Part XXVII

Derivatives and Integrals of Trigonometric Functions

Objective. *To compute derivatives and integrals involving trigonometric functions.*

Basic Identities

$\tan u = \dfrac{\sin u}{\cos u}$ $\qquad$ $\cot u = \dfrac{1}{\tan u} = \dfrac{\cos u}{\sin u}$

$\sec u = \dfrac{1}{\cos u}$ $\qquad$ $\csc u = \dfrac{1}{\sin u}$

$\sin^2 u + \cos^2 u = 1$ $\qquad$ $\sec^2 u - \tan^2 u = 1$ $\qquad$ $\csc^2 u - \cot^2 u = 1$

Limits

We have

- $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 0$
- $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x} = 0$

Derivatives

1. $\dfrac{d}{du}(\sin u) = \cos u$
2. $\dfrac{d}{du}(\cos u) = -\sin u$
3. $\dfrac{d}{du}(\tan u) = \sec^2 u = 1 + \tan^2 u$
4. $\dfrac{d}{du}(\cot u) = -\csc^2 u = -(1 + \cot^2 u)$
5. $\dfrac{d}{du}(\sec u) = \sec u \tan u$
6. $\dfrac{d}{du}(\csc u) = -\csc u \cot u$

Integrals

1. $\displaystyle\int \sin u\,du = -\cos u + C$

2. $\displaystyle\int \cos u\,du = \sin u + C$

3. $\displaystyle\int \sec^2 u\,du = \tan u + C$

4. $\displaystyle\int \csc^2 u\,du = -\cot u + C$

5. $\displaystyle\int \sec u \tan u\,du = \sec u + C$

6. $\displaystyle\int \csc u \cot u\,du = -\csc u + C$

7. $\displaystyle\int \tan u\,du = \ln\,|\sec u| + C$

8. $\displaystyle\int \cot u\,du = \ln\,|\sin u| + C$

Examples on limits

1. $\displaystyle\lim_{x\to 0}\frac{\sin 3x}{x} = 3\lim_{x\to 0}\frac{\sin 3x}{3x} = 3$

2. $\displaystyle\lim_{x\to 0}\frac{2x-\sin x}{x} = \lim_{x\to 0}\left(\frac{2x}{x}-\frac{\sin x}{x}\right) = \lim_{x\to 0}\frac{2x}{x} - \lim_{x\to 0}\frac{\sin x}{x} = 2-1=1$

3. $\displaystyle\lim_{x\to 0} x\cot x = \lim_{x\to 0}\frac{x\cos x}{\sin x} = \lim_{x\to 0}\frac{\cos x}{\left(\dfrac{\sin x}{x}\right)} = 1$

4. $\displaystyle\lim_{x\to 0}\frac{\sin x}{\tan 2x} = \lim_{x\to 0}\frac{\sin x\cos 2x}{\sin 2x} = \frac{1}{2}\lim_{x\to 0}\frac{\dfrac{\sin x}{x}\cos 2x}{\dfrac{\sin 2x}{2x}} = \frac{1}{2}\lim_{x\to 0}\cos 2x = \frac{1}{2}$

Examples on derivatives

Find the derivatives of the given functions.

1. $y = 3\cos(x^2)$

Solution. $y' = -3\,(2x)\sin(x^2) = -6x\sin(x^2)$

2. $y = 2x\sin^2 x$

Solution. $y' = 2\sin^2 x + 4x\sin x\cos x$

3. $y = 2\sec x - x^2 \tan x$

Solution. $y' = 2\sec x \tan x - 2x\tan x - x^2\sec^2 x$

4. $y = \dfrac{1-\cos x}{1+\sin x}$

Solution. $y' = \dfrac{(\sin x)(1+\sin x) - (\cos x)(1-\cos x)}{(1+\sin x)^2} = \dfrac{\sin x - \cos x + 1}{(1+\sin x)^2}$

5. $y = \cot x + x\csc^2 x$

Solution. $y' = -\csc^2 x + \csc^2 x - 2x\csc x \csc x \cot x = -2x\csc^2 x \cot x$

6. $y = \ln(\cos(x^2))$

Solution. $y' = \dfrac{-\sin(x^2)(2x)}{\cos(x^2)} = -2x\tan(x^2)$

7. $y = e^{\cos t}$

Solution. $y' = -e^{\cos t}\sin t$

8. $y = \tan(e^x)$

Solution. $y' = e^x \sec^2(e^x)$

9. $y = \ln(\sec x + \tan x)$

Solution. $y' = \dfrac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \dfrac{\sec x(\tan x + \sec x)}{\sec x + \tan x} = \sec x$

10. $y = \sqrt{\cos x}$

Solution. $y' = -\dfrac{\sin x}{2\sqrt{\cos x}}$

11. $y = \sin(\cos x)$

Solution. $y' = -\sin x \cos(\cos x)$

12. $y = 1 + \cot^2(2x)$

Solution. $y' = (2\cot(2x))(-\csc^2(2x))(2) = -4\csc^2(2x)\cot(2x)$

13. $y = \dfrac{1-\cos t}{\csc t}$

Solution. We can use the quotient rule as in Example 4 above. However, it is better to rewrite the function as

$$y = \frac{1-\cos t}{1/\sin t} = \sin t - \sin t \cos t$$

so that $y' = \cos t - \cos^2 t + \sin^2 t$.

Examples on integrals

Find the given integrals.

1. $\int \sin 2x \ dx$

Solution. Let $u = 2x$. Then $du = 2dx$ and

$$\int \sin 2x \ dx = \frac{1}{2}\int \sin u \ du = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos 2x + C$$

2. $\int \sqrt{\sin t} \cos t \ dt$

Solution. Let $u = \sin t$. Then $du = \cos t \ dt$ and

$$\int \sqrt{\sin t}\cos t \ dt = \int \sqrt{u}du = \frac{2}{3}u\sqrt{u} + C = \frac{2}{3}(\sin t)\sqrt{\sin t} + C$$

3. $\int \frac{\sin x}{(1-\cos x)^4} \ dx$

Solution. Let $u = 1 - \cos x$. Then $du = \sin x \ dx$ and

$$\int \frac{\sin x}{(1-\cos x)^4} \ dx = \int \frac{du}{u^4} = \int u^{-4} du = \frac{u^{-3}}{-3} + C = -\frac{1}{3(1-\cos x)^3} + C$$

4. $\int \frac{dx}{\cos^2(3x)}$

Solution. Let $u = 3x$. Then $du = 3dx$ and

$$\int \frac{dx}{\cos^2(3x)} = \int \sec^2(3x) \ dx = \frac{1}{3}\int \sec^2 u \ du = \frac{1}{3}\tan u + C = \frac{1}{3}\tan 3x + C$$

5. $\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx$

Solution. Let $u = \sqrt{x}$. Then $du = \frac{dx}{2\sqrt{x}}$ and

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx = \int \frac{\sin u}{u}(2u)\,du = -2\cos u + C = -2\cos\sqrt{x} + C$$

6. $\int r\sin(r^2) \ dr$

Solution. Let $u = r^2$. Then $du = 2rdr$ and

$$\int r\sin(r^2) \ dr = \int r\sin u \ \frac{du}{2r} = \frac{1}{2}\int \sin u \ du = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos(r^2) + C$$

7. $\int \frac{\sin(\ln x)}{x} \ dx$

Solution. Let $u = \ln x$. Then $du = \dfrac{dx}{x}$ and

$$\int \frac{\sin(\ln x)}{x}\ dx = \int \frac{\sin u}{x}\ x du = \int \sin u\ du = -\cos u + C = -\cos(\ln x) + C$$

8. $\displaystyle\int x \sec x^2 \tan x^2 dx$

Solution. Let $u = x^2$. Then $du = 2xdx$ and

$$\int x \sec x^2 \tan x^2\ dx = \frac{1}{2}\int \sec u \tan u\ du = \frac{\sec u}{2} + C = \frac{\sec x^2}{2} + C$$

9. $\displaystyle\int r \sin r\ dr$

Solution. We use integration by parts. Let $u = r$, $dv = \sin r\ dr$. Then $du = dr$, $v = -\cos r$ and

$$\int r \sin r\ dr = -r\cos r + \int \cos r\ dr = \sin r - r\cos r + C$$

10. $\displaystyle\int e^x \sin x\ dx$

Solution. We use integration by parts.
Let $u = e^x$, $dv = \sin x\ dx$. Then $du = e^x dx$, $v = -\cos x$ and

$$\int e^x \sin x\ dx = -e^x \cos x + \int e^x \cos x\ dx$$

For $\displaystyle\int e^x \cos x\ dx$ we again use integration by parts.
Let $m = e^x$, $dn = \cos x\ dx$. Then $dm = e^x dx$, $n = \sin x$ and

$$\int e^x \cos x\ dx = e^x \sin x - \int e^x dx \sin x\ dx$$

Hence

$$\int e^x \sin x\ dx = -e^x \cos x + e^x \sin x - \int e^x dx \sin x\ dx$$

i.e.

$$2\int e^x \sin x\ dx = -e^x \cos x + e^x \sin x + C$$

which gives

$$\int e^x \sin x\ dx = \frac{1}{2}e^x(\sin x - \cos x) + C_1$$

Exercises

1. Find

(a) $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{6x}$

(b) $\lim_{x \to 0} \dfrac{\sin 2x}{\tan 6x}$

(c) $\lim_{x \to 0} \dfrac{\cos x - 1}{\sin x}$

(d) $\lim_{x \to 0} \dfrac{\tan 3x}{2x^3 - x}$

2. Differentiate

(a) $x \cos x + 2 \tan x$

(b) $e^{\sin x} (\cos x + \sec x)$

(c) $\dfrac{\sin t}{1 + \tan t}$

(d) $\ln(\sin x) + \csc(\ln x)$

3. Evaluate

(a) $\displaystyle\int x^3 \sec^2(x^4 + 2)\ dx$

(b) $\displaystyle\int \frac{\tan(\sqrt{x})}{\sqrt{x}}\ dx$

(c) $\displaystyle\int \frac{\csc^2 x}{1 + \cot x}\ dx$

(d) $\displaystyle\int_0^{\pi} (\sin x - \cos x)\ dx$

(e) $\displaystyle\int_0^{\pi/2} x \cos x\ dx$