In what follows we consider Laplace integrals of the type in which $\phi(t) = t^\lambda g(t)$, where $g(t)$ possesses Taylor series expansion about $t=0$ with $g(0) \neq 0$ and λ is real and $\lambda > -1$. This ensure convergence of the integral at $t=0$. If $g(t)=0$ having a zero of order r at $t=0$ then it has a factor t^r and we re-define $g(t)$ such that now $g(t) \neq 0$ at $t=0$ while t^r is absorbed in t^λ.

<u>Watson Lemma</u>:
$$\int_0^T e^{-xt} t^\lambda g(t)\,dt \sim \sum_{n=0}^{\infty} \frac{g^{(n)}(0)\,\Gamma(\lambda+n+1)}{n!\, x^{\lambda+n+1}}$$
(x real, +ve, T some +ve number) as $x \to \infty$.

<u>Proof</u>. $|g(t)| < K e^{ct}$.

Since $g(t)$ has a Taylor series expansion about $t=0$,
$$g(t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} t^n = \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{N} a_n t^n + r_N(t) \quad (1)$$
where $|r_N(t)| < L\, t^{N+1}$ for $|t| < R$

and $a_n = \dfrac{g^{(n)}(0)}{n!}$.

We shall consider the radius of convergence $R > T$ but this is not necessary. In the integral, we use (1) to get
$$f(x) = \int_0^T e^{-xt} t^\lambda \sum_{n=0}^{N} a_n t^n + \int_0^T e^{-xt} t^\lambda r_N(t)\,dt. \quad — (2)$$

We now concentrate on the second integral on the R.H.S
$$\left| \int_0^T e^{-xt} t^\lambda r_N(t)\,dt \right| < L \int_0^T e^{-xt} t^{\lambda+N+1}\,dt \quad — (3)$$

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Put $\tau = xt$ $\quad (d\tau = x\,dt)$ on R.H.S

$$\text{R.H.S} = \int_0^T e^{-xt}\, t^{\lambda+N+1}\, dt = x^{-(\lambda+N+2)} \int_0^{xT} e^{-\tau}\, \tau^{\lambda+N+1}\, d\tau$$

$$= x^{-(\lambda+N+2)} \left[\int_0^{\infty} e^{-\tau}\, t^{\lambda+N+1}\, d\tau - \int_{xT}^{\infty} e^{-\tau}\, \tau^{\lambda+N+1}\, d\tau \right]$$

$$= x^{-(\lambda+N+2)}\, \Gamma(\lambda+N+2) - \underbrace{x^{-(\lambda+N+2)} \int_{xT}^{\infty} e^{-\tau}\, \tau^{\lambda+N+1}\, d\tau}$$

Again consider second integral on R.H.S. Put

$\tau = x\,T(1+u)$ $\qquad (d\tau = x\,T\,du)$

$$x^{-(\lambda+N+2)} \int_{xT}^{\infty} e^{-\tau}\, t^{\lambda+N+1}\, d\tau = T^{\lambda+N+2}\, e^{-xT} \int_0^{\infty} e^{-xTu} (1+u)^{\lambda+N+1}\, du$$

$$< T^{\lambda+N+2}\, e^{-xT} \int_0^{\infty} e^{-xTu}\, e^{(\lambda+N+1)u}\, du$$

$$\left(\text{Using } (1+u)^a < e^{au},\ a>0,\ u>0 \right)$$

$$= T^{\lambda+N+2}\, e^{-xT}\, \frac{1}{xT - (\lambda+N+1)}$$

$$= T^{\lambda+N+2}\, e^{-xT}\, \frac{1}{xT} \left(1 + \frac{\lambda+N+1}{xT} + \cdots \right)$$

$$\sim T^{\lambda+N+1}\, \frac{e^{-xT}}{x},$$

Going back to R.H.S $\int(3)$ we see

$$\int_0^T e^{-xt}\, t^{\lambda+N+1}\, dt = x^{-(\lambda+N+2)}\, \Gamma(\lambda+N+2) + o\left(e^{-xT}\right)$$

$$= x^{-(\lambda+N+2)}\, \Gamma(\lambda+N-2) + o\left(x^{-(\lambda+N+2)}\right)$$

as $x \to \infty$.

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Using this estimate in (3)

$$\int_0^T e^{-xt} t^\lambda r_N(t)\,dt = O(x^{-(\lambda+N+2)}) \text{ as } x\to\infty.$$

Thus (2) becomes

$$\int_0^T e^{-xt} t^\lambda \sum_{n=0}^{\infty} a_n t^n\,dt = \sum_{n=0}^{N} a_n \left[\left\{\int_0^\infty - \int_T^\infty\right\} e^{-xt} t^{\lambda+n}\,dt\right]$$

$$= \sum_{n=0}^{N} a_n \Gamma(\lambda+n+1)\, x^{-(\lambda+n+1)} + O(x^{-(\lambda+N+1)})$$

as $x\to\infty$.

Thus

$$f(x) = \sum_{n=0}^{N} a_n \Gamma(\lambda+n+1)\, x^{-(\lambda+n+1)} + O(x^{-(\lambda+N+2)})$$

which gives the required asymptotic formula

$$\int_0^T e^{-xt} t^\lambda g(t)\,dt \sim \sum_{n=0}^{\infty} \frac{g^n(0)\,\Gamma(\lambda+n+1)}{n!\; x^{\lambda+n+1}} \text{ as } x\to\infty.$$

Remark: (1) This result can also be shown for complex z as long as we take $|\arg z| < \pi/2$. Proof can be seen in Copson (1965: Asymptotic expansions, Cambridge University Press)

(2) The upper limit T can be replaced by ∞ without requiring the radius of convergence R to be infinite