5.6 Substitution and Area between Curves.

Thm①: If g' is continuous on $[a,b]$ and f is cont. on the range of $g(x)=u$, then $\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$

EX1 Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$

Let $u = x^3+1 \Rightarrow du = 3x^2\,dx$. If $x=-1 \Rightarrow u=0$,

$x=1 \Rightarrow u=2$

$$\Rightarrow I = \int_0^2 u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}}\Big|_0^2 = \frac{2}{3}\sqrt{8} = \frac{4\sqrt{2}}{3}.$$

EX2 Evaluate

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot\theta\,\csc^2\theta\,d\theta.$$

Let $u = \cot\theta \Rightarrow du = -\csc^2\theta\,d\theta$

$$\Rightarrow I = \int_1^0 -u\,du = \int_0^1 u\,du = \frac{u^2}{2}\Big|_0^1$$

$$= \frac{1}{2}.$$

Theorem: Let f be cont. on $[-a,a]$, then

① If f is an Even, $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

② " " " odd, $\int_{-a}^{a} f(x)\,dx = 0$

EX3 : Find the value of

a) $$\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4}\,dx .$$

$$f(x) = \frac{\tan x}{1+x^2+x^4} \Rightarrow f(-x) = \frac{\tan(-x)}{1+(-x)^2+(-x)^4}$$

$$= \frac{-\tan x}{1+x^2+x^4} = -f(x)$$

$\therefore$ $f(x)$ is odd.

$$\Rightarrow \int_{-1}^{1} \frac{\tan x}{1+x^2+x^4}\,dx = 0$$

b) $$\int_{-1}^{1} \frac{x^2+\tan x}{1+x^2}\,dx$$

$$I = \int_{-1}^{1} \frac{x^2}{1+x^2}\,dx + \int_{-1}^{1} \cancelto{0}{\frac{\tan x}{1+x^2}}\,dx$$

$$= 2\int_{0}^{1} \frac{x^2}{1+x^2}\,dx = 2\int_{0}^{1}\left(1 - \frac{1}{1+x^2}\right)dx$$

$$= 2\left[\, x - \tan^{-1}x \Big|_0^1 \right] = 2\left[1 - \frac{\pi}{4}\right]$$

$$= 2 - \frac{\pi}{2} .$$

c) $$\int_{-1}^{1} \frac{x^2+\tan^{-1}x}{1+x^2}\,dx \quad (\text{EFS}).$$

EX4 Find $\int_{-2}^{2} x^4 \left(x e^{-x^2} + 5 \right) dx$.

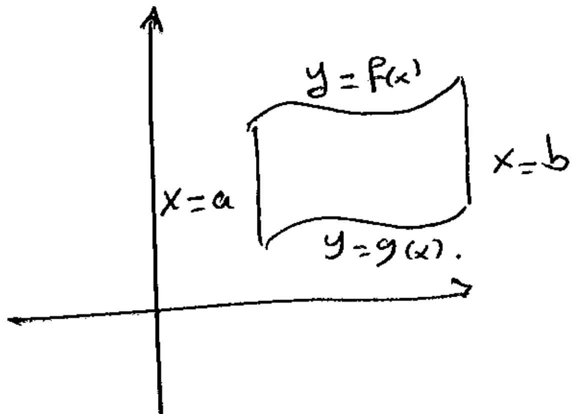
$$I = \cancel{\int_{-2}^{2} x^5 e^{-x^2} dx}^{\,0} + \int_{-2}^{2} 5x^4 dx$$

$$= 10 \int_{0}^{2} x^4 dx = 10 \left(\frac{x^5}{5} \Big|_0^2 \right) = 2(32) = 64.$$

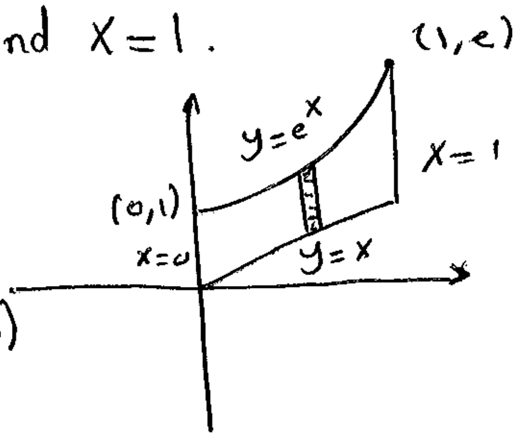
Area between Curves.

Case I : How to find the area that lies between two curves $y = f(x)$ and $y = g(x)$ and vertical lines $x = a$, $x = b$, where $f(x)$ and $g(x)$ are cont. and $f(x) \geq g(x)$ for all $x \in [a,b]$.

$$\text{Area} = \int_{a}^{b} [f(x) - g(x)] \, dx.$$



EX5 Find the Area of the region bounded by $y = e^x$, $y = x$ and the Lines $x = 0$ and $x = 1$.



$$\text{Area} = \int_{0}^{1} (e^x - x) \, dx$$

$$= e^x - \frac{x^2}{2} \Big|_0^1 = \left(e - \tfrac{1}{2}\right) - (1 - 0)$$

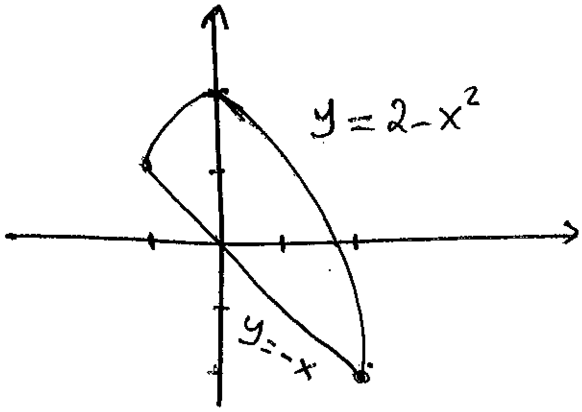
$$= e - \tfrac{3}{2}$$

EX5 Find the Area of the region enclosed by the parabola $y = 2 - x^2$ and the Line $y = -x$.

Soln: Find the intersection points.

$$2 - x^2 = -x \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$$

$$\Rightarrow x = -1, \quad x = 2.$$



$$\therefore \text{Area} = \int_{-1}^{2} (2-x^2) - (-x)\, dx$$

$$= \int_{-1}^{2} (-x^2 + x + 2)\, dx$$

$$= -\frac{x^3}{3} + \frac{x^2}{2} + 2x \Big|_{-1}^{2}$$

$$= \left(-\frac{8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right)$$

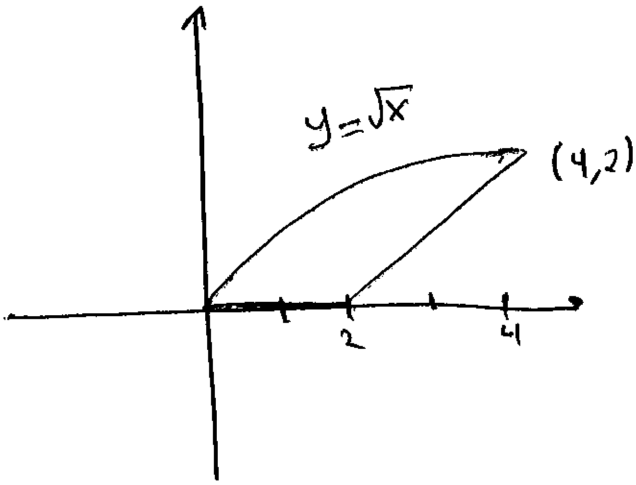
$$= -3 + 8 - \frac{1}{2} = 5 - \frac{1}{2} = \frac{9}{2}.$$

EX6 Find the area of the region in the first quadrant bounded by $y = \sqrt{x}$ and below by the x-axis and $y = x - 2$.

Soln: $\sqrt{x} = x - 2 \Rightarrow x = x^2 - 4x + 4$

$$\Rightarrow x^2 - 5x + 4 = 0 \Rightarrow (x-1)(x-4) = 0$$

$$\Rightarrow \begin{array}{l} x = 1 \text{ rejected} \\ x = 4 \checkmark \end{array}$$



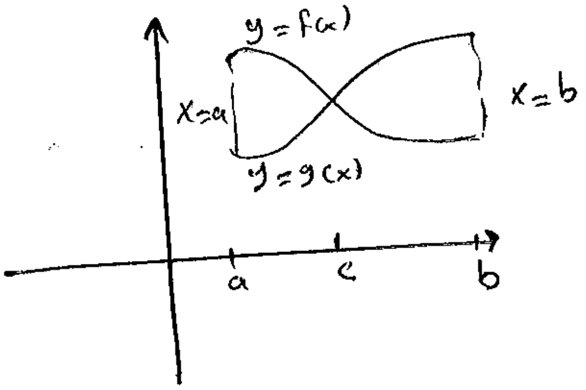
$$\text{Area} = \int_0^2 (\sqrt{x} - 0)\,dx + \int_2^4 \sqrt{x} - (x-2)\,dx$$

$$= \frac{2}{3}x^{\frac{3}{2}}\Big|_0^2 + \frac{2}{3}x^{\frac{3}{2}}\Big|_2^4 - \frac{x^2}{2}\Big|_2^4 + 2x\Big|_2^4$$

$$= \cancel{\frac{2}{3}\sqrt{8}} + \frac{16}{3} - \cancel{\frac{2}{3}\sqrt{8}} - 8 + 2 + 8 - 4$$

$$= \frac{16}{3} - 2 = \frac{10}{3}.$$

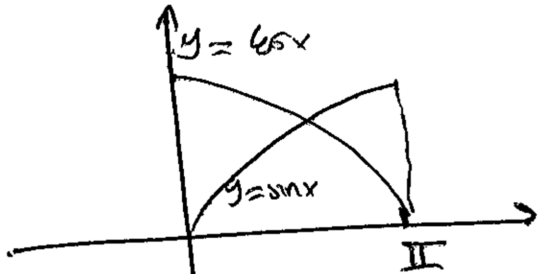
Case II :



$$\text{Area} = \int_a^b |f(x) - g(x)|\,dx = \int_a^c [f(x) - g(x)]\,dx + \int_c^b [g(x) - f(x)]\,dx$$

EX7 Find the Area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \frac{\pi}{2}$.

<u>Soln</u>: $\sin x = \cos x \Rightarrow x = \frac{\pi}{4}$.



<u>Soln</u>:

$$\text{Area} = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)\, dx$$

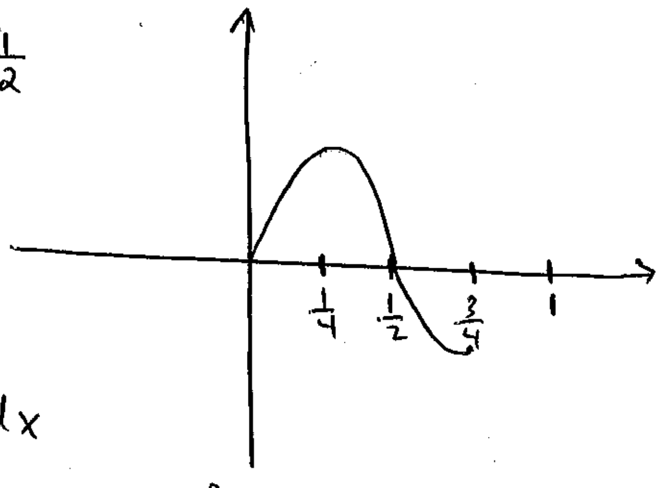
$$= 2\sqrt{2} - 2.$$

EX8 Find the area of the region enclosed by the curve $y = \pi \sin(2\pi x)$ and the x-axis between $x = 0$ and $x = \frac{3}{4}$.

$$\pi \sin(2\pi x) = 0 \Rightarrow 2\pi x = n\pi, \quad n = 0, \pm 1, \dots$$

$$\Rightarrow x = \frac{n}{2}$$

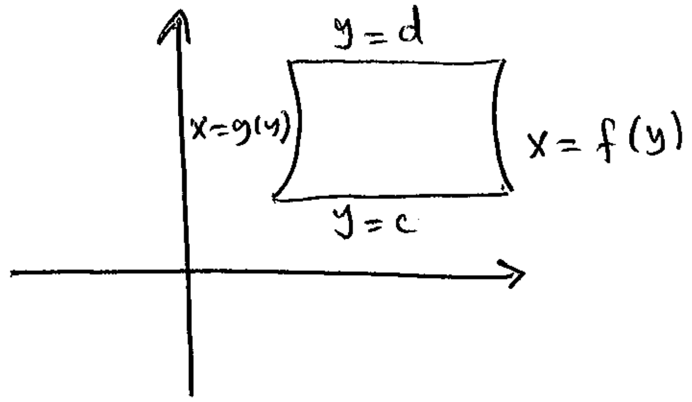
$$\therefore x = \frac{1}{2}$$



$$\text{Area} = \int_0^{\frac{1}{2}} \pi \sin(2\pi x)\, dx + \int_{\frac{1}{2}}^{\frac{3}{4}} 0 - \pi \sin(2\pi x)\, dx$$

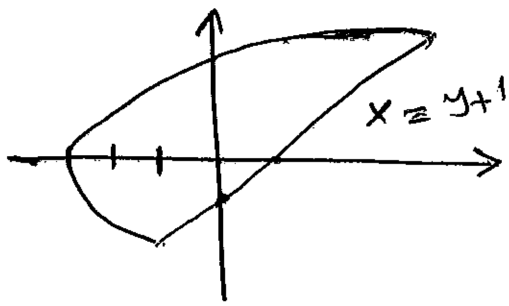
$$= -\frac{1}{2}\cos(2\pi x)\Big|_0^{\frac{1}{2}} + \frac{1}{2}\cos(2\pi x)\Big|_{\frac{1}{2}}^{3/4} = -\frac{1}{2}(-1-1) + \frac{1}{2}(0+1) = 3/2$$

Case III :



$$\text{Area} = \int_c^d [f(y) - g(y)]\, dy .$$

EX 9 Find the area enclosed by the Line $y = x - 1$ and the parabola $y^2 = 2x + 6$.



$y = x - 1 \Rightarrow x = y + 1$.

$y^2 = 2x + 6 \Rightarrow x = \dfrac{y^2 - 6}{2}$.

$$\therefore \frac{y^2 - 6}{2} = y + 1 \Rightarrow y^2 - 2y - 8 = 0$$

$$\Rightarrow (y - 4)(y + 2) = 0$$

$$\Rightarrow y = 4 , \ y = -2$$

$\therefore$ Intersection points $(5, 4)$, $(-1, -2)$

Method I

$$\text{Area} = \int_{-2}^{4} (y+1) - \left(\frac{y^2 - 6}{2} \right) dy$$

$$= \int_{-2}^{4} \left(y + 1 - \frac{y^2}{2} + 3 \right) dy = \frac{y^2}{2} + y - \frac{y^3}{6} + 3y \Big|_{-2}^{4}$$

$$= 18 .$$

Method II

$$\text{Area} = \int_{-3}^{-1} \sqrt{2x+6} - \left(-\sqrt{2x+6}\right) dx + \int_{-1}^{5} \left[\sqrt{2x+6} - (x-1)\right] dx$$

$$= 18 .$$

EX 10 Find the Area of the region in the right-half plane bounded by the curves $y = 2x-1$, $y=x^2$ and $y=-x$.

Soln: $y = x^2$, $y = 2x-1$

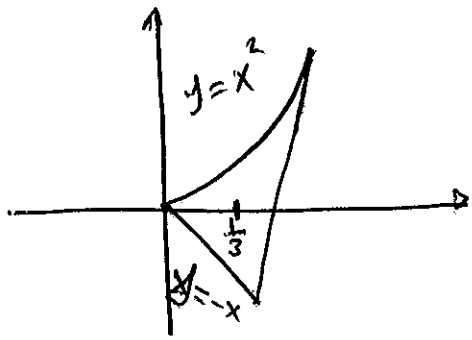
$\Rightarrow x^2 = 2x-1$ or $x^2 - 2x + 1 = 0 \Rightarrow (x-1)^2 = 0$

$\Rightarrow \boxed{x=1}$

$y = x^2$, $y = -x$

$\therefore \quad x^2 = -x \Rightarrow x^2 + x = 0$

$\Rightarrow x = 0, \; x = -1$



$y = 2x-1$, $y = -x$

$\Rightarrow 2x - 1 = -x$ or $3x = 1$ or $x = \frac{1}{3}$

$$\text{Area} = \int_{0}^{\frac{1}{3}} x^2 - (-x)\, dx + \int_{\frac{1}{3}}^{1} x^2 - (2x-1)\, dx$$