SECTION 15.5 Series Solutions of Differential Equations

Power Series Solution of a Differential Equation • Approximation by Taylor Series

Power Series Solution of a Differential Equation

We conclude this chapter by showing how power series can be used to solve certain types of differential equations. We begin with the general **power series solution** method.

Recall from Chapter 8 that a power series represents a function f on an interval of convergence, and that you can successively differentiate the power series to obtain a series for f', f'', and so on. These properties are used in the power series solution method demonstrated in the first two examples.

EXAMPLE 1 Power Series Solution

Use a power series to solve the differential equation $y' - 2y = 0$.

Solution Assume that $y = \Sigma a_n x^n$ is a solution. Then, $y' = \Sigma n a_n x^{n-1}$. Substituting for y' and $-2y$, you obtain the following series form of the differential equation. (Note that, from the third step to the fourth, the index of summation is changed to ensure that x^n occurs in both sums.)

$$y' - 2y = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} 2a_n x^n$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} 2a_n x^n$$

Now, by equating coefficients of like terms, you obtain the **recursion formula** $(n+1)a_{n+1} = 2a_n$, which implies that

$$a_{n+1} = \frac{2a_n}{n+1}, \qquad n \geq 0.$$

This formula generates the following results.

a_0	a_1	a_2	a_3	a_4	a_5	. . .
a_0	$2a_0$	$\dfrac{2^2 a_0}{2}$	$\dfrac{2^3 a_0}{3!}$	$\dfrac{2^4 a_0}{4!}$	$\dfrac{2^5 a_0}{5!}$	. . .

Using these values as the coefficients for the *solution* series, you have

$$y = \sum_{n=0}^{\infty} \frac{2^n a_0}{n!} x^n = a_0 \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = a_0 e^{2x}.$$

In Example 1, the differential equation could be solved easily without using a series. The differential equation in Example 2 cannot be solved by any of the methods discussed in previous sections.

EXAMPLE 2 Power Series Solution

Use a power series to solve the differential equation $y'' + xy' + y = 0$.

Solution Assume that $\sum_{n=0}^{\infty} a_n x^n$ is a solution. Then you have

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad xy' = \sum_{n=1}^{\infty} n a_n x^n, \qquad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Substituting for y'', xy', and y in the given differential equation, you obtain the following series.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = -\sum_{n=0}^{\infty} (n+1)a_n x^n$$

To obtain equal powers of x, adjust the summation indices by replacing n by $n + 2$ in the left-hand sum, to obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n = -\sum_{n=0}^{\infty} (n+1)a_n x^n.$$

By equating coefficients, you have $(n+2)(n+1)a_{n+2} = -(n+1)a_n$, from which you obtain the recursion formula

$$a_{n+2} = -\frac{(n+1)}{(n+2)(n+1)} a_n = -\frac{a_n}{n+2}, \qquad n \geq 0,$$

and the coefficients of the solution series are as follows.

$$a_2 = -\frac{a_0}{2} \qquad\qquad a_3 = -\frac{a_1}{3}$$

$$a_4 = -\frac{a_2}{4} = \frac{a_0}{2 \cdot 4} \qquad\qquad a_5 = -\frac{a_3}{5} = \frac{a_1}{3 \cdot 5}$$

$$a_6 = -\frac{a_4}{6} = -\frac{a_0}{2 \cdot 4 \cdot 6} \qquad\qquad a_7 = -\frac{a_5}{7} = -\frac{a_1}{3 \cdot 5 \cdot 7}$$

$$\vdots \qquad\qquad \vdots$$

$$a_{2k} = \frac{(-1)^k a_0}{2 \cdot 4 \cdot 6 \cdots (2k)} = \frac{(-1)^k a_0}{2^k (k!)} \qquad\qquad a_{2k+1} = \frac{(-1)^k a_1}{3 \cdot 5 \cdot 7 \cdots (2k+1)}$$

Thus, you can represent the general solution as the sum of two series—one for the even-powered terms with coefficients in terms of a_0 and one for the odd-powered terms with coefficients in terms of a_1.

$$\begin{aligned} y &= a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \cdots\right) + a_1\left(x - \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} - \cdots\right) \\ &= a_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k (k!)} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{3 \cdot 5 \cdot 7 \cdots (2k+1)} \end{aligned}$$

The solution has two arbitrary constants, a_0 and a_1, as you would expect in the general solution of a second-order differential equation.

Approximation by Taylor Series

A second type of series solution method involves a differential equation *with initial conditions* and makes use of Taylor series, as given in Section 8.10.

EXAMPLE 3 Approximation by Taylor Series

Use a Taylor series to find the series solution of

$$y' = y^2 - x$$

given the initial condition $y = 1$ when $x = 0$. Then, use the first six terms of this series solution to approximate values of y for $0 \le x \le 1$.

Solution Recall from Section 8.10 that, for $c = 0$,

$$y = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdot \cdot \cdot .$$

Because $y(0) = 1$ and $y' = y^2 - x$, you obtain the following.

$$\begin{aligned}
& & y(0) &= 1\\
y' &= y^2 - x & y'(0) &= 1\\
y'' &= 2yy' - 1 & y''(0) &= 2 - 1 = 1\\
y''' &= 2yy'' + 2(y')^2 & y'''(0) &= 2 + 2 = 4\\
y^{(4)} &= 2yy''' + 6y'y'' & y^{(4)}(0) &= 8 + 6 = 14\\
y^{(5)} &= 2yy^{(4)} + 8y'y''' + 6(y'')^2 & y^{(5)}(0) &= 28 + 32 + 6 = 66
\end{aligned}$$

Therefore, you can approximate the values of the solution from the series

$$\begin{aligned}
y &= y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5 + \cdot \cdot \cdot\\
&= 1 + x + \frac{1}{2}x^2 + \frac{4}{3!}x^3 + \frac{14}{4!}x^4 + \frac{66}{5!}x^5 + \cdot \cdot \cdot .
\end{aligned}$$

Using the first six terms of this series, you can compute values for y in the interval $0 \le x \le 1$, as shown in the table at the left.

x	y
0.0	1.0000
0.1	1.1057
0.2	1.2264
0.3	1.3691
0.4	1.5432
0.5	1.7620
0.6	2.0424
0.7	2.4062
0.8	2.8805
0.9	3.4985
1.0	4.3000

EXERCISES FOR SECTION 15.5

In Exercises 1–6, verify that the power series solution of the differential equation is equivalent to the solution found using the techniques in Sections 5.7 and 15.1–15.4.

1. $y' - y = 0$
2. $y' - ky = 0$
3. $y'' - 9y = 0$
4. $y'' - k^2y = 0$
5. $y'' + 4y = 0$
6. $y'' + k^2y = 0$

In Exercises 7–10, use power series to solve the differential equation and find the interval of convergence of the series.

7. $y' + 3xy = 0$
8. $y' - 2xy = 0$
9. $y'' - xy' = 0$
10. $y'' - xy' - y = 0$

In Exercises 11 and 12, find the first three terms of each of the power series representing independent solutions of the differential equation.

11. $(x^2 + 4)y'' + y = 0$
12. $y'' + x^2y = 0$

In Exercises 13 and 14, use Taylor's Theorem to find the series solution of the differential equation under the specified initial conditions. Use n terms of the series to approximate y for the given value of x and compare the result with the approximation given by Euler's Method for $\Delta x = 0.1$.

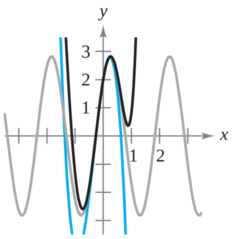
13. $y' + (2x - 1)y = 0, y(0) = 2, n = 5, x = \frac{1}{2},$
14. $y' - 2xy = 0, y(0) = 1, n = 4, x = 1$

15. ***Investigation*** Consider the differential equation $y'' + 9y = 0$ with initial conditions $y(0) = 2$ and $y'(0) = 6$.

(a) Find the solution of the differential equation using the techniques of Section 15.3.

(b) Find the series solution of the differential equation.

(c) The figure shows the graph of the solution of the differential equation and the third-degree and fifth-degree polynomial approximations of the solution. Identify each.



16. Consider the differential equation $y'' - xy' = 0$ with the initial conditions $y(0) = 0$ and $y'(0) = 2$. (See Exercise 9.)

(a) Find the series solution satisfying the initial conditions.

(b) Use a graphing utility to graph the third-degree and fifth-degree series approximations of the solution. Identify the approximations.

(c) Identify the symmetry of the solution.

In Exercises 17 and 18, use Taylor's Theorem to find the series solution of the differential equation under the specified initial conditions. Use n terms of the series to approximate y for the given value of x.

17. $y'' - 2xy = 0$, $y(0) = 1$, $y'(0) = -3$, $n = 6$, $x = \frac{1}{4}$

18. $y'' - 2xy' + y = 0$, $y(0) = 1$, $y'(0) = 2$, $n = 8$, $x = \frac{1}{2}$

In Exercises 19–22, verify that the series converges to the given function on the indicated interval. (*Hint:* Use the given differential equation.)

19. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x, \ (-\infty, \infty)$

Differential equation: $y' - y = 0$

20. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos x, \ (-\infty, \infty)$

Differential equation: $y'' + y = 0$

21. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n + 1} = \arctan x, \ (-1, 1)$

Differential equation: $(x^2 + 1)y'' + 2xy' = 0$

22. $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n + 1)} = \arcsin x, \ (-1, 1)$

Differential equation: $(1 - x^2)y'' - xy' = 0$

23. Find the first six terms in the series solution of Airy's equation $y'' - xy = 0$.

REVIEW EXERCISES FOR CHAPTER 15

In Exercises 1–4, classify the differential equation according to type and order.

1. $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

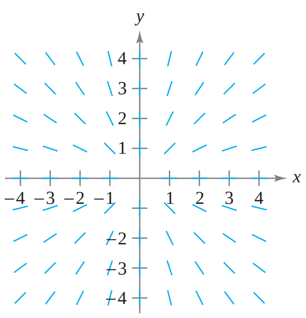
2. $yy'' = x + 1$

3. $y'' + 3y' - 10 = 0$

4. $(y'')^2 + 4y' = 0$

In Exercises 5 and 6, use the given differential equation and its direction field.

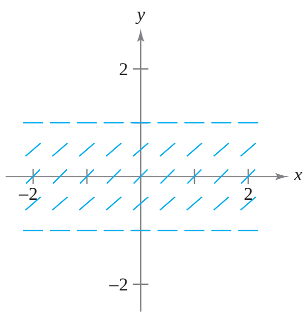
5. $\dfrac{dy}{dx} = \dfrac{y}{x}$



(a) Sketch several solution curves for the differential equation on the direction field.

(b) Find the general solution of the differential equation. Compare the result with the sketches from part (a).

6. $\dfrac{dy}{dx} = \sqrt{1 - y^2}$



(a) Sketch several solution curves for the differential equation on the direction field.

(b) When is the rate of change of the solution greatest? When is it least?

(c) Find the general solution of the differential equation. Compare the result with the sketches from part (a).

In Exercises 7–10, match the differential equation with its solution.

Differential Equation	*Solution*
7. $y' - 4 = 0$	(a) $y = C_1e^{2x} + C_2e^{-2x}$
8. $y' - 4y = 0$	(b) $y = 4x + C$
9. $y'' - 4y = 0$	(c) $y = C_1 \cos 2x + C_2 \sin 2x$
10. $y'' + 4y = 0$	(d) $y = Ce^{4x}$

In Exercises 11–32, find the general solution of the first-order differential equation.

11. $\dfrac{dy}{dx} - \dfrac{y}{x} = 2 + \sqrt{x}$

12. $\dfrac{dy}{dx} + xy = 2y$

13. $y' - \dfrac{2y}{x} = \dfrac{y'}{x}$

14. $\dfrac{dy}{dx} - 3x^2y = e^{x^3}$

15. $\dfrac{dy}{dx} - \dfrac{y}{x} = \dfrac{x}{y}$

16. $\dfrac{dy}{dx} - \dfrac{3y}{x^2} = \dfrac{1}{x^2}$

17. $(10x + 8y + 2)\,dx + (8x + 5y + 2)\,dy = 0$

18. $(y + x^3 + xy^2)\,dx - x\,dy = 0$

19. $(2x - 2y^3 + y)\,dx + (x - 6xy^2)\,dy = 0$

20. $3x^2y^2\,dx + (2x^3y + x^3y^4)\,dy = 0$

21. $dy = (y \tan x + 2e^x)\,dx$

22. $y\,dx - \left(x + \sqrt{xy}\right)dy = 0$

23. $(x - y - 5)\,dx - (x + 3y - 2)\,dy = 0$

24. $y' = 2x\sqrt{1 - y^2}$

25. $x + yy' = \sqrt{x^2 + y^2}$

26. $xy' + y = \sin x$

27. $yy' + y^2 = 1 + x^2$

28. $2x\,dx + 2y\,dy = (x^2 + y^2)\,dx$

29. $(1 + x^2)\,dy = (1 + y^2)\,dx$

30. $x^3yy' = x^4 + 3x^2y^2 + y^4$

31. $xy' - ay = bx^4$

32. $y' = y + 2x(y - e^x)$

In Exercises 33–40, find the particular solution of the differential equation that satisfies the boundary condition.

33. $y' - 2y = e^x$
$y(0) = 4$

34. $y' + \dfrac{2y}{x} = -x^9y^5$
$y(1) = 2$

35. $x\,dy = (x + y + 2)\,dx$
$y(1) = 10$

36. $ye^{xy}dx + xe^{xy}\,dy = 0$
$y(-2) = -5$

37. $(1 + y)\ln(1 + y)\,dx + dy = 0$
$y(0) = 2$

38. $(2x + y - 3)\,dx + (x - 3y + 1)\,dy = 0$
$y(2) = 0$

39. $y' = x^2y^2 - 9x^2$
$y(0) = \dfrac{3(1 + e)}{1 - e}$

40. $2xy' - y = x^3 - x$
$y(4) = 2$

In Exercises 41 and 42, find the orthogonal trajectories of the given family and sketch several members of each family.

41. $(x - C)^2 + y^2 = C^2$

42. $y - 2x = C$

43. ***Snow Removal*** Assume that the rate of change in the number of miles s of road cleared per hour by a snowplow is inversely proportional to the height h of snow.

(a) Write and solve the differential equation to find s as a function of h.

(b) Find the particular solution if $s = 25$ miles when $h = 2$ inches and $s = 12$ miles when $h = 10$ inches $(2 \le h \le 15)$.

44. ***Growth Rate*** Let x and y be the sizes of two internal organs of a particular mammal at time t. Empirical data indicate that the relative growth rates of these two organs are equal, and hence we have

$$\frac{1}{x}\frac{dx}{dt} = \frac{1}{y}\frac{dy}{dt}.$$

Solve this differential equation, writing y as a function of x.

45. ***Population Growth*** The rate of growth in the number N of deer in a state park varies jointly over time t as N and $L - N$, where $L = 500$ is the estimated limiting size of the herd. Write N as a function of t if $N = 100$ when $t = 0$ and $N = 200$ when $t = 4$.

46. ***Population Growth*** The rate of growth in the number N of elk in a game preserve varies jointly over time t (in years) as N and $300 - N$ where 300 is the estimated limiting size of the herd.

(a) Write and solve the differential equation for the population model if $N = 50$ when $t = 0$ and $N = 75$ when $t = 1$.

(b) Use a graphing utility to graph the direction field of the differential equation and the particular solution of part (a).

(c) At what time is the population increasing most rapidly?

(d) If 400 elk had been placed in the preserve initially, use the direction field to describe the change in the population over time.

47. ***Slope*** The slope of a graph is given by $y' = \sin x - 0.5y$. Find the equation of the graph if the graph passes through the point $(0, 1)$. Use a graphing utility to graph the solution.

48. ***Investment*** Let $A(t)$ be the amount in a fund earning interest at an annual rate r compounded continuously. If a continuous cash flow of P dollars per year is withdrawn from the fund, the rate of change of A is given by the differential equation

$$\frac{dA}{dt} = rA - P$$

where $A = A_0$ when $t = 0$. Solve this differential equation for A as a function of t.

49. ***Investment*** A retired couple plans to withdraw P dollars per year from a retirement account of \$500,000 earning 10% compounded continuously. Use the result of Exercise 48 and a graphing utility to graph the function A for each of the following continuous annual cash flows. Use the graphs to describe what happens to the balance in the fund for each of the cases.

(a) $P = \$40{,}000$

(b) $P = \$50{,}000$

(c) $P = \$60{,}000$

50. ***Investment*** Use the result of Exercise 48 to find the time necessary to deplete a fund earning 14% interest compounded continuously if $A_0 = \$1{,}000{,}000$ and $P = \$200{,}000$.

In Exercises 51–54, find the particular solution of the differential equation that satisfies the initial conditions. Use a graphing utility to graph the solution.

	Differential Equation	*Initial Conditions*
51.	$y'' - y' - 2y = 0$	$y(0) = 0, y'(0) = 3$
52.	$y'' + 4y' + 5y = 0$	$y(0) = 2, y'(0) = -7$
53.	$y'' + 2y' - 3y = 0$	$y(0) = 2, y'(0) = 0$
54.	$y'' + 2y' + 5y = 0$	$y(1) = 4, y(2) = 0$

In Exercises 55–60, find the general solution of the second-order differential equation.

55. $y'' + y = x^3 + x$

56. $y'' + 2y = e^{2x} + x$

57. $y'' + y = 2\cos x$

58. $y'' + 5y' + 4y = x^2 + \sin 2x$

59. $y'' - 2y' + y = 2xe^x$

60. $y'' + 2y' + y = \dfrac{1}{x^2e^x}$

In Exercises 61–64, find the particular solution of the differential equation that satisfies the initial conditions.

	Differential Equation	*Initial Conditions*
61.	$y'' - y' - 6y = 54$	$y(0) = 2, y'(0) = 0$
62.	$y'' + 25y = e^x$	$y(0) = 0, y'(0) = 0$
63.	$y'' + 4y = \cos x$	$y(0) = 6, y'(0) = -6$
64.	$y'' + 3y' = 6x$	$y(0) = 2, y'(0) = \frac{10}{3}$

Vibrating Spring **In Exercises 65 and 66, describe the motion of a 64-pound weight suspended on a spring. Assume that the weight stretches the spring $\frac{4}{3}$ feet from its natural position.**

65. The weight is pulled $\frac{1}{2}$ foot below the equilibrium position and released.

66. The weight is pulled $\frac{1}{2}$ foot below the equilibrium position and released. The motion takes place in a medium that furnishes a damping force of magnitude $\frac{1}{8}$ speed at all times.

67. ***Investigation*** The differential equation

$$\frac{8}{32}y'' + by' + ky = \frac{8}{32}F(t), \quad y(0) = \frac{1}{2}, \quad y'(0) = 0$$

models the motion of a weight suspended on a spring.

(a) Solve the differential equation and use a graphing utility to graph the solution for each of the assigned quantities for b, k, and $F(t)$.

(i) $b = 0, k = 1, F(t) = 24\sin \pi t$

(ii) $b = 0, k = 2, F(t) = 24\sin\left(2\sqrt{2}\,t\right)$

(iii) $b = 0.1, k = 2, F(t) = 0$

(iv) $b = 1, k = 2, F(t) = 0$

(b) Describe the effect of increasing the resistance to motion b.

(c) Explain how the motion of the object would change if a stiffer spring (increased k) were used.

(d) Matching the input and natural frequencies of a system is known as resonance. In which case of part (a) does this occur, and what is the result?

68. ***Think About It*** Explain how you can find a particular solution of the differential equation

$$y'' + 4y' + 6y = 30$$

by observation.

In Exercises 69 and 70, find the series solution of the differential equation.

69. $(x - 4)y' + y = 0$

70. $y'' + 3xy' - 3y = 0$