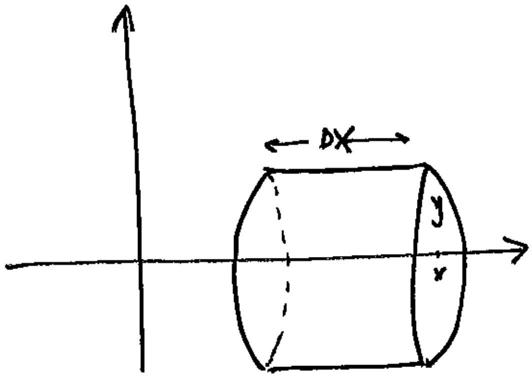
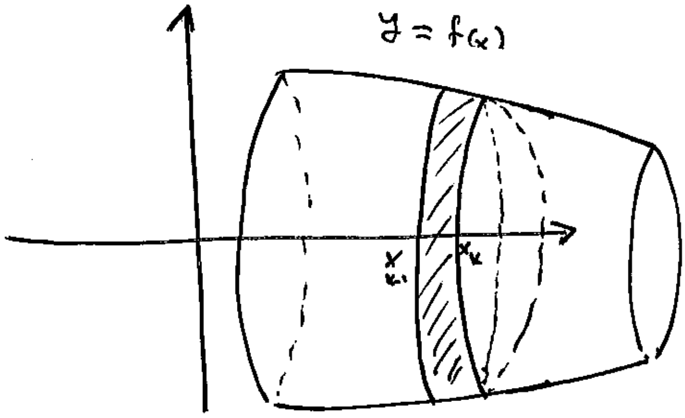
6.4 Areas of Surfaces of Revolution

Idea:



Surface Area $= 2\pi y \, \Delta x$

Question: How to find the Area of the surface generated by revolving the graph of a nonnegative function $y = f(x)$, $a \le x \le b$ about x-axis ?



$$S_k \approx 2\pi \cdot \frac{f(x_{k-1}) + f(x_k)}{2} \cdot \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

$$\approx 2\pi \; f(c_k) \sqrt{1 + (f'(c_k))^2} \; \Delta x_k$$

$$S \approx \sum_{k=1}^{n} 2\pi \; f(c_k) \sqrt{1 + (f'(c_k))^2} \; \Delta x_k$$

$$S = \lim_{\|P\| \to 0} \sum_{k=1}^{n} 2\pi f(c_k) \sqrt{1 + (f'(c_k'))^2} \, \Delta x_k$$

$$= \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx$$

Definion: If the function $f(x) \geq 0$ is continuously differentiable on $[a,b]$, the area of the surface generated by revolving the graph of $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx$$

EX1 Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$, about the x-axis.

Soln: $\frac{dy}{dx} = \frac{1}{\sqrt{x}} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{x} = \frac{x+1}{x}$

$$\therefore S = \int_1^2 2\pi (2\sqrt{x}) \cdot \sqrt{\frac{x+1}{x}} \, dx$$

$$= 4\pi \int_1^2 \sqrt{x+1} \, dx = 4\pi \left((x+1)^{3/2} \cdot \frac{2}{3} \Big|_1^2 \right)$$

$$= \frac{8\pi}{3} \left(3^{3/2} - 2^{3/2} \right) = \frac{8\pi}{3} \left(3\sqrt{3} - 2\sqrt{2} \right).$$

* Surface Area for Revolution about the y-axis.

If $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the graph of $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi\, g(y) \sqrt{1 + (g'(y))^2}\, dy .$$

EX2 The Line segment $x = 1 - y$, $0 \leq y \leq 1$, is revolved about the y-axis. Find the surface area of the generated Cone.

<u>Soln</u>:

$$S = \int_0^1 2\pi (1-y) \sqrt{1 + (-1)^2}\, dy$$

$$= 2\pi\sqrt{2} \int_0^1 1 - y\, dy$$

$$= 2\pi\sqrt{2} \left(y - \frac{y^2}{2} \Big|_0^1 \right) = \pi\sqrt{2} .$$

EX3 Find the area of the surface generated by revolving the curve $y = \sqrt{2x - x^2}$, $1/2 \leq x \leq 3/2$ about the x-axis.

$$\frac{dy}{dx} = \frac{2 - 2x}{2\sqrt{2x - x^2}} = \frac{1 - x}{\sqrt{2x - x^2}}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{(1-x)^2}{2x-x^2} = \frac{2x-x^2+1-2x+x^2}{2x-x^2}$$

$$= \frac{1}{2x-x^2}$$

$$\therefore \quad S = \int_{\frac{1}{2}}^{\frac{3}{2}} 2\pi \sqrt{2x-x^2} \cdot \sqrt{\frac{1}{2x-x^2}}\, dx$$

$$= 2\pi \int_{\frac{1}{2}}^{\frac{3}{2}} dx = 2\pi .$$

END.