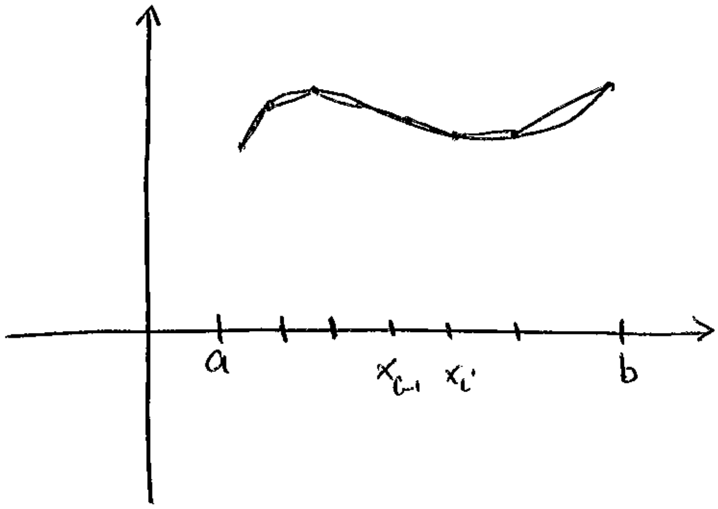
6.3 Arc Length

Target: To find the length of a curve $y = f(x)$

$a \le x \le b$.



$$L_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$L \approx \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

by M.V.T on $[x_{i-1}, x_i]$, there is $x_i^* \in (x_{i-1}, x_i)$

s.t $\dfrac{\Delta y_i}{\Delta x_i} = f'(x_i^*) \Rightarrow \Delta y_i = f'(x_i^*)\,\Delta x_i$

$$\therefore \quad L \approx \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2}\,\Delta x_i$$

$$\therefore \quad L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2}\,\Delta x_i$$

$$\Rightarrow \quad L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$$

Case I : $y = f(x)$, $a \le x \le b$

If $f'(x)$ is continuous on $[a,b]$, then the Length of the curve of $y = f(x)$ from $A = (a, f(a))$ to $B = (b, f(b))$ is

$$L = \int_a^b \sqrt{1+(f'(x))^2}\, dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

EX1 Find the Length of the curve

$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1 , \quad 0 \le x \le 1$$

<u>Soln</u> :
$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}\, x^{\frac{1}{2}} = 2\sqrt{2}\, x^{\frac{1}{2}}$$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + 8x$$

$$\therefore L = \int_0^1 \sqrt{1+8x}\, dx = \frac{1}{8}(1+8x)^{3/2} \cdot \frac{2}{3} \Big|_0^1$$

$$= \frac{1}{24}\left[9^{3/2} - 1 \right] = \frac{1}{12}(26) = \frac{13}{6} .$$

EX2 Find the Length of the curve of $y = \frac{x^3}{12} + \frac{1}{x}$

$1 \le x \le 4$

$$y' = \frac{x^2}{4} - \frac{1}{x^2} \Rightarrow (y')^2 = \left(\frac{x^2}{4}\right)^2 - 2\left(\frac{x^2}{4}\right)\left(\frac{1}{x^2}\right) + \left(\frac{1}{x^2}\right)^2$$

$$= \left(\frac{x^2}{4}\right)^2 - \frac{1}{2} + \left(\frac{1}{x^2}\right)^2$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = \left(\frac{x^2}{4}\right)^2 + \frac{1}{2} + \left(\frac{1}{x^2}\right)^2$$

$$= \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2$$

$$\therefore\ L = \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx = \frac{x^3}{12} - \frac{1}{x}\Big|_1^4$$

$$= \left(\frac{64}{12} - \frac{1}{4}\right) - \left(\frac{1}{12} - 1\right)$$

$$= \frac{63}{12} + \frac{3}{4} = \frac{21}{4} + \frac{3}{4}$$

$$= \frac{24}{4} = 6.$$

EX3 Find the Length of the curve

$$y = \frac{1}{2}\left(e^x + e^{-x}\right), \quad 0 \le x \le 2.$$

$$y' = \frac{1}{2}\left(e^x - e^{-x}\right) \Rightarrow (y')^2 = \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right)$$

$$= \frac{1}{4}e^{2x} - \frac{1}{2} + \frac{1}{4}e^{-2x}$$

$$\Rightarrow 1 + (y')^2 = \frac{1}{4}e^{2x} + \frac{1}{2} + \frac{1}{4}e^{-2x}$$

$$= \frac{1}{4}\left(e^x + e^{-x}\right)^2$$

$$\therefore\ L = \int_0^2 \frac{1}{2}\left(e^x + e^{-x}\right) dx = \frac{1}{2}\left(e^x - e^{-x}\Big|_0^2\right)$$

$$= \frac{1}{2}\left(e^2 - e^{-2} - 0\right) = \frac{1}{2}\left(e^2 - e^{-2}\right).$$

Case II: If $x = g(y)$, $c \le y \le d$.

If $g'(y)$ is continuous on $[c,d]$, then the Length of $x = g(y)$ from $A = (g(c), c)$ to $B = (g(d), d)$ is

$$L = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1+(g'(y))^2}\, dy.$$

Remark: In both Formulas, we need $f'(x)$, $g'(y)$ to be continuous on the given interval.

EX4 Find the Length of the curve $y = \left(\frac{x}{2}\right)^{2/3}$ from $x=0$ to $x=2$.

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-\frac{1}{3}} \cdot \frac{1}{2}$$ is not continuous at $x=0$

So, we can not use the first formula.

$$y = \left(\frac{x}{2}\right)^{2/3} \Rightarrow x = 2y^{3/2}, \quad 0 \le y \le 1$$

$$\Rightarrow \frac{dx}{dy} = 3y^{\frac{1}{2}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = 9y$$

$$\Rightarrow 1+\left(\frac{dx}{dy}\right)^2 = 1+9y \Rightarrow L = \int_0^1 \sqrt{1+9y}\, dy$$

$$= \frac{1}{9}\left[(1+9y)^{3/2} \cdot \frac{2}{3} \Big|_0^1 \right] = \frac{2}{27}\left[10^{3/2} - 1\right]$$

$$= \frac{2}{27}\left[10\sqrt{10} - 1\right].$$

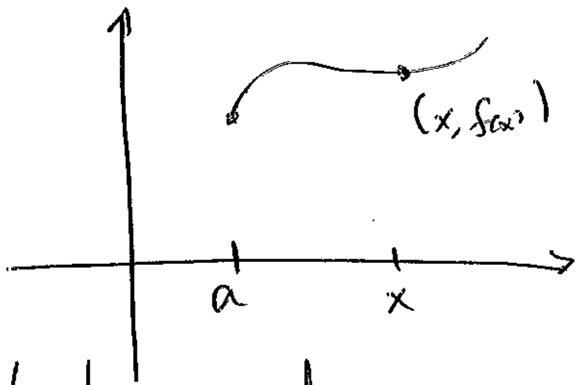
EX5 Write an integral to find the arc length of the parabola $x = y^2$ from $(0,0)$ to $(1,1)$.

$$\frac{dx}{dy} = 2y \Rightarrow \left(\frac{dx}{dy}\right)^2 + 1 = 1 + 4y^2.$$

$$L = \int_0^1 \sqrt{1+4y^2}\, dy.$$

* Arc Length function $S(x)$.

$$S(x) = \int_a^x \sqrt{1+(f'(t))^2}\, dt$$



where $(a, f(a))$ is the starting point.

EX6 Find the arc length function for the curve $f(x) = \frac{x^3}{12} + \frac{1}{x}$. Taking $A = (1, \frac{13}{12})$ as the starting point.

$$f'(x) = \frac{x^2}{4} - \frac{1}{x^2} \Rightarrow 1 + (f'(x))^2 = 1 + \left(\frac{x^2}{4}\right)^2 - \frac{1}{2} + \left(\frac{1}{x^2}\right)^2$$

$$\Rightarrow 1 + (f'(x))^2 = \left(\frac{x^2}{4}\right)^2 + \frac{1}{2} + \left(\frac{1}{x^2}\right)^2 = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2$$

$$\therefore S(x) = \int_1^x \left(\frac{t^2}{4} + \frac{1}{t^2}\right) dt$$

$$= \frac{t^3}{12} - \frac{1}{t}\Big|_1^x = \left(\frac{x^3}{12} - \frac{1}{x}\right) - \left(\frac{1}{12} - 1\right)$$

$$= \frac{x^3}{12} - \frac{1}{x} + \frac{11}{12}.$$

b) Use the result in part (a) to find the Length of the curve $y = f(x)$ from $A = \left(1, \frac{13}{12}\right)$ to $B = \left(4, \frac{67}{12}\right)$.

$$L = S(4) = \frac{64}{12} - \frac{1}{4} + \frac{11}{12} = \frac{64 - 3 + 11}{12} = \frac{72}{12} = 6$$

END.