} \Rightarrow G(u) = \ln \left(\frac{3u+2}{3u-2} \right)^{\frac{1}{2}} = \frac{1}{2} \ln \left(\frac{3u+2}{3u-2} \right)$
 $\therefore G(u) = \frac{1}{2} [\ln(3u+2) - \ln(3u-2)]$
 $G'(u) = \frac{1}{2} \left[\frac{3}{3u+2} - \frac{3}{3u-2} \right] = \frac{1}{2} \left[\frac{9u-6-9u+6}{9u^2-4} \right]$
 $= \frac{1}{2} \left[\frac{-12}{9u^2-4} \right] = \frac{-6}{9u^2-4}$

Ex.4, 245, $f(x) = \log_{10}(2+\sin x)$

$$f'(x) = \frac{0 + \cos x}{2 + \sin x} \cdot \frac{1}{\ln(10)}$$

$$= \frac{\cos x}{(2 + \sin x) \ln(10)}$$

(2)

Ex. 6, Find $f'(x)$ if $f(x) = \ln|x|$ See 3.8

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$f(x) = \ln|x| = \begin{cases} \ln(x), & x > 0 \\ \ln(-x), & x < 0 \end{cases} \quad f(x) \text{ not cont. at } x=0$$

$$\text{Dom } f(x) = \mathbb{R} - \{0\} = (-\infty, 0) \cup (0, \infty)$$

$$f'(x) = \begin{cases} \frac{1}{x} \text{ if } x > 0 \\ \frac{-1}{-x} = \frac{1}{x} \text{ if } x < 0 \end{cases} \Rightarrow f'(x) = \frac{1}{x} \text{ if } x \neq 0$$

In general $\frac{d}{dx} \ln|x| = \frac{1}{x}$

Ex. Find y' if $y = \ln|x^2-5|$

$$y' = \frac{2x}{x^2-5}.$$

*Logarithmic Differentiation : See the steps page 247.

For complex expressions, it is much easier if we take the \ln , then diff.

Ex. Use logarithmic differentiation:

Q. 36, $y = \sqrt{x} e^{x^2} (x^2+1)^{10}$ Take \ln for both sides:

$$\ln y = \ln [x^{\frac{1}{2}} e^{x^2} (x^2+1)^{10}]$$

$$\ln y = \frac{1}{2} \ln x + x^2 + 10 \ln(x^2+1)$$

$$\frac{1}{y} \cdot y' = \frac{1}{2} \cdot \frac{1}{x} + 2x + 10 \cdot \frac{2x}{x^2+1} = \frac{1}{2x} + 2x + \frac{20x}{x^2+1}$$

$$\Rightarrow y' = y \left[\frac{1}{2x} + 2x + \frac{20x}{x^2+1} \right] = \sqrt{x} e^{x^2} (x^2+1)^{10} \left[\frac{1}{2x} + 2x + \frac{20x}{x^2+1} \right]$$

Q. 44, $y = \ln x \Rightarrow \ln y = \ln(x^{\ln x}) = \ln x \cdot \ln x = (\ln x)^2$

$\frac{1}{y} \cdot \frac{dy}{dx} = 2(\ln x) \cdot \frac{1}{x} \Rightarrow \frac{dy}{dx} = y \cdot \frac{2}{x} \ln x = x^{\ln x} \left(\frac{2 \ln x}{x} \right)$

Q. 47, Find y' if $y = \ln(x^2+y^2)$

$$\frac{y'}{y} = \frac{2x+2y \cdot y'}{x^2+y^2} \Rightarrow (x^2+y^2)y' = 2x + 2yy'$$

$$x^2y' + y^2y' - 2yy' = 2x$$

$$y'(x^2+y^2-2y) = 2x \Rightarrow y' = \frac{2x}{x^2+y^2-2y}.$$

* The Number e as a limit

For $f(x) = \ln x$, $f'(x) = \frac{1}{x} \Rightarrow f'(1) = \frac{1}{1} = 1$

But, $f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{\ln(1+h) - \ln(1)}{h}$
 $= \lim_{h \rightarrow 0} \frac{1}{h} \ln(1+h) = \lim_{h \rightarrow 0} \ln(1+h)^{\frac{1}{h}} = 1$

$\therefore e = \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} \approx 2.7182818$
Because: $e = e' = e^{\lim_{x \rightarrow 0} \ln(1+x)^{\frac{1}{x}}} = \lim_{x \rightarrow 0} e^{\ln(1+x)^{\frac{1}{x}}} = \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}}$.

NOTE. If $n = \frac{1}{x}$, as $x \rightarrow 0$, $n \rightarrow \infty$

$\therefore e = \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$

Q-52, Show that $\lim_{n \rightarrow \infty} (1 + \frac{x}{n})^n = e^x$ for any $x > 0$
249

Let $\frac{1}{m} = \frac{x}{n} \Rightarrow m = \frac{n}{x}$ and $n = xm$.
as $n \rightarrow \infty$, $m \rightarrow \infty$

$\therefore \lim_{n \rightarrow \infty} (1 + \frac{x}{n})^n = \lim_{m \rightarrow \infty} (1 + \frac{1}{m})^{xm}$
 $= \lim_{m \rightarrow \infty} [(1 + \frac{1}{m})^m]^x = \left[\lim_{m \rightarrow \infty} (1 + \frac{1}{m})^m \right]^x = e^x$.

The End

* Hyperbolic Functions *Objectives

1. To define hyperbolic functions
2. = consider the = identities
3. = define = derivatives
4. = consider inverse hyperbolic functions and their derivatives

Def. Hyperbolic functions:

1. $\sinh x = \frac{e^x - e^{-x}}{2}$

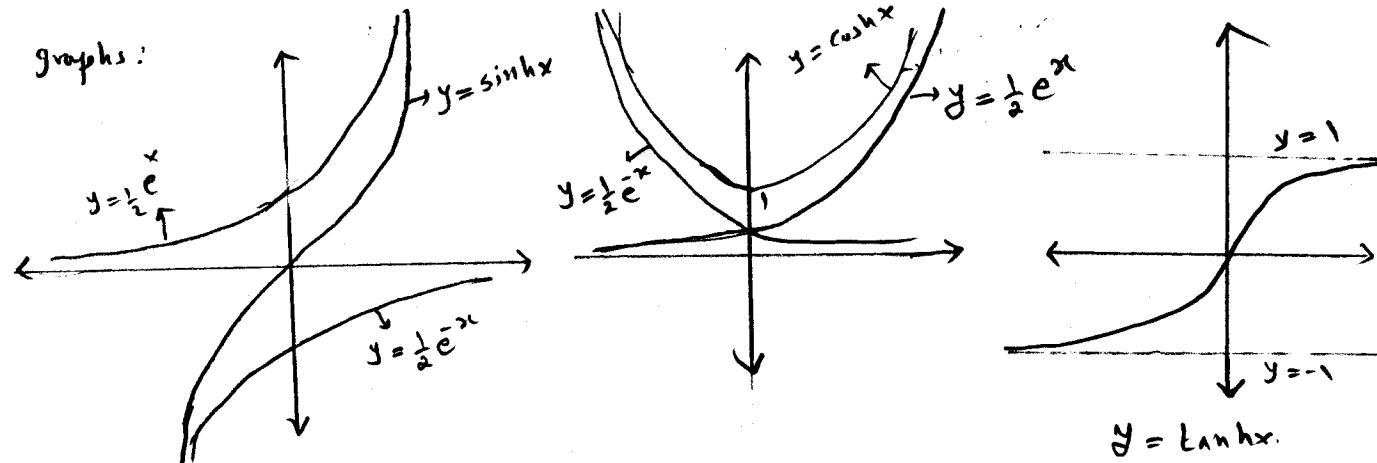
2. $\cosh x = \frac{e^x + e^{-x}}{2}$

3. $\tanh x = \frac{\sinh x}{\cosh x}$

4. $\operatorname{csch} x = \frac{1}{\sinh x}$

5. $\operatorname{sech} x = \frac{1}{\cosh x}$

6. $\operatorname{coth} x = \frac{\cosh x}{\sinh x}$

* Some graphs:

- Notes:
1. For $y = \sinh x$: $\operatorname{Dom}(y) = (-\infty, +\infty)$, $\operatorname{Range}(y) = (-\infty, +\infty)$
 2. For $y = \cosh x$: $\operatorname{Dom}(y) = (-\infty, +\infty)$, $\operatorname{Range}(y) = [1, \infty)$
 3. For $y = \tanh x$: $\operatorname{Dom}(y) = (-\infty, +\infty)$, $\operatorname{Range}(y) = (-1, 1)$.

Ex. Find the numerical value of:

Q.2, a) $\tanh(0) = \frac{\sinh(0)}{\cosh(0)} = \frac{(e^0 - e^0)/2}{(e^0 + e^0)/2} = \frac{0}{1} = 0$

b) $\tanh(1) = \frac{\sinh(1)}{\cosh(1)} = \frac{(e^1 - e^{-1})/2}{(e^1 + e^{-1})/2} = \frac{1}{2} \left(\frac{e - \frac{1}{e}}{e + \frac{1}{e}} \right) = \frac{e^2 - 1}{2e} \cdot \frac{2e}{e^2 + 1} = \frac{e^2 - 1}{e^2 + 1} \approx 0.76159$

* Hyperbolic Identities

1. $\sinh(-x) = -\sinh x$ odd

2. $\cosh(-x) = \cosh(x)$ even

3. $\cosh^2 x - \sinh^2 x = 1$

4. $1 - \tanh^2 x = \operatorname{sech}^2 x$

5. $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

6. $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

Ex. Prove the identity

Q.10, 255, $\cosh x + \sinh x = e^x$

Proof, $\cosh x - \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = \frac{2e^x}{2} = e^x = R.H.S$

Q.13, 255, $\coth^2 x - 1 = \operatorname{csch}^2 x$

$$\coth^2 x - 1 = \frac{\cosh^2 x}{\sinh^2 x} - 1 = \frac{\cosh^2 x}{\sinh^2 x} - \frac{\sinh^2 x}{\sinh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\sinh^2 x} = \frac{1}{\sinh^2 x} = \operatorname{csch}^2 x$$

Q.16, 255, $\cosh 2x = \cosh^2 x + \sinh^2 x$.

$$\cosh 2x = \cosh(x+x) = \cosh x \cosh x + \sinh x \sinh x = \cosh^2 x + \sinh^2 x$$

Q.21, 255, If $\tanh x = \frac{4}{5}$, find the values of other hyperbolic functions

$$\tanh x = \frac{4}{5} > 0 \Rightarrow \boxed{\coth x = \frac{1}{\tanh x} = \frac{5}{4}}$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x \Rightarrow 1 - \frac{16}{25} = \operatorname{sech}^2 x \Rightarrow \operatorname{sech}^2 x = \frac{9}{16} \Rightarrow \boxed{\operatorname{sech} x = \frac{3}{4}} (x > 0)$$

$$\Rightarrow \boxed{\cosh x = \frac{5}{3}}$$

$$\cosh^2 x - \sinh^2 x = 1 \Rightarrow \frac{25}{9} - \sinh^2 x = 1 \Rightarrow \sinh^2 x = \frac{16}{9} \Rightarrow \boxed{\sinh x = \frac{4}{3}}$$

$$\operatorname{csch} x = \frac{3}{4}$$

*Derivatives of hyperbolic functions

$$1. \frac{d}{dx}(\sinh x) = \cosh x \quad 2. \frac{d}{dx}(\cosh x) = \sinh x \quad 3. \frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$4. \frac{d}{dx}(\operatorname{coth} x) = -\operatorname{csch} x \coth x \quad 5. \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$6. \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch}^2 x$$

Note, $\frac{d}{dx}[\sinh(g(x))] = \cosh(g(x)) \cdot g'(x)$ which is valid for others.

Ex. Find the derivative:

Q.30, 255, $f(x) = \tanh(4x) \Rightarrow f'(x) = 4 \operatorname{sech}^2 4x$.

Q.36, 255, $f(t) = e^t \operatorname{sech} t$

$$\begin{aligned} f'(t) &= e^t \cdot (-\operatorname{sech} t \tanh t) + \operatorname{sech} t \cdot e^t \\ &= e^t \operatorname{sech} t (-\tanh t + 1) \\ &= e^t \operatorname{sech} t (1 - \tanh t). \end{aligned}$$

Q.38, $f(t) = \ln(\sinh t)$

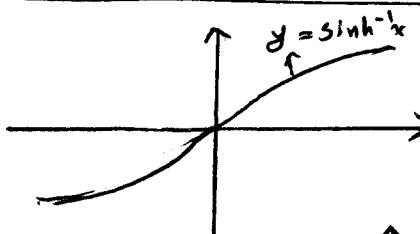
255 $f'(t) = \frac{(\sinh t)'}{\sinh t} = \frac{\cosh t}{\sinh t} = \coth t$

Q.40, $y = \sinh(\cosh x)$

$y' = \cosh(\cosh x) \cdot \sinh(x)$.

*Inverse Hyperbolic Functions:

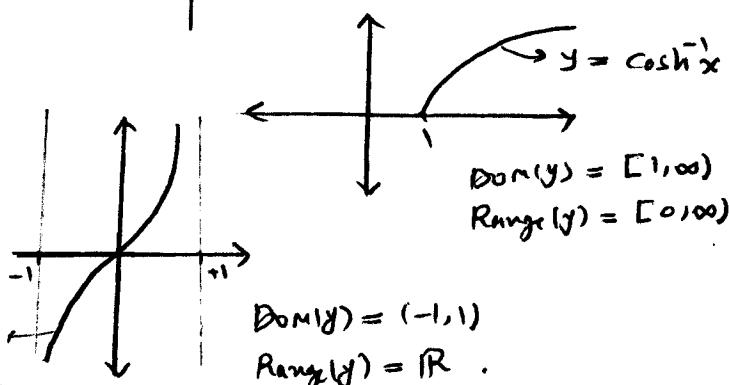
1. $y = \sinh^{-1}x \Leftrightarrow \sinhy = x$



$$y = \sinh^{-1}x$$

Dom(y) = \mathbb{R}
Range(y) = \mathbb{R}

2. $y = \cosh^{-1}x \Rightarrow \cosh y = x$



3. $y = \tanh^{-1}x \Leftrightarrow \tanh y = x$

$$\text{Dom}(y) = (-1, 1)$$

Range(y) = \mathbb{R} .

Def: 1. $\sinh^{-1}x = \ln(x + \sqrt{x^2+1})$, $x \in \mathbb{R}$

2. $\cosh^{-1}x = \ln(x + \sqrt{x^2-1})$, $x \geq 1$

3. $\tanh^{-1}x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$, $-1 < x < 1$.

Ex-3, Show that $\sinh^{-1}x = \ln(x + \sqrt{x^2+1})$

253

Let $y = \sinh^{-1}x \Rightarrow x = \sinhy \Rightarrow \frac{e^y - e^{-y}}{2} = x$

$$e^y - e^{-y} = 2x$$

$$e^y - 2x - e^{-y} = 0 \quad (\text{multiply by } e^y)$$

$$e^{2y} - 2xe^y - 1 = 0 \Rightarrow (e^y)^2 - (2e^y)x - 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = \frac{2x \pm 2\sqrt{x^2 + 1}}{2} = x \pm \sqrt{x^2 + 1}$$

But $e^y > 0$ and $x - \sqrt{x^2 + 1} < 0 \Rightarrow e^y = x + \sqrt{x^2 + 1}$

$$\Rightarrow y = \ln(x + \sqrt{x^2 + 1})$$

*Derivatives of Inverse Hyperbolic Func.

1. $\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}$

2. $\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}}$

3. $\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}$

4. $\frac{d}{dx}(\operatorname{csch}^{-1}x) = -\frac{1}{|x|\sqrt{x^2+1}}$

5. $\frac{d}{dx}(\operatorname{sech}^{-1}x) = -\frac{1}{x\sqrt{1-x^2}}$

6. $\frac{d}{dx}(\operatorname{coth}^{-1}x) = -\frac{1}{1-x^2}$.

See 3.9

Ex.4 Prove that $\frac{d}{dx} (\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$.

Let $y = \sinh^{-1} x \Rightarrow \sinh y = x$

$$\cosh y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y}, \quad (\cosh^2 y - \sinh^2 y = 1) \\ \Rightarrow \cosh y = \sqrt{1 + \sinh^2 y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}.$$

Q.R.: $\frac{d}{dx} \sinh^{-1}(x) = \frac{d}{dx} \ln(x + \sqrt{x^2+1}) = \frac{1 + \frac{ex}{2\sqrt{x^2+1}}}{x + \sqrt{x^2+1}}$

$$= \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}} \cdot \frac{1}{x + \sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}$$

NOTE: $\frac{d}{dx} \sinh^{-1}(g(x)) = \frac{1}{\sqrt{1+(g(x))^2}} \cdot g'(x)$, and similarly for others

Q.42: $y = x^2 \sinh^{-1}(2x)$

$$\begin{aligned} \frac{dy}{dx} &= x^2 \cdot \frac{1}{\sqrt{1+(2x)^2}} \cdot 2 + \sinh^{-1}(2x) \cdot 2x \\ &= 2x \left[\frac{x}{\sqrt{1+4x^2}} + \sinh^{-1}(2x) \right] \end{aligned}$$

Q.47: $y = \coth^{-1} \sqrt{x^2+1}$

$$\frac{dy}{dx} = \frac{1}{1-(\sqrt{x^2+1})^2} \cdot \frac{2x}{2\sqrt{x^2+1}} = \frac{1}{x-x^2-1} \cdot \frac{x}{\sqrt{x^2+1}} = -\frac{1}{x\sqrt{x^2+1}}.$$

Ex.5: Find $\frac{d}{dx} (\tanh^{-1}(\sin x))$

$$= \frac{1}{1-(\sin x)^2} \cdot (\sin x)' = \frac{\cos x}{1-\sin^2 x} = \frac{\cos x}{\cos^2 x} = \frac{1}{\cos x} = \sec x$$

The End

* Related Rates *

Objective: To solve an application on differentiation using the chain rule.

To solve an application on related rates, take care of the following

1. Sketch the graph—if possible—and put all unknown variables on it
2. Find a relationship between the variables.
3. Differentiate with respect to time all variables using the chain rule

Q.4: If $x^2 + y^2 = 25$ and $\frac{dy}{dt} = 6$, find $\frac{dx}{dt}$ when $y=4$.
260

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow \frac{dx}{dt} = -\frac{2y}{2x} \frac{dy}{dt} = -\frac{y}{x} \frac{dy}{dt}$$

$$\text{When } y=4 \Rightarrow x^2 = 25 - 16 = 9 \Rightarrow x = \pm \sqrt{9} = \pm 3.$$

$$\therefore \frac{dx}{dt} \Big|_{y=4} = -\frac{4}{\pm 3} \cdot 6 = \pm 8$$

Ex.1: Let the volume be $V \Rightarrow \frac{dv}{dt} = 100 \text{ cm}^3/\text{s}$
260

$$= = \text{radius} = r \Rightarrow \frac{dr}{dt} = ? \text{ when } 2r=50 \Rightarrow r=25 \text{ cm.}$$

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$$

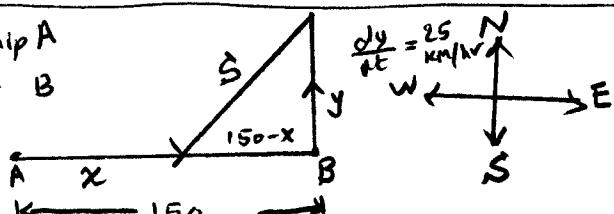
$$\Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2} \frac{dv}{dt} \Rightarrow \frac{dr}{dt} \Big|_{r=25} = \frac{1}{4\pi(25)^2} \cdot 100 = \frac{1}{25\pi} \text{ cm/s}$$

∴ The radius is increasing at the rate of $\frac{1}{25\pi}$ cm/s.

Q.10: Let X be the distance moved by ship A
260

$y =$ $=$ $=$ $=$ $=$ between
the two ships

$$S^2 = (150-x)^2 + y^2$$



$$\frac{dx}{dt} = +35 \text{ km/hr}$$

$$2S \frac{ds}{dt} = 2(150-x) \cdot \frac{dx}{dt} + 2y \frac{dy}{dt} \Rightarrow S \frac{ds}{dt} = (150-x)(-\frac{dx}{dt}) + y \frac{dy}{dt}$$

$$\text{When } t=4 \text{ hrs} \Rightarrow x = (4)(35) = 140 \text{ km}, y = (4)(25) = 100 \text{ km}$$

$$S^2 = (150-140)^2 + (100)^2 = 100 + 10000 = 10100 \Rightarrow S = \sqrt{10100} = \sqrt{101} \sqrt{100} = 10\sqrt{101}$$

$$\therefore 10\sqrt{101} \frac{ds}{dt} = (150-140)(-35) + 100(25) = -350 + 2500 = 2150 \Rightarrow \frac{ds}{dt} = \frac{2150}{10\sqrt{101}} = \frac{215}{\sqrt{101}} \text{ Km/hr.}$$

Q.19, Let V be the volume of the water in the tank
 h = level $\Rightarrow r = \frac{1}{3}h$

$$V = \frac{1}{3}\pi r^2 h, \quad \frac{dV}{dt} = C - 10,000 \text{ cm}^3/\text{min.}$$

$$\frac{dh}{dt} = 20 \text{ cm/min.}$$

We want the value C | $h=2$ $\Rightarrow 200 \text{ cm}$

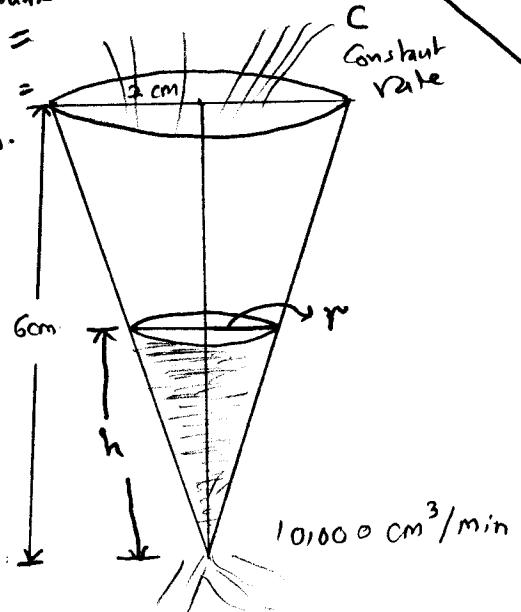
$$V = \frac{1}{3}\pi r^2 h \quad \text{But by similar triangles}$$

$$\frac{r}{h} = \frac{2}{6} = \frac{1}{3} \Rightarrow 3r = h \quad r = \frac{1}{3}h.$$

$$\therefore V = \frac{1}{3}\pi \left(\frac{1}{3}h\right)^2 h = \frac{1}{27}\pi h^3$$

$$\frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt}$$

$$C - 10,000 = \frac{1}{9}\pi(200)^2 \cdot (20) \Rightarrow C = 10000 + \frac{800,000\pi}{9} \text{ cm}^3/\text{min.}$$

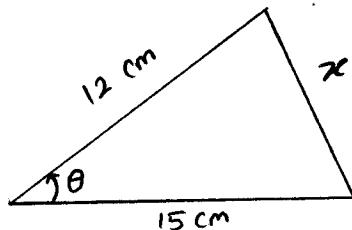


Q.26, Let θ be the angle between the two known sides

$$\frac{d\theta}{dt} = 2^\circ/\text{min.} = \frac{2\pi}{180} = \frac{\pi}{90} \text{ rad/min.}$$

Let x be the other side

$$\text{We want } \frac{dx}{dt} \mid \theta = 60^\circ$$



By Law of Cosines:

$$x^2 = (12)^2 + (15)^2 - 2(12)(15) \cos \theta$$

$$x^2 = 369 - 360 \cos \theta$$

$$2x \frac{dx}{dt} = 0 - 360(-\sin \theta) \frac{d\theta}{dt} = 360 \sin \theta \frac{d\theta}{dt}$$

$$\text{when } \theta = 60^\circ, x^2 = 369 - 360 \cos(60^\circ) = 189 \Rightarrow x = \sqrt{189} = 3\sqrt{21}$$

$$\therefore 3\sqrt{21} \frac{dx}{dt} = 180 \sin(60^\circ) \cdot \frac{\pi}{90} = \frac{2\pi\sqrt{3}}{2} = \pi\sqrt{3}$$

$$\therefore \frac{dx}{dt} = \frac{\pi\sqrt{3}}{3\sqrt{21}} = \frac{\pi\sqrt{3}}{3\sqrt{3}\cdot\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{7}\pi}{21} \text{ cm/min.}$$

Ex.4, Let x be the distance between C and A
 $y = \Rightarrow = = = = B$

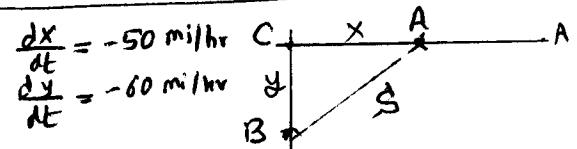
$s = \Rightarrow = = = A \Rightarrow B$

$$s^2 = x^2 + y^2, \text{ want } \frac{ds}{dt} \mid x=4, y=-3$$

$$2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \Rightarrow s \frac{ds}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt}$$

$$\Rightarrow s \frac{ds}{dt} = (-3)(-50) + (4)(-60) \Rightarrow \frac{ds}{dt} = -\frac{39}{s} = -78 \text{ mil/hr}$$

The cars approaches each other at rate of 78 mil/hr



$$s^2 = (4)^2 + (-3)^2 = 25$$

$$\Rightarrow s = 5$$

Q.31, 264, Let x be the distance moved by the bottom of the ladder
 $\theta = \angle$ between the top of the ladder
and the wall

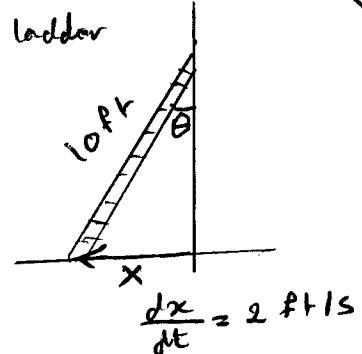
Want

$$\frac{d\theta}{dt} \Big|_{\theta=\frac{\pi}{4}}$$

$$\sin \theta = \frac{x}{10} \Rightarrow \cos \theta \frac{d\theta}{dt} = \frac{1}{10} \frac{dx}{dt}$$

$$\Rightarrow \cos \frac{\pi}{4} \cdot \frac{d\theta}{dt} = \frac{1}{10} \quad (2) = \frac{1}{5}$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{1}{5} \cdot \frac{1}{\sqrt{2}} = \frac{2}{5\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{5\sqrt{2}} = \frac{\sqrt{2}}{5} \text{ rad/s.}$$



The End -

*Linear Approximations and Differentials*Objectives:

1. To define the linear approximation and solve problems
2. $\Rightarrow \Rightarrow \Rightarrow$ differentials $\Rightarrow \Rightarrow \Rightarrow$

Def: The linear approximation or tangent line approximation of f at a is:

$$f(x) \approx f(a) + f'(a)(x-a)$$

The linear function is: $L(x) = f(a) + f'(a)(x-a)$

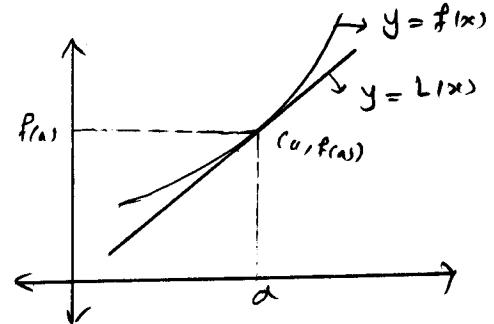
$$y - y_1 = f'(x_1)(x - x_1)$$

$$y - f(a) = f'(a)(x-a)$$

$$y = f(a) + f'(a)(x-a)$$

$$\text{or } f(x) \approx f(a) + f'(a)(x-a) = L(x)$$

$$f(x) \approx L(x)$$



Ex. Find the linearization $L(x)$ of f at a .

Q.5 267. $f(x) = x^3$, $a=1$ $L(x) = f(x) + f'(a)(x-a)$

$$f'(x) = 3x^2 \Rightarrow f'(1) = 3(1)^2 = 3 \Rightarrow f(1) = (1)^3 = 1$$

$$\therefore L(x) = f(a) + f'(a)(x-a)$$

$$L(x) = 1 + 3(x-1) = 1 + 3x - 3 = 3x - 2.$$

$$\therefore f(x) \approx 3x - 2.$$

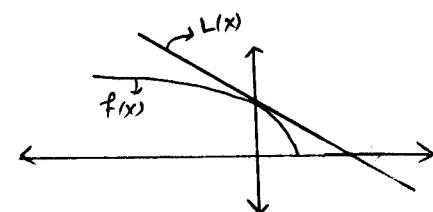
Q.6 267 (i) Find the linear approximation: $f(x) = \sqrt{1-x}$ at $a=0$.

$$f'(x) = \frac{-1}{2\sqrt{1-x}} \Rightarrow f'(a) = f'(0) = \frac{-1}{2\sqrt{1-0}} = \frac{-1}{2}$$

$$f(a) = f(0) = \sqrt{1-0} = 1$$

$$\therefore L(x) = f(a) + f'(a)(x-a)$$

$$= 1 + (-\frac{1}{2})(x-0) = 1 - \frac{1}{2}x \Rightarrow \sqrt{1-x} \approx 1 - \frac{1}{2}x.$$



(ii) Use it to approximate $\sqrt{0.9}$ and $\sqrt{0.99}$.

$$1) \sqrt{0.9} = \sqrt{1-0.1} = \sqrt{0.9} \approx 1 - \frac{1}{2}(0.1) = 1 - 0.05 = 0.95$$

$$2) \sqrt{0.99} = \sqrt{1-0.01} = 1 - \frac{1}{2}(0.01) = 1 - 0.005 = 0.995.$$

Q.14, verify the linear approximation at $a=0$. Determine the values of x such that it is accurate to within 0.1 given

$$e^x = 1+x, a=0$$

$$f(x) = e^x \rightarrow f(0) = e^0 = 1$$

$$f'(x) = e^x \rightarrow f'(0) = e^0 = 1 \Rightarrow f(x) \approx f(a) + f'(a)(x-a)$$

$$= f(0) + f'(0)(x-0)$$

$$= 1 + 1(x-0) = 1+x.$$

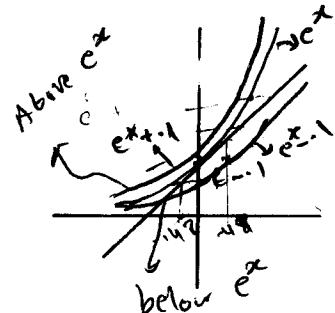
$$\therefore e^x \approx 1+x$$

NOTE: The accuracy is $|f(x) - L(x)| < \text{error}$

$$|e^x - (1+x)| < 0.1 \Rightarrow -0.1 < e^x - (1+x) < 0.1$$

$$-e^{-0.1} < -(1+x) < 1 - e^{-0.1} \Rightarrow e^{-0.1} < 1+x < e^{-0.1} + 1$$

$$\Rightarrow -0.483 < x < 0.416 \quad \text{by the graph of the tangent.}$$



Differentials

Def. The differential dy for the function $y = f(x)$ is given by

$$\frac{dy}{dx} = f'(x) \Rightarrow dy = f'(x) dx.$$

Geometric Meaning:

Let $P(x, f(x))$ and $R(x+dx, f(x+dx))$ be two points of f .

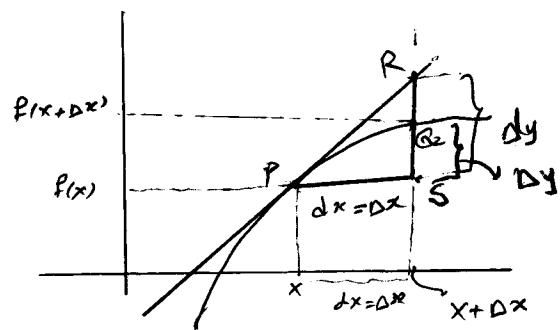
Let $dx = \Delta x$. Then,

$$\Delta y = f(x+dx) - f(x)$$

The tangent slope $= f'(x)$: $SR = dy = f'(x) dx$.

The linear approximation of f at a can be written as:

$$f(a+dx) \approx f(a) + dy \quad (\Delta y \approx dy = f'(x) dx)$$



Ex. a) Find dy b) Evaluate dy for the given x, dx .

Q.21, 268: $y = x^2 + 2x \Rightarrow x = 3, dx = \frac{1}{2}$.

(a) $dy = f'(x) dx = (2x+2) dx$

(b) When $x=3, dx = \frac{1}{2} \Rightarrow dy = (2(3)+2)(\frac{1}{2}) = 4$

Q.22, 268: $y = e^{x/4} \Rightarrow x = 0, dx = 0.1$

(a) $dy = \frac{1}{4} e^{x/4} dx$

(b) $dy = \frac{1}{4} e^0 (0.1) = \frac{1}{4}(1)(0.1) = \frac{1}{40} = 0.025$

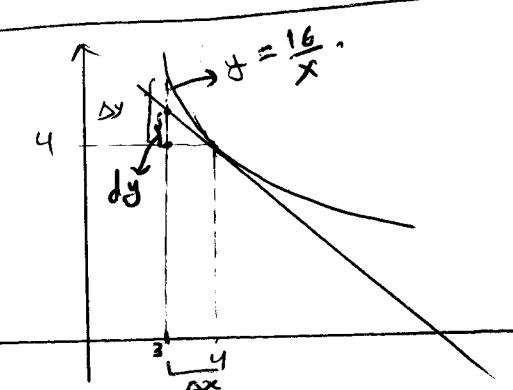
Ex. Compare $\Delta y, dy$, where $dx = \Delta x$.

Q.30, 268: $y = \frac{16}{x} \Rightarrow x = 4, \Delta x = -1$

a) $\Delta y = f(x+dx) - f(x) = f(4-1) - f(4)$
 $= f(3) - f(4) = \frac{16}{3} - \frac{16}{4} = \frac{4}{3}.$

b) $dy = -\frac{16}{x^2} dx$

When $x=4, dx = \Delta x = -1 \Rightarrow dy = -\frac{16}{(4)^2} \cdot (-1) = 1$



Ex. Use differentials (or linear approximation) to estimate,

Q.32, 268: $\sqrt{99.8}$ Let $y = \sqrt{x} \Rightarrow dy = \frac{1}{2\sqrt{x}} dx$

when $x = 100, dx = -0.2 \Rightarrow dy = \frac{1}{2\sqrt{100}} (-0.2) = -\frac{1}{10} = -0.1$

$$\begin{aligned} f(a+dx) &\approx f(a) + dy \\ &= f(100) + dy \\ &= 10 + (-0.1) = 9.99 \end{aligned}$$

Ex.5: $r = 21 \text{ cm}$ with a possible error of at most 0.05 cm

What is the maximum error in using r to compute the volume of the sphere

$$V = \frac{4}{3} \pi r^3 \quad r = 21, \text{ error } dr = \Delta r = 0.05, \text{ then}$$

$$dV = 4\pi r^2 dr$$

$$dV = 4\pi(21)^2(0.05) \approx 277$$

∴ The maximum error in the calculated volume is about 277 cm^3 .

Def. The relative error is of $y = f(x)$ is: $\frac{\Delta y}{y} \approx \frac{dy}{y}$

Ex. For Ex.5. The relative error is:

$$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3} \pi r^3} = \frac{3 dr}{r}.$$

Relative error
in the volume

↓ Relative error in the
radius.

$$\text{Relative error in radius} = \frac{dr}{r} = \frac{0.05}{21} \approx 0.0024$$

Def. The percentage error = relative error * 100%.

$$\text{For Ex.5} \text{ percentage error} = 0.0024 * 100\% = 0.24\%.$$

$$\text{Relative error in volume} = \frac{dr}{V} = 3 \frac{dr}{r} = 3(0.0024) = 0.007$$

$$\text{The percentage error in volume} = 0.007 * 100\% = 0.7\%.$$

Q-41: edge = 30 cm with an error of 0.1 cm. Use differentials to estimate:
268 i) maximum possible error ii) relative error iii) percentage error in

a) The volume of the cube : let x be the length of the cube edge.

$$\text{i)} \quad V = x^3 \Rightarrow dV = 3x^2 dx, \quad dx = \Delta x = 0.1$$

$$\text{ii)} \quad \text{Max. error in volume} = dV = 3(30)^2(0.1) = 270 \text{ cm}^3.$$

$$\text{iii)} \quad \text{Relative error} = \frac{dV}{V} = \frac{3x^2 dx}{x^3} = 3 \frac{dx}{x} = 3 \frac{(0.1)}{30} = 0.01$$

$$\text{iii)} \quad \text{Percentage error in volume} = 0.01 * 100\% = 1\%.$$

b) The surface area of the cube : let S be the surface area.

$$\text{i)} \quad S = 6x^2 \Rightarrow dS = 12x dx, \quad x = 30, \quad dx = 0.1$$

$$\text{ii)} \quad \text{MAX. error in } S = dS = 12(30)(0.1) = 36 \text{ cm}^2$$

$$\text{iii)} \quad \text{Relative error in } S \approx \frac{dS}{S} = \frac{12x dx}{6x^2} = 2 \frac{dx}{x} = 2 \frac{(0.1)}{30} = 0.006$$

$$\text{iii)} \quad \text{Percentage error in } S = 0.006 * 100\% = 0.6\%.$$

The End