Some Applications of Plurisubharmonic Functions to Orbits of Real Reductive Groups

H. Azad

Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals,
Dhahran 31261, Saudi Arabia

and

J.J. Loeb

Department de Mathematiques, Université D'Angers, 49045 Angers, France

INTRODUCTION

The principal aim of this paper is to prove the following results. Section 1 explains the terminology involved; applications are given in Section 4.

THEOREM 1. *Let $G \subset GL(n, \mathbb{R})$ be a real reductive group with Cartan decomposition*

$$\mathsf{g} = \mathsf{k} \oplus \mathsf{p}, \quad \mathsf{g} \text{ being the Lie algebra of } G.$$

Let $G^{\mathbb{C}}$ be the subgroup of $GL(n, \mathbb{C})$ with Lie algebra $\mathsf{g} \oplus i\mathsf{g}$ and $\tilde{K}$ the subgroup of $G^{\mathbb{C}}$ whose Lie algebra is $\mathsf{k} \oplus i\mathsf{p}$.

Let $\Omega^{\mathbb{C}}$ be a complex homogeneous space for $G^{\mathbb{C}}$ and φ a $\tilde{K}$-invariant strictly plurisubharmonic function on $\Omega^{\mathbb{C}}$.

If Ω is a G-orbit in $\Omega^{\mathbb{C}}$ and $f = \varphi|\Omega$ has a critical point, then f is proper, Ω is closed in $\Omega^{\mathbb{C}}$ and the critical set of f is a single K-orbit, K being the subgroup of G whose Lie algebra is k. Moreover, the function f achieves its minimum value on its critical set.

THEOREM 2. *Using the notations of Theorem 1, if*

(i) G operates on a real vector space V and $G^{\mathbb{C}}$ on $V^{\mathbb{C}} = V \bigotimes_{\mathbb{R}} \mathbb{C}$ by complexification and N is the norm square function on $V^{\mathbb{C}}$ obtained from a $\tilde{K}$-invariant hermitian inner product on $V^{\mathbb{C}}$ which is real-valued on V, then for all $v \in V$ one has

$$N(gv) = N(\overline{g}v), \quad \forall\, g \in G^{\mathbb{C}},$$

where $\overline{g}$ denotes the complex conjugate of g.

(ii) If φ is a differentiable function on $G^{\mathbb{C}}$ such that $\varphi(g) = \varphi(\overline{g})$, then $\varphi|G$ has the identity e as a critical point $\Leftrightarrow$ φ has e as a critical point.

(iii) If $v \in V$ and $\Omega = G \cdot v$, $\Omega^{\mathbb{C}} = G^{\mathbb{C}} \cdot v$, then v is a critical point of $N|\Omega \Leftrightarrow v$ is a critical point of $N|\Omega^{\mathbb{C}}$ (N as in (i)).

The following theorem reduces the study of orbits of real reductive groups to that of their complexifications (see (4.2) infra).

THEOREM 3. *Let G be a Lie group, σ an automorphism of finite order of G and H a σ-invariant closed subgroup of G. Let $(G/H)_\sigma$ denote the set of fixed points of σ in G/H. If $\xi \in (G/H)_\sigma$ then the G_σ°-orbit of ξ is the connected component of $(G/H)_\sigma$ which contains ξ. Hence, the components are single G_σ°-orbits.*

As in [1] and [2], the proofs use elementary convexity properties of plurisubharmonic functions and a basic result of Mostow [14, Thm. 3], which itself is essentially a convexity result: it is a consequence of the convexity of norm of Jacobi fields on manifolds of nonpositive curvature (see Appendix).

The definition of reductive groups given in this paper is different from that in e.g. [11, p. 384]. However, it is sufficient for the problems considered here and leads to substantial simplifications in proofs. Our results are independent of Refs [3, 4]. Related results are given in D. Luna [13] and in Richardson-Slodowy [17] to which this paper owes much. Plurisubharmonic functions continue to play a role in group theory and important applications have been made by A.T. Huckleberry and his school [see e.g. 9] and by K.H. Neeb [16]. The group-theoretic aspects of plurisubharmonic functions have also been applied by Uwe Helmke [7, 8] to problems of system and control engineering and there is interest in the analogue of results of [1, 2] for real reductive groups.

1. REDUCTIVE GROUPS

Standard references for reductive groups are Borel - Harish-Chandra [4] and Springer [19]. In this paper, we will take the following as a working definition. All groups and subgroups will henceforth be Lie. G° will denote the connected component of G, G' its commutator and $Z(G)$ its center.

DEFINITION. A connected subgroup G of $GL(n, \mathbb{R})$ is reductive if its Lie algebra g has a decomposition

$$\mathsf{g} = \mathsf{k} \oplus \mathsf{p}$$

where

(i) $[\mathsf{k}, \mathsf{k}] \subset \mathsf{k}$, $[\mathsf{k}, \mathsf{p}] \subset \mathsf{p}$, $[\mathsf{p}, \mathsf{p}] \subset \mathsf{k}$

and

(ii) the Lie group $\tilde{K}$ of $GL(n, \mathbb{C})$ whose Lie algebra is $\tilde{\mathsf{k}} = \mathsf{k} \oplus i\mathsf{p}$ is compact.

PROPOSITION 1.1 (i) *The group G is a closed subgroup of $GL(n, \mathbb{R})$ and $G = KP$, where $K = \langle \exp(X) : X \in \mathsf{k} \rangle$ and $P = \exp(\mathsf{p})$. Moreover G is homeomorphic to $K \times P$ and K is maximal compact in G.*

(ii) There is a $\tilde{K}$-invariant hermitian inner product on $\mathbb{C}^n$ which is real-valued on $\mathbb{R}^n$ and consequently an orthonormal basis for $\mathbb{R}^n$ remains an orthonormal basis for $\mathbb{C}^n$.

Proof. (i) Since $\mathsf{g} \oplus i\mathsf{g} = \tilde{\mathsf{k}} \oplus i\tilde{\mathsf{k}}$ and the group $\tilde{K}$ is compact, it follows that $G^{\mathbb{C}}$ is the Zariski closure of $\tilde{K}$ [20, § 8], so $G^{\mathbb{C}}$ is closed in $GL(n, \mathbb{C})$. Therefore the group G is also closed, as it is the connected component of $G^{\mathbb{C}}_{\sigma}$, σ being complex conjugation in $GL(n, \mathbb{C})$. Similarly, $K = \langle \exp(X) : X \in \mathsf{k} \rangle = \tilde{K}_{\sigma}$ is also closed. Moreover, as $\tilde{K}^{\mathbb{C}} = G^{\mathbb{C}} = \tilde{K} \exp(i\tilde{\mathsf{k}})$ and $\tilde{K} \exp(i\tilde{\mathsf{k}})$ is homeomorphic to $\tilde{K} \times \exp(i\tilde{\mathsf{k}})$ [20, § 8], we also have $G = KP$, where $P = \exp(\mathsf{p})$.

By choosing a hermitian $\tilde{K}$-invariant inner product on $\mathbb{C}^n$, it is clear that $\exp(i\mathsf{k})$ is represented by positive hermitian matrices and therefore $\tilde{K}$ is maximal compact in $G^{\mathbb{C}}$ and K is maximal compact in G.

(ii) [Cf. 17] Let $V = \mathbb{R}^n, V^{\mathbb{C}} = \mathbb{R}^n \bigotimes_{\mathbb{R}} \mathbb{C}$. Let σ denote complex conjugation in $V^{\mathbb{C}}$ as well as in $GL(V^{\mathbb{C}})$ and J the real endomorphism of $V^{\mathbb{C}}$ induced by multiplication by i. The endomorphism J centralizes $G^{\mathbb{C}}$, hence also $\tilde{K}$. From $\sigma g \sigma^{-1} = g^{\sigma}$ we see that σ normalizes $\tilde{K}$. Therefore $\tilde{K}$ is normalized by the group generated by J and σ. From $\sigma J \sigma^{-1} = J^{-1}$ we see that the latter group is finite. Hence the group generated by $\tilde{K}, J$ and σ is compact. Select an inner product R on $V^{\mathbb{C}}$, considered as a real space, that is invariant under this group. If W is a subspace of $V^{\mathbb{C}}$ that is J and σ-invariant, then its orthogonal complement $W^{\perp}$ is also J and σ-invariant. Therefore we can find a basis $e_1, \ldots, e_n$ of V such that

$$V^{\mathbb{C}} = \langle e_1, Je_1 \rangle \perp \cdots \perp \langle e_n, Je_n \rangle.$$

Hence $e_1, \ldots, e_n$ is an orthonormal basis of $V^{\mathbb{C}}$ for the hermitian form $H(\xi, \eta) = R(\xi, \eta) + iR(J\xi, \eta)$ and H is real-valued on V.

Remark. Since $\tilde{K} = Z(\tilde{K})\, \tilde{K}'$ with $Z(\tilde{K}) \cap \tilde{K}'$ finite, we have $\tilde{K} = \Big(Z(\tilde{K})\Big)^{\circ} \tilde{K}'$. Therefore $G^{\mathbb{C}} = Z(G^{\mathbb{C}})^{\circ}(G^{\mathbb{C}})'$ and $G = (Z(G))^{\circ} G'$; also $Z(G)$ consists of semisimple elements. In particular, if $Z(G)$ is finite, then in any linear representation ρ of G in $GL(V)$, the image $\rho(G)$ is also reductive. On the other hand, if $Z(G)$ is infinite, its image in a representation may no longer be reductive. We shall therefore consider only those representations in which the connected component of the center is represented as a reductive group.

2. PRELIMINARY LEMMAS

A plurisubharmonic (briefly psh) function on an n-dimensional complex manifold M is a function f whose complex hessian matrix $[(\partial^2 f / \partial z_i \partial \overline{z}_j)]$, in a system of local holomorphic coordinates $z_1, \ldots, z_n$ is positive semi-definite. And f is strictly plurisubharmonic (briefly spsh), if its complex hessian is strictly positive definite. In other words, f is spsh if the hermitian form $(Lf)_p$ defined by $(Lf)_p(u, v) = (\partial\overline{\partial} f)(p)(u, \overline{v})$, $p \in M$, $u, v \in T^{1,0}_p(M)$, is positive definite.

The following lemma is basic. A version of it already occurs in [1, 12]. We make no assumptions about reductivity or compactness of the groups involved.

LEMMA 2.1. *Let $\tilde{G}$ be a complex Lie group, $\tilde{K}$ a subgroup of $\tilde{G}$ and $\tilde{P} = \{\exp(X) : X \in i \operatorname{Lie}(\tilde{K})\}$ with $\tilde{G} = \tilde{K}\tilde{P}$. Let G be a subgroup of $\tilde{G}$ with $G = KP$, where K is a subgroup of $\tilde{K}$ and $P = \{\exp(Y) : Y \in \mathsf{m}\}$, with m a subspace of $i \operatorname{Lie}(\tilde{K})$.*

Let φ be a $\tilde{K}$-invariant strictly plurisubharmonic function on a complex homogeneous space $\Omega^{\mathbb{C}}$ of $\tilde{G}$ and Ω a G-orbit in $\Omega^{\mathbb{C}}$. If $f = \varphi|\Omega$ has a critical point, then the critical set of f is a single K-orbit and f achieves its absolute minimum there. Moreover, if ξ is a critical point of f then the stabilizer G_ξ of ξ in G factorizes as $G_\xi = K_\xi P_\xi$, where K_ξ is the stabilizer of ξ in K and

$$\begin{aligned} P_\xi &= \{(\exp Y) : Y \in \mathsf{p}, (\exp Y)\xi = \xi\} \\ &= \{\exp Y) : Y \in \mathsf{p}, (\exp tY)\xi = \xi \ \ \forall t \in \mathbb{R}\}. \end{aligned}$$

Proof. Let ξ be a critical point of f and η another critical point of f. By $\tilde{K}$-invariance, we may assume that $\eta = \exp(X)\xi$ for some $X \in \mathsf{m} \subset i \ \operatorname{Lie}(\tilde{K})$.

Consider the function $g(z) = \varphi(\exp(zX) \cdot \xi)$, $z \in \mathbb{C}$. As φ is $\tilde{K}$-invariant, we have $g(x+iy) = g(x)$. Since g is subharmonic, $\Delta g \geq 0$ implies $g''(x) \geq 0$. So g is convex and it achieves its absolute minimum at any critical point. Since $x = 0$ and $x = 1$ are critical points of $g(x)$, we see that $g(x)$ is constant for $0 \leq x \leq 1$. Now $g(z) = g(\operatorname{Re}(z))$, so the function g is constant on the strip $0 \leq$ Re $z \leq 1$. Therefore $\varphi(\gamma(z)) = g(z) =$ constant on $0 \leq$ Re $z \leq 1$, where $\gamma(z) = \exp(zX) \cdot \xi$. Hence

$$(\partial\overline{\partial}\varphi)(\gamma(z))(\gamma'(z), \overline{\gamma'(z)}) = 0, \quad 0 \leq \operatorname{Re} z \leq 1$$

and since $i\partial\overline{\partial}\varphi$ is positive definite, we must have $\gamma'(z) \equiv 0$. Hence $\gamma(z)$ is constant, so $\gamma(0) = \xi = \gamma(1) = \eta$.

By the same argument, if $g = k\exp(Y) \in G$ with $Y \in \mathsf{m} \subset i$ Lie $(\tilde{K})$ and $g\xi = \xi$, then $\exp(tY) \cdot \xi = \xi$, $\ 0 \leq t \leq 1$, so $k \cdot \xi = \xi$. But if $\exp(tY) \cdot \xi = \xi$, then $\exp(-tY) \cdot \xi = \xi$, so the entire 1-parameter subgroup $\{\exp(tY)\}_{t \in \mathbb{R}}$ stabilizes ξ. Hence G_ξ has the factorization claimed above.

LEMMA 2.2. *If $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function whose restriction to each line through the origin is convex and has the origin as its only critical point, then $\lim\limits_{\|x\| \to \infty} f(x) = +\infty$.*

For a proof, see [2].

3. PROOFS OF MAIN RESULTS

We shall use the notation set up in Section 1 without further comment.

Proof of Theorem 1. Let $\Omega^{\mathbb{C}}$ be a complex homogeneous space for $G^{\mathbb{C}}$, φ a $\tilde{K}$-invariant spsh function on $\Omega^{\mathbb{C}}$ whose restriction f to a G-orbit Ω has a critical point ξ. By Lemma

2.1, the stabilizer G_ξ of ξ factorizes as $G_\xi = K_\xi P_\xi$, where $K_\xi = G_\xi \cap K$ and

$$\begin{aligned} P_\xi &= \{\exp(Y) : Y \in \mathsf{p}, \exp(Y)\cdot\xi = \xi\} \\ &= \{\exp(Y) : Y \in \mathsf{p},\ (\exp tY)\cdot\xi = \xi \quad \forall\, t \in \mathbb{R}\}. \end{aligned}$$

So $P_\xi = \exp(\mathsf{q})$ where $\mathsf{q} = \{Y : Y \in \mathsf{p},\ (\exp tY)\cdot\xi = \xi \quad \forall t \in \mathbb{R}\}$. From the characterization of q given it follows that $\mathsf{q} = \mathsf{p} \cap \mathsf{g}_\xi$, where g_ξ denotes the Lie algebra of G_ξ. Therefore q is a $G_\xi \cap K = L$ invariant subspace of p and $[\mathsf{q},\mathsf{q}] \subset \mathsf{g}_\xi \cap \mathsf{k} = \mathsf{k}_\xi$, hence $[Z,[X,Y]] \subset \mathsf{q}$ for all $X, Y, Z \in \mathsf{q}$. Let $H = G_\xi,\ L = K_\xi$, so $H = L\exp(\mathsf{q})$, and q is an L-invariant subspace of p such that $[X,[X,Y]] \in \mathsf{q}\ \ \forall\, X, Y \in \mathsf{q}$.

Using the notations of Section 1, the Lie algebra g has the Cartan decomposition $\mathsf{g} = \mathsf{k} \oplus \mathsf{p}$ and $G = K\exp(\mathsf{p})$. By part (ii) of Proposition 1.1, there is a hermitian inner product on $\mathbb{C}^n$ which is invariant under the compact group $\tilde{K}$ and an orthonormal basis of $\mathbb{R}^n$ remains an orthonormal basis of $\mathbb{C}^n$ over $\mathbb{C}$. Taking matrices relative to this basis, we see that k is represented by real skew-symmetric matrices and p by symmetric matrices. So the form $B(X,Y) = \mathrm{Tr}(XY)$ is nondegenerate on g. It is negative definite on k and positive definite on p. Let q' be the orthogonal complement of q in p relative to B. Then the argument in Mostow [14, Thm 3, p. 40] is directly applicable to this situation and one has the generalized polar decomposition

$$G = K\exp(\mathsf{q}')\exp(\mathsf{q})$$

with uniqueness of expressions. Since $H = L\exp(\mathsf{q})$, this gives immediately $G/H \cong K \underset{L}{\times} \exp(\mathsf{q}')$.

The rest of the argument is similar to that in [2]; we reproduce it here for completeness.

Fix $v \in \mathsf{q}',\ v \neq 0$. Consider the function

$$g_v(t) = \varphi(\exp tv \cdot \xi), \qquad (t \in \mathbb{R}).$$

As in Lemma 2.1, the function g_v is convex and it has $t = 0$ as a critical point. If g_v had another critical point $t_0 \neq 0$, then by the argument in Lemma 2.1, $\exp(tv)\cdot\xi$ would equal ξ for all $t \in \mathbb{R}$, contradicting the fact that $k \underset{L}{\times} v \mapsto k\exp(v)\xi\ (k \in K,\ \ v \in \mathsf{q}')$ is a bijection. Consider the function $F(v) = \varphi(\exp v\cdot\xi)$, $(v \in \mathsf{q}')$. By what has just been shown, the function F satisfies all the hypotheses of Lemma 2.2 and so $\lim\limits_{\|v\|\to\infty} F(v) = +\infty$. To show that $f = \varphi|(G\cdot\xi)$ is proper, we have to show that the sublevel sets $f \leq c\ \ (c \in \mathbb{R})$ are compact.

Let $\{k_n\exp(v_n)\cdot\xi\}$ be a sequence in $G\cdot\xi$ with $k_n \in K$ and $v_n \in \mathsf{q}'$. Since f is K-invariant, we have $F(v_n) = f(\exp v_n\cdot\xi) = \varphi(\exp v_n\cdot\xi) \leq c$. Since $\lim\limits_{\|v\|\to\infty} F(v) = +\infty$, we see that the sequence $\{v_n\}$ must be bounded. Extracting convergent subsequences of $\{k_n\}$ and $\{v_n\}$ we see that the sequence $\{k_n\exp(v_n)\cdot\xi\}$ contains a convergent subsequence. Hence the sublevel sets $f \leq c$ are compact and f is proper. The remaining assertions follow at once from Lemma 2.1.

Proof of Theorem 2. (i) The group G operates on V. Let $\pi : G \to GL(V)$ be the corresponding representation. Let $X_1, \ldots, X_r$ be a basis of the Lie algebra of G. The complexification $\pi(G)^{\mathbb{C}}$ of $\pi(G)$ in $GL(V^{\mathbb{C}})$ is generated by the complex 1-parameter subgroups $\exp(z\pi(X_k))$, $z \in \mathbb{C}$, $1 \leq k \leq r$, and clearly

$$[\exp(z\pi(X_k))]^- = \exp(\overline{z}\pi(X_k)).$$

Hence complex conjugation leaves $(\pi(G))^{\mathbb{C}}$ stable, and one has $[\pi(g) \cdot v]^- = \overline{\pi(g)}\,\overline{v} = \pi(\overline{g})\overline{v}$, for $g \in G$, $v \in V^{\mathbb{C}}$. Since the given hermitian inner product on $V^{\mathbb{C}}$ is real-valued on V, we have $N(\overline{v}) = N(v)$ for all $v \in V^{\mathbb{C}}$. From this it follows that $N(gv) = N(\overline{gv}) = N(\overline{g}v)$ for $g \in G^{\mathbb{C}}$ and $v \in V$. This proves part (i).

(ii) The map $(X, Y) \mapsto \exp(X + iY)$ $X, Y \in \mathfrak{g}$ gives local coordinates at the identity e. Now if $\psi(X, Y) = \varphi(\exp(X + iY))$, then $\psi(X, Y) = \psi(X, -Y)$, which clearly implies (ii).

(iii) This is a direct consequence of parts (i) and (ii).

Proof of Theorem 3. For the proof, it is useful to define tangent spaces of arbitrary subsets of a manifold. For a subset Z of a manifold M and a point $p \in Z$, the tangent space $T_p(Z)$ to Z at p is the subspace of $T_p(M)$ spanned by the vectors $\gamma'(0)$ where $\gamma : I \to M$, I an interval containing 0, is a differentiable curve with $\gamma(0) = p$ and whose trace lies in Z

Let K be a compact group of transformations of M whose fixed point set M_K is non-empty. Fix a K-invariant Riemannian metric on M. For $p \in M_K$ choose a geodesic ball B_p centered at p with the property that any two points in B_p can be joined by a unique geodesic in B_p: in other words, B_p is a geodesic strongly convex ball [6, p. 34]. Hence if r, s are points in $B_p \cap M_K$, then the geodesic segment joining r with s lies entirely in $B_p \cap M_K$. Therefore $\exp_p^{-1}(B_p \cap M_K)$ is homeomorphic to an open ball in $(T_p(M))_K$. The same argument shows that $\dim T_p(M_K) = \dim (T_p(M))_K$. Similarly if $q \in M_K$ and $B_p \cap B_q \neq \phi$ then for all $r \in B_p \cap B_q \cap M_K$ we have $\dim T_r(M_K) = \dim T_p(M_K) = \dim T_q(M_K)$. Thus the set $\{r \in M_K :\ r$ can be joined to p by a continuous curve $\{\gamma(t)\}_{t \in I} \subset M_K$ with $\dim T_{\gamma(t)}(M_K) = \dim T_p(M_K)$ for all $t \in I\}$ is both open and closed, hence it is the component of M_K which contains p, and this component is invariant under any connected group operating on M_K.

Now take $M = G/H$, with $\sigma(H) = H$ and K the group generated by σ. We have $M_K = M_\sigma$, where $\sigma(gH) = \sigma(g)\sigma(H) = \sigma(g)H$, $\forall\, g \in G$. Thus $\sigma(g \cdot m) = \sigma(g) \cdot \sigma(m)$, so if $\sigma(m) = m$ and $g \cdot m = m$, then $\sigma(g) \cdot m = m$. Therefore the stabilizer G_m of m is σ-invariant if $\sigma(m) = m$. For such an m, consider the isomorphism $i : G/G_m \tilde{\to} G \cdot m$. We have $i \circ \sigma = \sigma \circ i$. Let $\xi_0 = eG_m$. The component of $(G/G_m)_\sigma$ containing ξ_0 is mapped to the component of $(G \cdot m)_\sigma = M_\sigma$ containing m. We have already seen that if C is a component of M_σ and $z \in C$, then the dimension of $T_z(C)$ is the same as the dimension of $(T_z(M))_\sigma$ and it is constant as z varies over C. Now $\dim T_m(M_\sigma) = \dim T_{\xi_0}(G/G_m)_\sigma$ which in turn equals the dimension of $(\mathfrak{g}/\mathfrak{g}_m)_\sigma \cong \dfrac{\mathfrak{g}_\sigma}{(\mathfrak{g}_m)_\sigma}$, as σ is of finite order: here $\mathfrak{g}_m$ is the Lie algebra of the stabilizer G_m of m in G. Hence the G_σ° orbit of ξ_0 in $(G/G_m)_\sigma$ is open in the component containing ξ_0. Recalling that the component of $(G/G_m)_\sigma$ containing ξ_0 is mapped to the component of $(G \cdot m)_\sigma = M_\sigma$ containing m by the homeomorphism $i : G/G_m \cong G \cdot m$, we see that the G_σ°-orbit of m is open in the component C containing

m. Therefore all the orbits of G_σ° in C are open, hence they are also closed. So C consists of just one G_σ° orbit. This completes the proof of the theorem.

4. APPLICATIONS

In this section, $G, G^{\mathbb{C}}, N$ etc. have the same meaning as in the statement of Theorem 1.

4.1. (Birkes [3]) *If G operates on a real vector space V and $G^{\mathbb{C}}$ on $V^{\mathbb{C}}$ and a G-orbit of $p \in V$ is closed, then the $G^{\mathbb{C}}$-orbit of p in $V^{\mathbb{C}}$ is also closed.*

Proof. Let Ω be a closed G-orbit in V. Since the function N is proper, $N|\Omega$ achieves its minimum, say at q. By Theorem 1, the function $\varphi(g) = N(gq)$ $(q \in G^{\mathbb{C}})$ has a critical point at e, so by the Kempf-Ness theorem [10] or by [2], the orbit $\Omega^{\mathbb{C}} = G^{\mathbb{C}} \cdot q = G^{\mathbb{C}} \cdot p$ is closed.

4.2. (Borel - Harish-Chandra [4]). *If $v \in V$ and $\Omega^C = G^{\mathbb{C}} \cdot v$, then $\Omega^{\mathbb{C}} \cap V$ is a union of equidimensional orbits of G, each of which is a component of $\Omega^{\mathbb{C}} \cap V$.*
In particular, if $\Omega^{\mathbb{C}}$ is closed, then all G-orbits in $\Omega^{\mathbb{C}} \cap V$ are closed.

Proof. For this result, G need not be reductive. It suffices, by Theorem 3, to show that all G-orbits in $\Omega^{\mathbb{C}} \cap V$ have the same dimension. Now if $v \in V$ and $X, Y \in \mathrm{Lie}(G)$ with $(X + iY)v = 0$, then $Xv = 0$, $Yv = 0$, so all G-orbits in $G^{\mathbb{C}}v \cap V$ have the same dimension.

Remark. In [4] it is shown, using an argument from real algebraic geometry, that $G^{\mathbb{C}}v \cap V$ $(v \in V)$ is a finite union of G-orbits.

4.3. (Richardson–Slodowy [17]). *If N restricted to a G-orbit Ω has a critical point, then Ω is closed.*

Proof. This follows immediately from Theorem 2 and (4.2).

4.4. *If φ is a $\tilde{K}$-invariant spsh function on $\Omega^{\mathbb{C}} = G^{\mathbb{C}} \cdot v$ $(v \in V)$ and $\varphi|(G \cdot v)$ has a critical point, then G_v is reductive and $G \cdot v$ is closed in $G^{\mathbb{C}} \cdot v$.*

Proof. This is a consequence of Theorem 1 and Lemma 2.1.

4.5. *If φ is a $\tilde{K}$-invariant spsh function defined in a neighbourhood of a G-orbit of $v \in V$ and $\varphi|G \cdot v$ has a critical point, then the G and the $G^{\mathbb{C}}$ orbits of v are closed.*

Proof. If $\varphi|G \cdot v$ has a critical point then $f = \varphi|\Omega$ is proper (Thm. 1) and since φ is defined in a neighbourhood of $G \cdot v$, the orbit $G \cdot v$ is closed in V; therefore by Theorem 1, $G^{\mathbb{C}} \cdot v$ is also closed in $V^{\mathbb{C}}$.

Remark. If we take a compact group K and V a representation of K, then all $K^{\mathbb{C}}$-

orbits of real points are closed in $V^{\mathbb{C}}$. The simplest example of a K-invariant spsh function which is critical along K-orbits but not along $K^{\mathbb{C}}$-orbits is given by $f(z) = (\log|z|) + |z|^2$ with $K = S^1, K^{\mathbb{C}} = \mathbb{C}^*$.

4.6. *If φ is a $\tilde{K}$-invariant spsh proper function on $V^{\mathbb{C}}$ such that $\varphi|G^{\mathbb{C}}v$ has a critical point ($v \in V$), then $\varphi|Gv$ also has a critical point.*

Proof. The assumptions imply that $G^{\mathbb{C}} \cdot v$ is closed in $V^{\mathbb{C}}$; hence by (4.2), $G \cdot v$ is closed in V so $\varphi|G \cdot v$ achieves its minimum value on $G \cdot v$.

4.7. *If $v \in V$, then $\overline{G \cdot v}$ contains a closed G-orbit.*

Proof. The function N is proper, so $N|\overline{G \cdot v}$ achieves its minimum value, say at p. Hence the G-orbit of p is closed.

APPENDIX

The proof of Theorem 1 relies on Mostow [14, 15]. A summary of the main ideas is given in A. Borel, Collected Works, Vol. 1, pp. 558–559. Mostow's theorem is also proved in Helgason [6, Thm. 1.4, p. 256]. The result is proved in these references for semisimple groups. However, the arguments are valid for reductive groups if one works with a suitable trace form instead of the Killing form, as is done in § 3 of this paper. One of the main technical points of Mostow [14, 15] is the distance increasing property of the exponential function on the space of symmetric matrices. As the space of positive symmetric $n \times n$ matrices is a homogeneous space of $GL(n, \mathbb{R})$ of non-positive curvature, this is a special case of the following result. This result is also proved in Helgason [6, Thm. 13.1, p. 73]. However, the following proof is more elementary, with its emphasis on convexity of the norm of Jacobi fields. The proof uses an idea similar to an idea in the proof of the Cartan-Hadamard theorem as given in [5, Lemma 3.2, p. 149].

Theorem. *If M is a complete Riemannian manifold of non-positive curvature then for all $p \in M$, $v \in T_p(M)$ and $w \in T_v(T_p(M))$, one has the inequality*

$$\|(d\exp_p(v))(w)\| \geq \|w\|.$$

Proof. We have $(d\exp_p(v))(w) = \left.\dfrac{d}{ds}\right|_{s=0} \exp_p(v + sw)$, so if we consider the variation $\alpha(t, s) = \exp_p(t(v + sw))$ of the geodesic $\gamma(t) = \exp_p(tv)$, then the corresponding Jacobi field $J(t) = \dfrac{\partial\alpha}{\partial s}(t, 0)$ vanishes at 0 and $J(1) = (d\exp_p(v))(w)$. Let $\psi(t) = \|J(t)\|$. We shall prove that ψ is differentiable for $t \neq 0$ and ψ is convex.

Now $\dfrac{\partial\alpha}{\partial s} = (d\exp_p(t(v+sw)))(tw)$, so $\left.\dfrac{\partial\alpha}{\partial s}\right|_{s=0} = (d\exp_p(tv))[tw]$, and therefore $J(t) = tV(t)$, where $V(t) = (d\exp_p(tv))(w)$. We have $J(0) = 0$, $J'(0) = w$ and the field J satisfies the equation

$$J''(t) = R(J(t), \gamma'(t))\gamma'(t),$$

R being the curvature tensor, and therefore $\langle J''(t), J(t)\rangle \geq 0$ as $\langle R(X,Y)X,Y\rangle \leq 0$ for all fields X, Y.

Let $\varphi(t) = \|J(t)\|^2$. Using $\langle J''(t), J(t)\rangle \geq 0$, we find that $\varphi''(t) \geq 0$. Therefore φ' is increasing and if $\varphi'(t_0) = 0$ for some $t_0 > 0$, then φ' would be identically 0 on $[0, t_0]$, taking into account that $\varphi'(0) = \langle J'(0), J(0)\rangle = 0$ as $J(0) = 0$. But then φ would also equal 0 on $[0, t_0]$, so $w = J'(0)$ would also equal 0. Assuming $w \neq 0$, we see that $J(t) = 0$ only at $t = 0$. Therefore $\psi(t) = \|J(t)\|$ is differentiable at $t \neq 0$.

Differentiating $\psi^2 = \|J\|^2$, using $J(t) = tV(t)$, we find that $\lim_{t\to 0} \psi'(t) = \|V(0)\| = \|w\|$.

Differentiating $\psi\psi' = \langle J, J'\rangle$ and simplifying we get $\psi^3\psi'' = \|J\|^2[\langle J, J''\rangle] + \|J\|^2\ \|J'\|^2 - \langle J, J'\rangle^2 \geq 0$, taking into account $\langle J, J''\rangle \geq 0$ and the Cauchy-Schwarz inequality. Therefore $\psi = \|J\|$ is convex and Taylor series gives $\psi(t) \geq \psi(a) + (t-a)\psi'(a)$ for $0 < a < t$.

Taking limits as $a \to 0^+$ we get $\psi(t) \geq \psi(0) + t\|w\| = t\|w\|$, so in particular $\psi(1) = \|J(1)\| = \|d(\exp_p(v))(w)\| \geq \|w\|$ $\square$.

Acknowledgment. The authors thank the referee heartily for his careful reading of the manuscript and for several suggestions which have been incorporated in the paper.

REFERENCES

1. H. Azad and J.J. Loeb, On a theorem of Kempf and Ness, *Indiana Univ. Math. J.* 39(1990), 61–65.

2. H. Azad and J.J. Loeb, Plurisubharmonic functions and the Kempf-Ness theorem, *Bull. Lond. Math. Soc.* 25(1993), 162–168.

3. D. Birkes, Orbits of linear algebraic groups, *Ann. of Math.* (2) 93(1971), 459–475.

4. A. Borel and Harish-Chandra, Arithmetic subgroups of algebraic groups, *Ann. of Math.* (2) 75(1962), 485–535.

5. M.P. Do Carmo, *Riemannian Geometry*, Birkhăuser, Boston, 1992.

6. S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, San Diego, California, 1978.

7. U. Helmke, Balanced realizations for linear systems: A variational approach, *SIAM J. Control and Optimization* 31(1993), 1–15.

8. U. Helmke and J.B. Moore, *Optimization and Dynamical Systems*, Springer–Verlag, London, 1994.

9. P. Heinzner and A.T. Huckleberry, Kählerian potentials and convexity properties of the moment map, *Invent. Math.* 126(1996), no. 1, 65–84.

10. G. Kempf and L. Ness, *The length of vectors in representation spaces*, Lecture Notes in Math. 732 (Springer, New York, 1978), 233-243.

11. A.W. Knapp, *Lie Groups beyond an Introduction*, Birkhäuser, Boston, Basel, Berlin, 1996.

12. J.J. Loeb, Pseudo-convexité des ouverts invariants et convexité géodésique dans certains espaces symétriques, Seminaire d'Analyse, Lecture Notes in Math. (ed. P. Lelong, P. Dolbeault and H. Skoda), Springer, New York, 1983.

13. D. Luna, Sur certaines operations différentiables des groupes de Lie, *Amer J. Math.* 97(1975), 172–181.

14. G.D. Mostow, Some new decomposition theorems for semisimple groups, *Memoirs Amer. Math. Soc.* no. 14 (1955), 31–54.

15. G.D. Mostow, Selfadjoint groups, *Ann. of Math.* (2) 62(1955), 44–54.

16. K.H. Neeb, On the complex and convex geometry of Olshanskii semigroups, *Ann. Inst. Fourier* 48:1(1998), 149–203.

17. R.W. Richardson and P.J. Slodowy, Minimum vectors for real reductive groups, *J. London Math. Soc.* (2) 42(1990), 409–429.

18. B. Shabat, *Introduction to Complex Analysis, II*, A.M.S., Providence, RI, 1992.

19. T.A. Springer, *Reductive Groups*, Proc. Symposia in Pure Math. 33(1), AMS, Providence (1972), 3–27.

20. R. Steinberg, *Lectures on Chevalley Groups*, Yale Univ. Press, Newhaven CT, 1967.