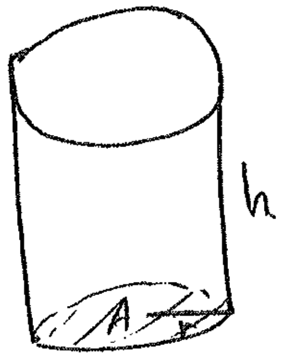
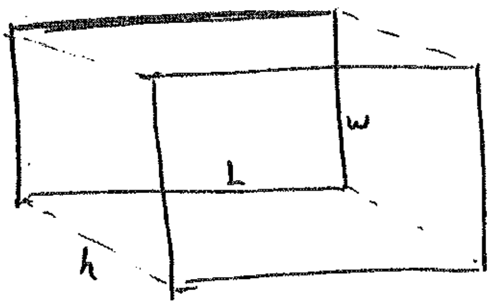
6.1 Volumes using cross-sections.

Volume of a cylinder: $V = A \times h$



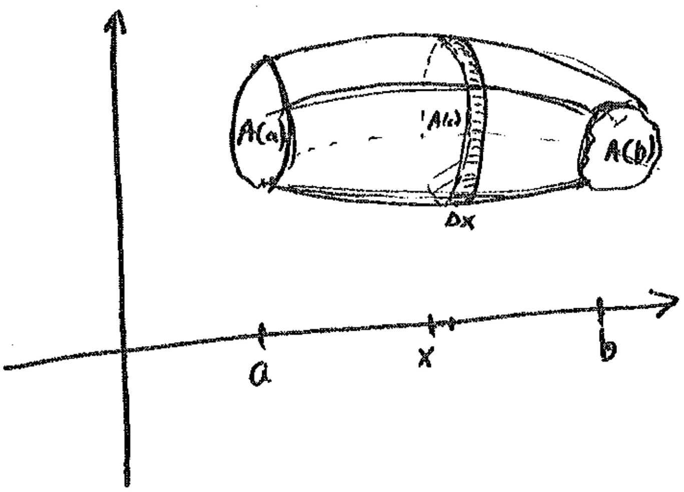
$V = \pi r^2 h$



$V = Lwh.$

Question: what if the solid region is Not a cylinder?

How to find the volume?



$$V \approx \sum_{i=1}^{n} A(x_i^*) \Delta x$$

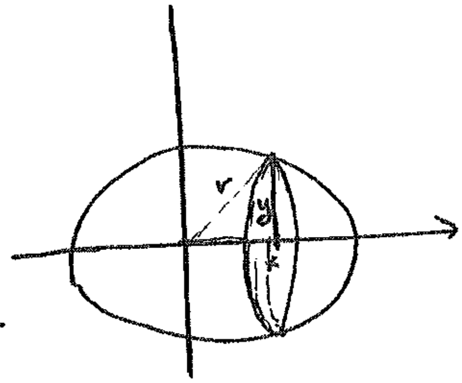
Volume: $V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*) \Delta x$

$$= \int_a^b A(x)\, dx$$

EX1 show that the volume of a sphere of radius r is

$V = \frac{4}{3}\pi r^3$

Soln: we set the sphere so that the center is at the origin.



$r^2 = x^2 + y^2$

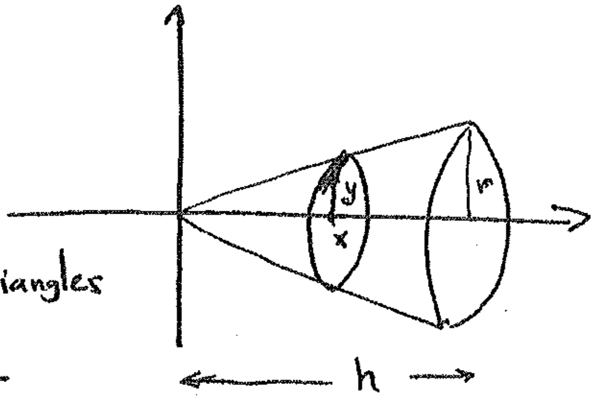
$\Rightarrow y^2 = r^2 - x^2 \Rightarrow y = \sqrt{r^2 - x^2}$

$A(x) = \pi(r^2 - x^2)$

$$V = \int_{-r}^{r} \pi(r^2 - x^2)\, dx = \frac{4}{3}\pi r^3$$

Q Show that the volume of Circular Cone with base radius r and height h is $V = \frac{1}{3}\pi r^2 h$.

<u>soln</u>:



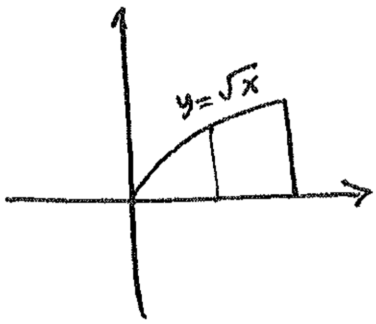
From the similar triangles we have $\frac{y}{x} = \frac{r}{h}$

$$\Rightarrow y = \frac{rx}{h} \Rightarrow A(x) = \pi\left(\frac{rx}{h}\right)^2$$

$$\Rightarrow V = \int_0^h \frac{\pi r^2 x^2}{h^2}\,dx = \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right|_0^h$$

$$= \frac{1}{3}\pi r^2 h.$$

EX2 Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

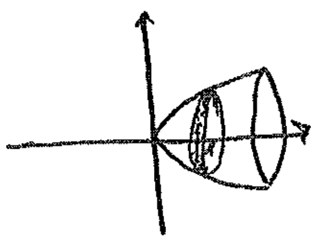


Cross section will be a disk.

$$A(x) = \pi(\sqrt{x})^2 = \pi x$$

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi x\,dx$$

$$= \pi \frac{x^2}{2}\Big|_0^1 = \frac{\pi}{2}.$$



EX3 Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$ and $x = 0$ about the y-axis.

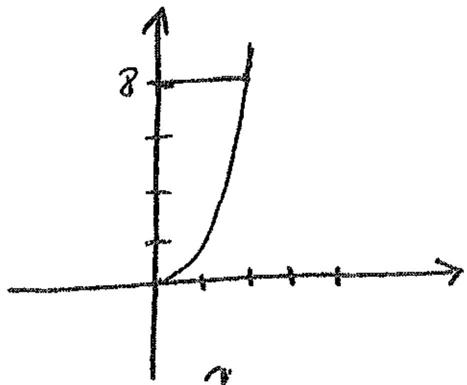
thickness $= \Delta y$

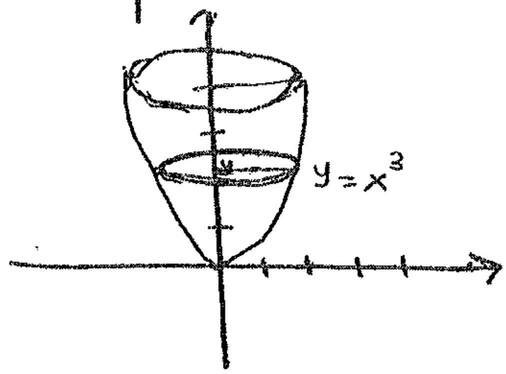
radius $= \sqrt[3]{y}$

$A(y) = \pi \sqrt[3]{y^2} = \pi y^{\frac{2}{3}}$

$$V = \int_0^8 A(y)\, dy = \int_0^8 \pi y^{\frac{2}{3}}\, dy$$

$$= \frac{96\pi}{5}.$$





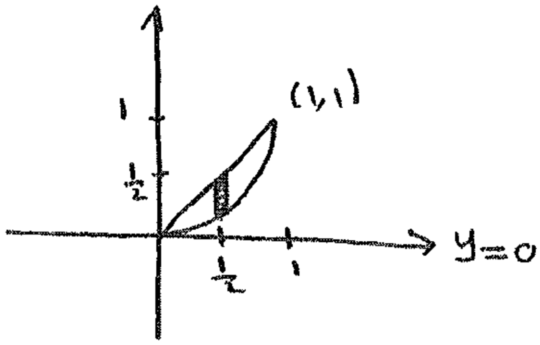
<u>Remark</u>: If we rotate a region bounded by two graphs, then the cross section will be a washer ◎

$$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$

$$V = A * \text{thickness} \quad (\text{Approximating cylinder})$$

$$\text{thicknes} = \begin{cases} \Delta x & \text{if the axis of revolution is horizontal} \\ \Delta y & \text{" " " " " vertical} \end{cases}$$

EX4 The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

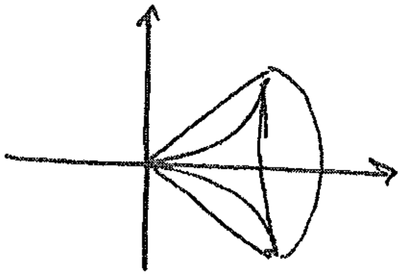


thickness $= \Delta x$

outer radius $= x$

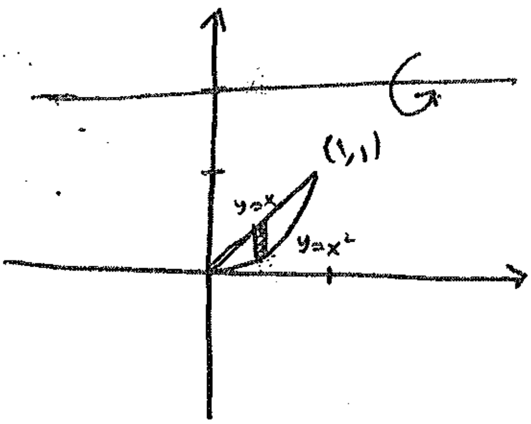
inner radius $= x^2$

$$A(x) = \pi\left[x^2 - (x^2)^2\right]$$



$$V = \pi\int_0^1 (x^2 - x^4)\,dx = \frac{2\pi}{15}$$

EX5 Find the volume of the solid obtained by rotating the region in EX4 about the Line $y = 2$.



thickness $= \Delta x$

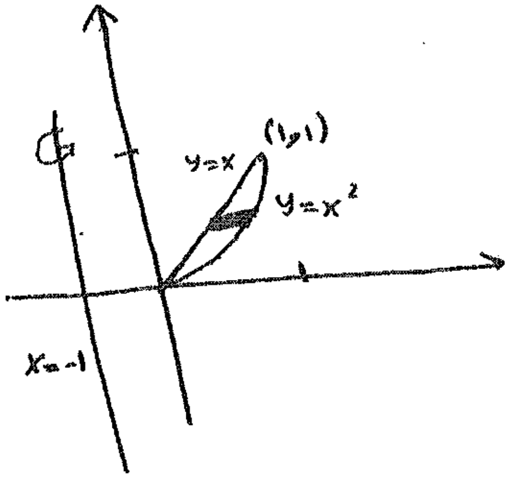
outer radius $= 2 - x^2$

inner radius $= 2 - x$

$$A(x) = \pi\left((2-x^2)^2 - (2-x)^2\right)$$

$$V = \pi\int_0^1 A(x)\,dx = \pi\int_0^1 \left((2-x^2)^2 - (2-x)^2\right)dx = \frac{8\pi}{15}$$

EX6 Find the volume of the solid obtained by rotating the region in EX4 about the line $x=-1$



thickness $= \Delta y$

outer radius $= \sqrt{y} + 1$

Inner radius $= y + 1$

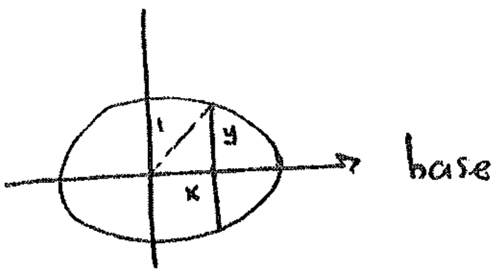
$$A(y) = \pi\left[(\sqrt{y}+1)^2 - (y+1)^2\right]$$

$$V = \pi\int_0^1 (\sqrt{y}+1)^2 - (y+1)^2 \, dy$$

$$= \frac{\pi}{2}.$$

EX7 A solid with a circular base of radius 1. If the parallel cross-section perpendicular to the base are equilateral triangles. Find the volume of the solid.

thickness : Δx

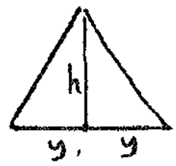


$A(x) = \frac{1}{2}$ base $*$ height

$y = \sqrt{1-x^2}$.

base $= 2\sqrt{1-x^2}$

$h = \sqrt{3}\, y = \sqrt{3}\sqrt{1-x^2}$



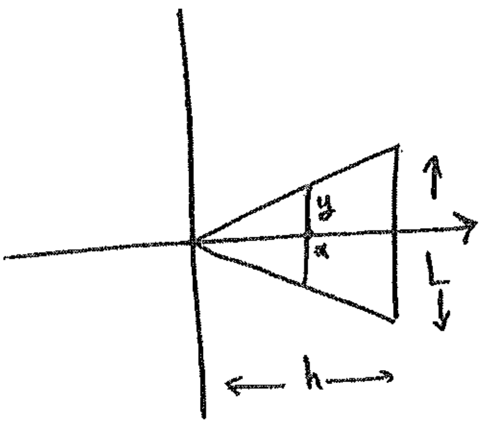
$\tan 60^\circ = \frac{h}{y}$

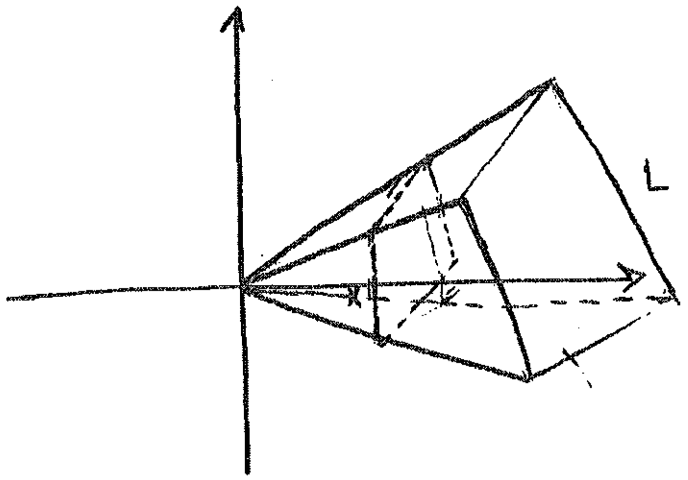
$h = y \tan 60^\circ = \sqrt{3}\, y$

$$A(x) = \frac{1}{2} \cdot 2\sqrt{1-x^2}\,\sqrt{3}\,\sqrt{1-x^2} = \sqrt{3}\,(1-x^2)$$

$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \sqrt{3}\,(1-x^2)\,dx = \frac{4\sqrt{3}}{3}.$$

<u>EX8</u> Find the Volume of a pyramid whose base is a square with side L and whose height is h.





let the Length of the square be $(2y)$

From the similar triangles

$$\frac{y}{x} = \frac{\frac{L}{2}}{h} \Rightarrow y = \frac{xL}{2h} \Rightarrow 2y = \frac{xL}{h}$$

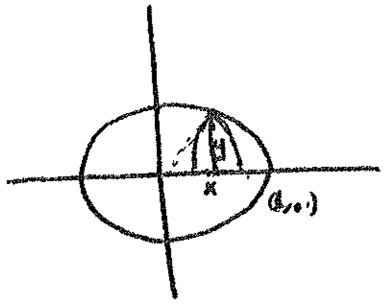
$$A(x) = \frac{L^2 x^2}{h^2} \Rightarrow V = \int_0^h \frac{L^2 x^2}{h^2}\,dx$$

$$= \frac{L^2 h}{3}.$$

OEQ(081)

A solid has a circular base of radius 1 and center (o, o)
If the cross-sections of the solid perpendicular to the x-axis are semicircles, then the volume of the solid is equal = ?

Soln:



$$x^2+y^2=1$$

$$y^2=1-x^2$$

$$A(x)=\frac{\pi}{2}\left(1-x^2\right)$$

$$V=\int_{-1}^{1}\frac{\pi}{2}\left(1-x^2\right)dx$$

$$=\pi\int_{0}^{1}1-x^2\,dx=\pi\left[x-\frac{x^3}{3}\right|_0^1$$

$$=\pi\left[1-\frac{1}{3}\right]=\frac{2\pi}{3}$$

END