

**King Fahd University of Petroleum and Minerals**  
**Department of Mathematics and Statistics**  
**Math 101-Calculus I**  
**Exam I**  
**Term (101)**

**Tuesday November 2, 2010**

**Allowed Time: 2 hours**

Name: Solutions

ID Number: \_\_\_\_\_

Section Number: \_\_\_\_\_ Serial Number: \_\_\_\_\_

**Instructions:**

1. Write neatly and legibly. You may lose points for messy work.
2. **Show all you work.** No points for answers without justification.
3. **Calculators and Mobiles are not allowed in this exam.**
4. Make sure that you have 6 pages of problems (**Total of 12 Problems**)

Page Total	Grade	Maximum Points
Page 1		16
Page 2		16
Page 3		18
Page 4		16
Page 5		16
Page 6		18
<b>Total</b>		<b>100</b>

1. (8-points) The displacement (in meters) of a particle moving in straight line is given by  $s(t) = t - \frac{1}{t}$ , where  $t$  is measured in seconds. Use limits to find the instantaneous velocity of the particle at  $t = 1$ .

$$\begin{aligned}
 v|_{t=1} &= \lim_{h \rightarrow 0} \frac{s(1+h) - s(1)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1+h - \frac{1}{1+h} - 0}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \frac{(1+2h+h^2-1)}{1+h} \\
 &= \lim_{h \rightarrow 0} \frac{h(2+h)}{h(1+h)} \\
 &= \lim_{h \rightarrow 0} \frac{2+h}{1+h} \\
 &= 2 \text{ m/sec}
 \end{aligned}
 \quad \left. \begin{array}{l} \text{3 pts} \\ \text{5 pts} \end{array} \right\}$$

2. (8-points) Use continuity to evaluate the limit

$$\begin{aligned}
 \lim_{x \rightarrow -3} \arctan \left( \frac{x^2 + 7x + 12}{x^2 + 5x + 6} \right) &= \lim_{x \rightarrow -3} \frac{(x+3)(x+4)}{(x+3)(x+2)} \\
 &= \lim_{x \rightarrow -3} \frac{x+4}{x+2} = -1
 \end{aligned}
 \quad \left. \begin{array}{l} \text{3 pts} \\ \text{2 pts} \end{array} \right\}$$

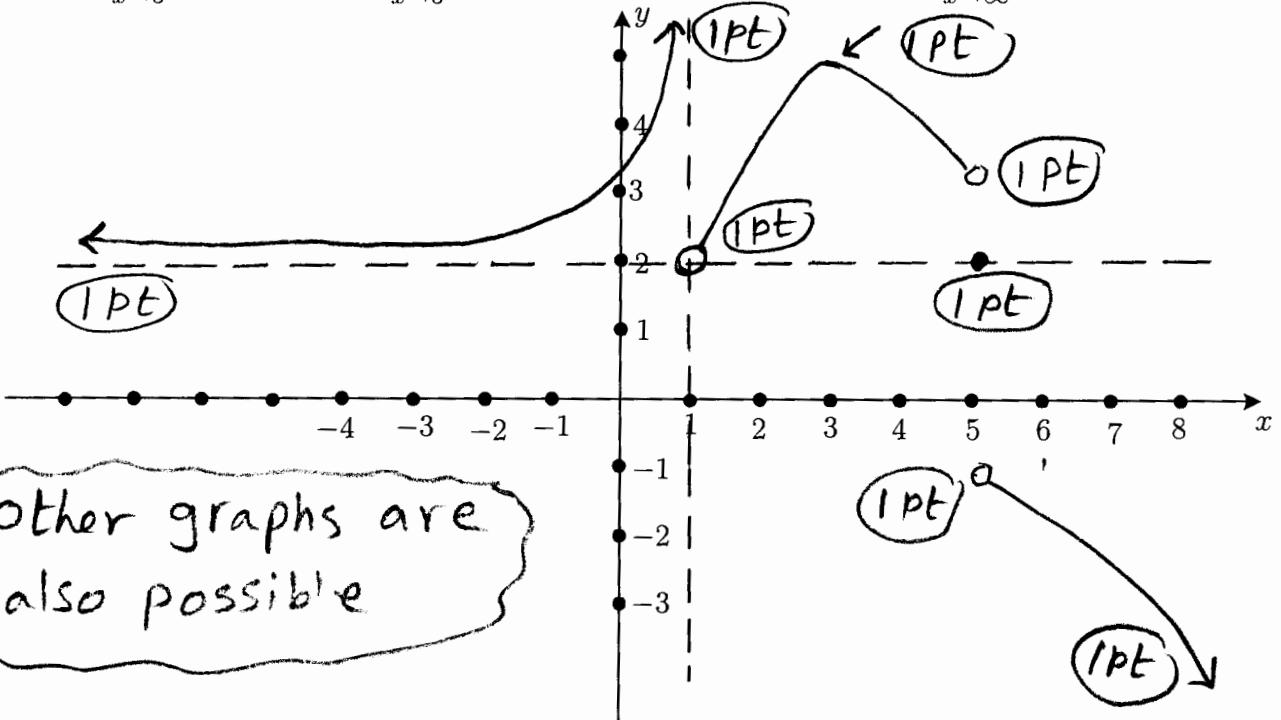
Also,  $\arctan$  is continuous at  $-1$  ...

$$\begin{aligned}
 \Rightarrow \lim_{x \rightarrow -3} \arctan \left( \frac{x^2 + 7x + 12}{x^2 + 5x + 6} \right) &= \arctan \left( \lim_{x \rightarrow -3} \frac{x^2 + 7x + 12}{x^2 + 5x + 6} \right) \\
 &= \arctan(-1) = -\frac{\pi}{4}
 \end{aligned}
 \quad \left. \begin{array}{l} \text{3 pts} \end{array} \right\}$$

3. (8-points) Sketch the graph of a function  $f$  that satisfies the following conditions

$$\lim_{x \rightarrow -\infty} f(x) = 2, \quad \lim_{x \rightarrow 1^-} f(x) = \infty, \quad \lim_{x \rightarrow 1^+} f(x) = 2, \quad f'(3) = 0,$$

$$\lim_{x \rightarrow 5^-} f(x) = 3, \quad \lim_{x \rightarrow 5^+} f(x) = -1, \quad f(5) = 2, \quad \lim_{x \rightarrow \infty} f(x) = -\infty.$$



4. (8-points) Use the Squeeze Theorem to show that  $\lim_{x \rightarrow 0^-} x^3 \sin \frac{\pi}{\sqrt[3]{x}} = 0$ .

We know that  $-1 \leq \sin \frac{\pi}{\sqrt[3]{x}} \leq 1, x \neq 0$ , 1 pt

and  $x \rightarrow 0^- \Rightarrow x < 0 \Rightarrow$

$$-x^3 \geq x^3 \sin \frac{\pi}{\sqrt[3]{x}} \geq x^3 \quad \text{--- } 2 \text{ pts}$$

(-2 pts if the above inequality is reversed)

But  $\lim_{x \rightarrow 0^-} (-x^3) = 0$  and  $\lim_{x \rightarrow 0^-} x^3 = 0$  2 pt

$\Rightarrow \lim_{x \rightarrow 0^-} x^3 \sin \frac{\pi}{\sqrt[3]{x}} = 0$  3 pts  
by the squeezing Theorem

5. Given the function  $f(x) = \frac{\sqrt{1+x^2} - \sqrt{1-x}}{x}$ .

(a) (3-points) Find the domain of  $f$  in interval notation.

Must have  $x \leq 1$  and  $x \neq 0$  } 3 pts  
 $\Rightarrow \text{domain } f = (-\infty, 0) \cup (0, 1]$  }

(b) (8-points) Find the horizontal asymptotes to the graph of  $f$ .

$$\begin{aligned} \lim_{x \rightarrow -\infty} f(x) &= \lim_{x \rightarrow -\infty} \frac{(1+x^2) - (1-x)}{x[\sqrt{1+x^2} + \sqrt{1-x}]} = \lim_{x \rightarrow -\infty} \frac{x(x+1)}{x[\sqrt{x^2+1} + \sqrt{1-x}]} \quad 3 \text{ pts} \\ &= \lim_{x \rightarrow -\infty} \frac{x(1 + \frac{1}{x})}{|x|[\sqrt{1 + \frac{1}{x^2}} + \sqrt{\frac{1}{x^2} - \frac{1}{x}}]} = \lim_{x \rightarrow -\infty} \frac{-(1 + \frac{1}{x})}{[\sqrt{1 + \frac{1}{x^2}} + \sqrt{\frac{1}{x^2} - \frac{1}{x}}]} \quad 3 \text{ pts} \\ &= -1 \quad \Rightarrow y = -1 \text{ is the only horizontal asymptote} \quad 2 \text{ pts} \end{aligned}$$

-2 pts if they calculate  $\lim_{x \rightarrow \infty} f(x)$

6. (7-points) Use limits to discuss the continuity of the greatest integer function  $f(x) = [x]$  on the interval  $[1, 2]$ .

- $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} [x] = 1 = f(1)$  } 2 pts  
 $\Rightarrow f$  is continuous from the right at 1
- For  $1 < c < 2 \Rightarrow \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} [x] = 1 = f(c)$  } 2 pts  
 $\Rightarrow f$  is continuous on the open interval  $(1, 2)$ .
- $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} [x] = 1 \neq f(2) = 2$  } 3 pts  
 $\Rightarrow f$  is discontinuous from the left at 2

7. (8-points) Let  $f(x) = 5x^3 - 4x^2 + 5$  and  $g(x) = x^3 + 2x^2 - 3x + 7$ . Use the Intermediate Value Theorem to show that the equation  $f(x) = g(x)$  has a solution between 1 and 2.

Let  $h(x) = f(x) - g(x) = 4x^3 - 6x^2 + 3x - 2$ . 2 pts

$h$  is continuous on the interval  $[1, 2]$ , 1 pt

and  $h(1) = -1$ ,  $h(2) = 12$  1 pt

Since  $-1 < 0 < 12$ , there is a number  $c$  in  $(1, 2)$  such that  $h(c) = 0$  by the Intermediate Value Theorem 2 pts

Thus, there is a root of the equation  $f(x) - g(x) = 0$ , or  $f(x) = g(x)$  in the interval  $(1, 2)$  2 pts

8. (8-points) Use the  $\epsilon, \delta$  definition of limit to prove that  $\lim_{x \rightarrow 6} \left( \frac{x}{4} + 3 \right) = \frac{9}{2}$ .

Given  $\epsilon > 0$ , we need  $\delta > 0$  such that ?

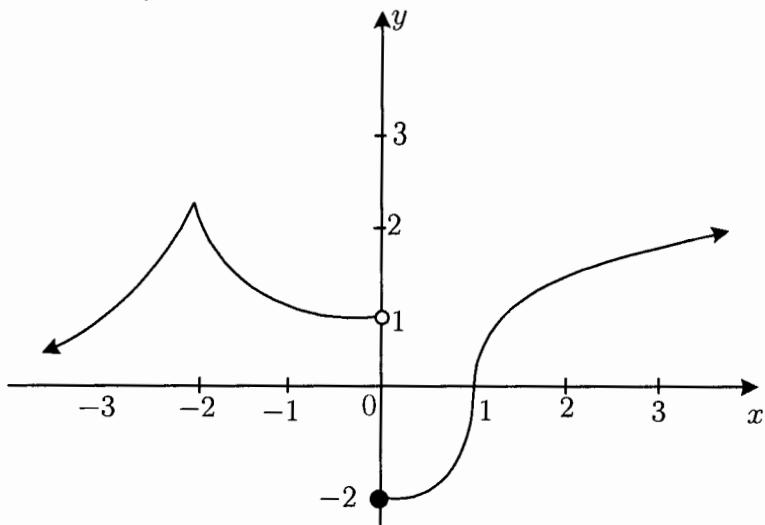
if  $0 < |x - 6| < \delta$ , then  $\left| \left( \frac{x}{4} + 3 \right) - \frac{9}{2} \right| < \epsilon$  5 pts

$$\Leftrightarrow \left| \frac{x}{4} - \frac{3}{2} \right| < \epsilon \Leftrightarrow |x - 6| < 4\epsilon$$

Thus if we choose  $0 < \delta \leq 4\epsilon$ , then the required result follows. 3 pts

( $\delta = 4\epsilon$  is also acceptable answer)

9. (6-points) Use the given graph of a function  $f$  to state **with reasons**, the numbers at which  $f$  is not differentiable.



$f$  is not differentiable at :

- 2 (corner)
  - 0 (discontinuity)
  - 1 (vertical tangent)
- } 2 pts each

10. (10-points) Let  $f(x) = \begin{cases} 5-x, & \text{if } x < 4 \\ \frac{1}{5-x}, & \text{if } x \geq 4 \end{cases}$ . Use limits to determine

whether  $f$  is differentiable or not at 4 [Hint: Find  $f'_-(4)$  and  $f'_+(4)$ ].

$$\begin{aligned} f'_-(4) &= \lim_{h \rightarrow 0^-} \frac{f(4+h) - f(4)}{h} = \lim_{h \rightarrow 0^-} \frac{(5-(4+h)) - 1}{h} \\ &= \lim_{h \rightarrow 0^-} \left( -\frac{h}{h} \right) = \lim_{h \rightarrow 0^-} (-1) = -1 \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} 4 \text{ pts}$$

$$\begin{aligned} f'_+(4) &= \lim_{h \rightarrow 0^+} \frac{f(4+h) - f(4)}{h} = \lim_{h \rightarrow 0^+} \frac{\frac{1}{5-(4+h)} - 1}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{\frac{1-(1-h)}{1-h}}{h} = \lim_{h \rightarrow 0^+} \frac{1}{1-h} = 1 \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} 4 \text{ pts}$$

$\Rightarrow f'_-(4) \neq f'_+(4)$  which means that  $f$  is not differentiable at 4 } 2 pts

11. (8-points) Find, if any, all the vertical asymptotes to the graph of the function  $f(x) = \frac{|x-1|}{x^3 - x^2 + x - 1}$ . Use limits to justify your answer.

$$f(x) = \frac{|x-1|}{x^2(x-1) + (x-1)} = \frac{|x-1|}{(x-1)(x^2+1)}$$

$\Rightarrow f$  is discontinuous at  $1 \notin \text{dom } f$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \frac{-(x-1)}{(x-1)(x^2+1)} = \lim_{x \rightarrow 1^-} \frac{-1}{x^2+1} = -\frac{1}{2}$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{(x-1)}{(x-1)(x^2+1)} = \lim_{x \rightarrow 1^+} \frac{1}{x^2+1} = \frac{1}{2}$$

$\Rightarrow f$  has no infinite discontinuity at 1

$\Rightarrow$  no vertical asymptotes to the graph of  $f$ .

12. (10-points) Find the values of  $a$  so that the given function is continuous or has a removable discontinuity

$$f(x) = \begin{cases} a(a+2), & \text{if } x = 1 \\ a^3x, & \text{if } x > 1 \\ 3a^2x^2 - 2ax, & \text{if } x < 1, \end{cases}$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) \Leftrightarrow 3a^2 - 2a = a^3 \Leftrightarrow a(a^2 - 3a + 2) = 0$$

$$\Leftrightarrow a(a-1)(a-2) = 0 \Leftrightarrow a = 0, 1, 2$$

For  $a=0$   $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = 0 = f(1)$

$f$  is continuous for  $a=0$

For  $a=1$   $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = 1 \neq f(1) = 3$

$\Rightarrow f$  has a removable discontinuity for  $a=1$

For  $a=2$   $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = 8 = f(2) \Rightarrow f$  is continuous for  $a=2$