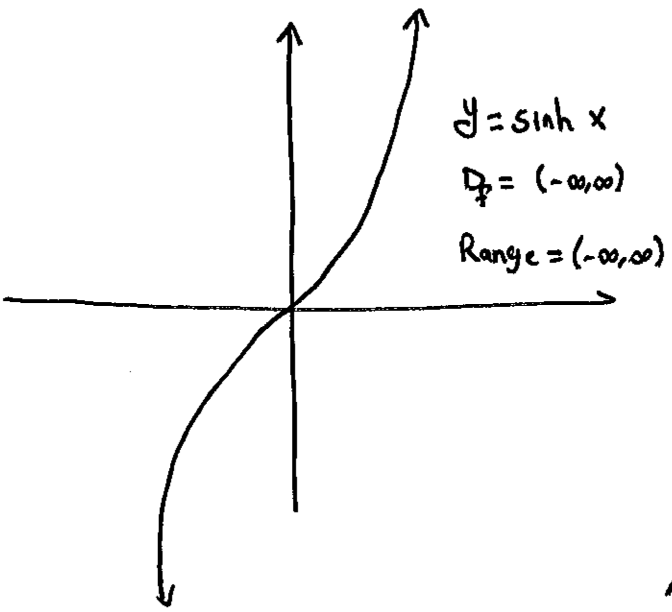
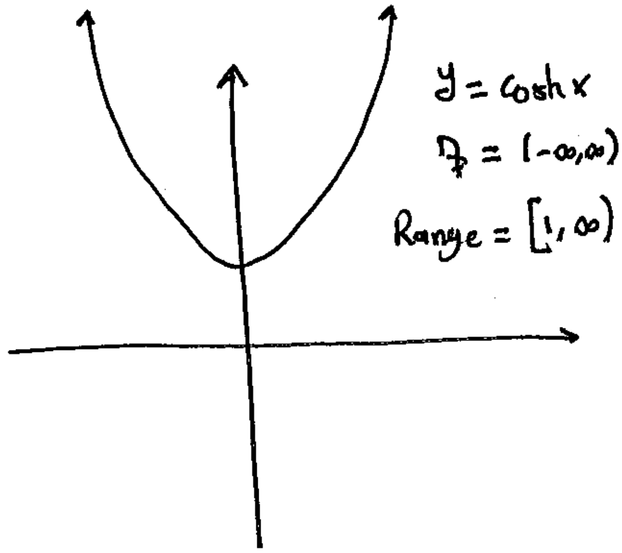
7.3 Hyperblic Functions

Def ①:

1. $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$ 2. $\cosh x = \dfrac{e^{x} + e^{-x}}{2}$ 3. $\tanh x = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

4. $\operatorname{csch} x = \dfrac{1}{\sinh x}$ 5. $\operatorname{sech} x = \dfrac{1}{\cosh x}$ 6. $\coth x = \dfrac{1}{\tanh x}$

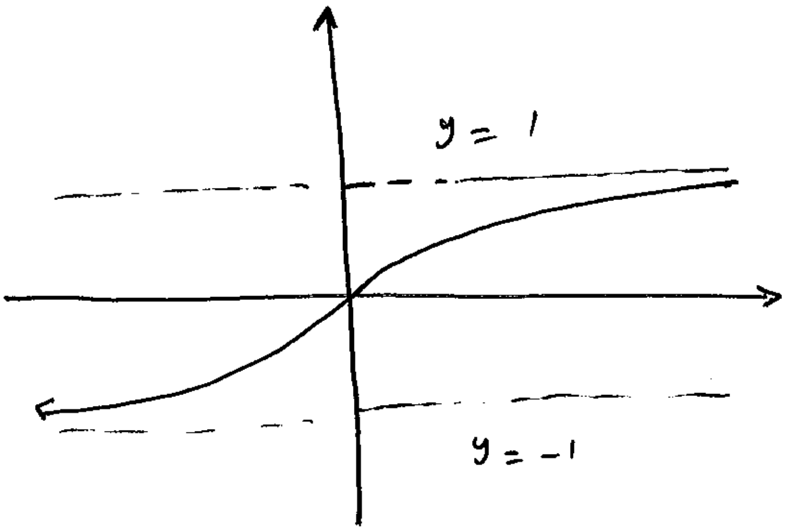




$y = \tanh x$

$D_f = (-\infty, \infty)$

Range $= (-1, 1)$



EX1 Find the Value of

a) $\tanh(0) = \dfrac{e^{0} - e^{-0}}{e^{0} + e^{-0}} = \dfrac{1-1}{1+1} = \dfrac{0}{2} = 0.$

b) $\tanh(\ln 3)$

$$= \frac{e^{\ln 3} - e^{-\ln 3}}{e^{\ln 3} + e^{-\ln 3}} = \frac{3 - \frac{1}{3}}{3 + \frac{1}{3}} = \frac{9-1}{9+1} = \frac{8}{10} = \frac{4}{5}.$$

c) $$\frac{1 - \tanh(\frac{1}{2})}{1 + \tanh(\frac{1}{2})} = \frac{1 - \frac{e^{\frac{1}{2}} - e^{-\frac{1}{2}}}{e^{\frac{1}{2}} + e^{-\frac{1}{2}}}}{1 + \frac{e^{\frac{1}{2}} - e^{-\frac{1}{2}}}{e^{\frac{1}{2}} + e^{-\frac{1}{2}}}} = \frac{(e^{\frac{1}{2}} + e^{-\frac{1}{2}}) - (e^{\frac{1}{2}} - e^{-\frac{1}{2}})}{(e^{\frac{1}{2}} + e^{-\frac{1}{2}}) + (e^{\frac{1}{2}} - e^{-\frac{1}{2}})}$$

$$= \frac{\cancel{2}\, e^{-\frac{1}{2}}}{\cancel{2}\, e^{\frac{1}{2}}} = \frac{1}{e}.$$

EX2 If $\cosh(\ln 2x) = 1$, then find $\cosh(2x)$.

<u>Soln</u>:

$$\cosh(\ln 2x) = 1 \Rightarrow \frac{e^{\ln 2x} + e^{-\ln 2x}}{2} = 1$$

$$\Rightarrow 2x + \frac{1}{2x} = 2$$

$$\Rightarrow (2x)^2 - 2(2x) + 1 = 0$$

$$\Rightarrow (2x-1)^2 = 0 \Rightarrow \boxed{2x = 1}$$

$$\therefore \cosh(2x) = \cosh(1)$$

$$= \frac{e + e^{-1}}{2} = \frac{e^2+1}{2e}.$$

Hyperbolic Identities

1) $\sinh(-x) = -\sinh x$

2) $\cosh(-x) = \cosh x$

3) $\cosh^2 x - \sinh^2 x = 1$

4) $1 - \tanh^2 x = \operatorname{sech}^2 x$

5) $\coth^2 x - 1 = \operatorname{csch}^2 x$

6) $\cosh^2 x = \dfrac{1 + \cosh 2x}{2}$

7) $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

EX3 show that $\cosh x + \sinh x = e^x$

$$L.H.S = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = \frac{2e^x}{2} = e^x = R.H.S.$$

EX4 If $\tanh x = \frac{4}{5}$ Find $\cosh x$.

Soln:

$$1 - \tanh^2 x = \operatorname{sech}^2 x \Rightarrow \operatorname{sech}^2 x = 1 - \frac{16}{25} = \frac{9}{25}$$

$$\therefore \operatorname{sech} x = \frac{3}{5} \quad (\operatorname{sech} x > 0)$$

$$\therefore \cosh x = \frac{5}{3}$$

Derivatives of hyperbolic functions

1) $\frac{d}{dx}(\sinh x) = \cosh x$

2) $\frac{d}{dx}(\cosh x) = \sinh x$

3) $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

4) $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

5) $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

6) $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

EX5 Find the derivative of

a) $y = \tanh(4x)$

$y' = 4 \operatorname{sech}^2(4x)$

b) $y = \sinh(\cosh x)$

$y' = \cosh(\cosh x) \cdot \sinh x$

c) $y = \tanh(\sqrt{1+x^2}) \Rightarrow y' = \operatorname{sech}^2(\sqrt{1+x^2}) \cdot \frac{\cancel{2}x}{\cancel{2}\sqrt{1+x^2}}$

d) $y = \ln(\sinh x)$

$$y' = \frac{\cosh x}{\sinh x} = \coth x.$$

Integral of hyperbolic functions.

1) $\int \sinh x \, dx = \cosh x + C$

2) $\int \cosh x \, dx = \sinh x + C$

3) $\int \operatorname{sech}^2 x \, dx = \tanh x + C$

4) $\int \operatorname{csch}^2 x \, dx = -\coth x + C$

5) $\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + C$

6) $\int \operatorname{csch} x \coth x \, dx = -\operatorname{csch} x + C$

EX6 Evaluate

a) $\int \coth(5x) \, dx = \int \frac{\cosh(5x)}{\sinh(5x)} \, dx$

let $u = \sinh(5x)$

$du = 5\cosh(5x)\, dx$

$$= \frac{1}{5} \int \frac{du}{u}$$

$$= \frac{1}{5} \ln|u| + C$$

$$= \frac{1}{5} \ln|\sinh(5x)| + C.$$

b) $\int_0^1 \sinh^2 x \, dx$

$$= \int_0^1 \frac{\cosh 2x - 1}{2} \, dx = \frac{1}{2}\int_0^1 \left(\cosh(2x) - 1\right) dx$$

$$= \frac{1}{2}\left[\frac{1}{2}\sinh(2x) - x \Big|_0^1\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\sinh(2) - 1\right]$$

$$= \frac{1}{4}\sinh(2) - \frac{1}{2}.$$

c) $\int_0^{\ln 2} 4e^x \sinh x \, dx$

$$= 4\int_0^{\ln 2} e^x \cdot \left(\frac{e^x - e^{-x}}{2}\right) dx$$

$$= 2\int_0^{\ln 2} \left(e^{2x} - 1\right) dx = 2\left[\frac{1}{2}e^{2x} - x \Big|_0^{\ln 2}\right]$$

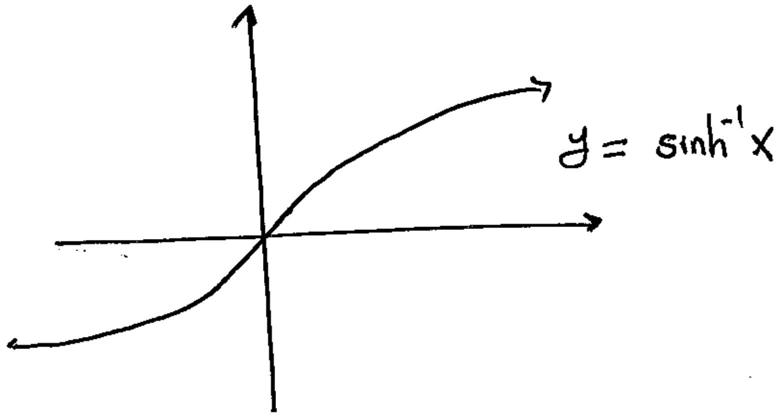
$$= e^{2x} - 2x \Big|_0^{\ln 2} = e^{2\ln 2} - 2\ln 2 - 1$$

$$= 4 - \ln 4 - 1$$

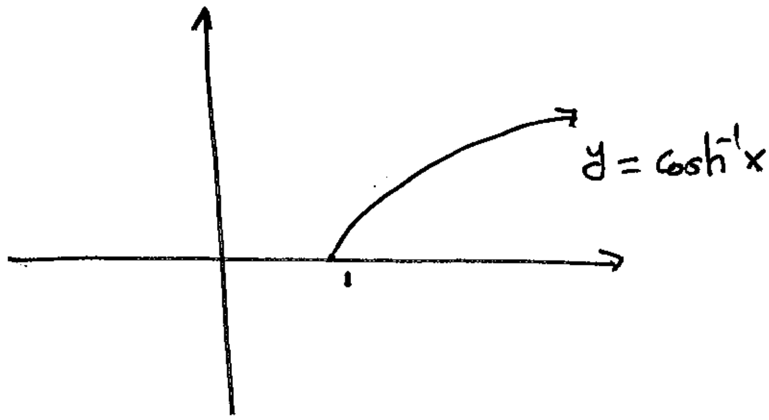
$$= 3 - \ln 4$$

Inverse Hyperbolic Functions

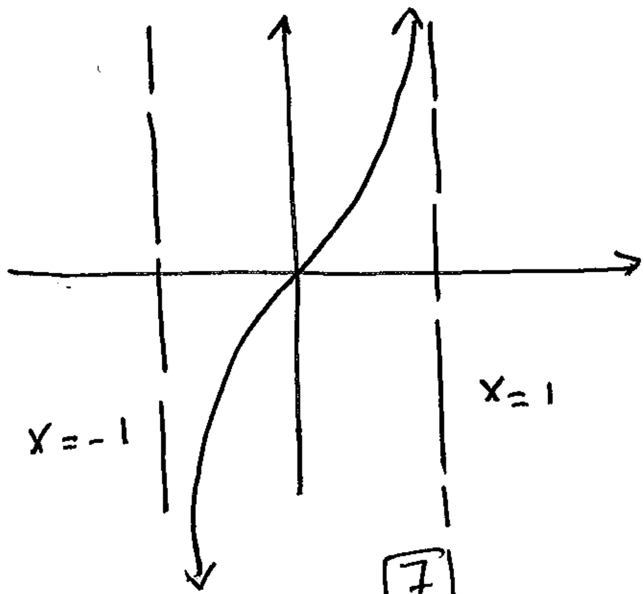
1) $y = \sinh^{-1} x \iff \sinh y = x$, $D_f = \text{Range } f = \mathbb{R}$



2) $y = \cosh^{-1} x \iff \cosh y = x$, $D_f = [1, \infty)$, $\text{Range} = [0, \infty)$



3) $y = \tanh^{-1} x \iff \tanh y = x$, $D_f = (-1, 1)$. $\text{Range} = \mathbb{R}$.



EX7 show that

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) \quad , x \in \mathbb{R}$$

<u>Soln</u>: let $\sinh^{-1} x = y \Leftrightarrow \sinh y = x$

$$\Rightarrow \frac{e^{y} - e^{-y}}{2} = x$$

$$\Rightarrow e^{y} - e^{-y} = 2x$$

$$\Rightarrow e^{2y} - (2x)\, e^{y} - 1 = 0$$

$$\therefore \quad e^{y} = \frac{2x \pm \sqrt{4x^2+4}}{2} = \frac{2x \pm 2\sqrt{x^2+1}}{2}$$

$$= x \pm \sqrt{x^2+1}$$

$$\Rightarrow e^{y} = x - \sqrt{x^2+1} \ (\text{rejected}) \ \text{or}$$

$$e^{y} = x + \sqrt{x^2+1} \Rightarrow y = \ln\left(x + \sqrt{x^2+1}\right).$$

Derivatives of Inverse Hyperbolic Functions

1) $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{1+x^2}} \quad , x \in \mathbb{R}.$

2) $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}} \quad , x > 1$

3) $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2} \quad , |x| < 1$

4) $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2} \quad , |x| > 1$

5) $\frac{d}{dx}\left(\operatorname{sech}^{-1}x\right) = -\frac{1}{x\sqrt{1-x^2}}$, $0 < x < 1$

6) $\frac{d}{dx}\left(\operatorname{csch}^{-1}x\right) = -\frac{1}{|x|\sqrt{x^2+1}}$, $x \neq 0$

EX8 Find $\frac{dy}{dx}$.

a) $y = x^2 \sinh^{-1}(2x)$

$$\frac{dy}{dx} = (2x)\sinh^{-1}(2x) + x^2 \cdot \frac{2}{\sqrt{1+(2x)^2}}$$

$$= 2x\left[\sinh^{-1}(2x) + \frac{x}{\sqrt{1+4x^2}}\right]$$

b) $y = \coth^{-1}\sqrt{x^2+1}$

$$\frac{dy}{dx} = \frac{1}{1-\left(\sqrt{x^2+1}\right)^2} \cdot \frac{2x}{2\sqrt{x^2+1}}$$

$$= \frac{1}{1-x^2-1} \cdot \frac{x}{\sqrt{x^2+1}}$$

$$= -\frac{1}{x\sqrt{x^2+1}}.$$

c) $y = x\sinh^{-1}\left(\frac{x}{3}\right) - \sqrt{9+x^2}$

$$y' = \sinh^{-1}\left(\frac{x}{3}\right) + x \cdot \frac{\frac{1}{3}}{\sqrt{1+\frac{x^2}{9}}} - \frac{\cancel{2}x}{\cancel{2}\sqrt{9+x^2}}$$

$$= \sinh^{-1}\left(\frac{x}{3}\right) + \frac{\cancel{\frac{1}{3}}x}{\cancel{\frac{1}{3}}\sqrt{9+x^2}} - \frac{x}{\sqrt{9+x^2}} = \sinh^{-1}\left(\frac{x}{3}\right)$$

d) $y = \ln(\cosh x) - \frac{1}{2}\tanh^2 x$

$$\frac{dy}{dx} = \frac{\sinh x}{\cosh x} - \tanh x \cdot \operatorname{sech}^2 x$$

$$= \tanh x - \tanh x \operatorname{sech}^2 x$$

$$= \tanh x \left(1 - \operatorname{sech}^2 x\right)$$

$$= \tanh x \cdot \tanh^2 x = \tanh^3 x.$$

Integrals of In terms of Inverse hyperbolic functions.

1) $$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \quad , \quad a > 0$$

2) $$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \quad , \quad x > a > 0$$

3) $$\int \frac{dx}{a^2-x^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + C & , \quad x^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right) + C & , \quad x^2 > a^2 \end{cases}$$

4) $$\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{x}{a}\right) + C \quad , \quad 0 < x < a$$

5) $$\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{x}{a}\right| + C \quad , \quad x \neq 0, \; a > 0$$

EX 9 Evaluate

$$\int_0^1 \frac{2}{\sqrt{3+4x^2}}\,dx$$

$$= \int_0^1 \frac{2}{\sqrt{3+(2x)^2}}\,dx \qquad \text{let } u = 2x$$
$$du = 2\,dx$$

$$= \int_0^2 \frac{du}{\sqrt{3+u^2}}$$

$$= \sinh^{-1}\left(\frac{u}{\sqrt{3}}\right)\Big|_0^2 = \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right).$$

EX 10 Prove that $\frac{d}{dx}\left(\sinh^{-1}x\right) = \frac{1}{\sqrt{1+x^2}}$

<u>Soln</u>: let $y = \sinh^{-1}x \Leftrightarrow \sinh y = x$

$$\Rightarrow \cosh y \cdot y' = 1$$

$$\Rightarrow y' = \frac{1}{\cosh y}, \qquad \cosh^2 y - \sinh^2 y = 1$$

$$\Rightarrow \cosh y = \sqrt{1+\sinh^2 y}$$
$$= \sqrt{1+x^2}$$

$$\therefore y' = \frac{1}{\sqrt{1+x^2}}$$

$$\therefore \frac{d}{dx}\left(\sinh^{-1}x\right) = \frac{1}{\sqrt{1+x^2}}.$$

END.