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ORIGINAL ARTICLE

# Exact solutions of some general nonlinear wave equations in elasticity

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**Abstract** A similarity analysis of a nonlinear wave equation in elasticity is studied; in particular, one with anharmonic corrections. The symmetry transformation give rise to exact solutions via the method of invariants. In some cases, graphical figure of the solutions are presented. Furthermore, we consider some cases wherein the velocities of the longitudinal and transversal plane waves are variable. Finally, a brief discussion on how a symmetry analysis on a perturbation of the elasticity equation can be pursued.

**Keywords** Similarity analysis · Exact solutions · Nonlinear elasticity equations

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## 1. Introduction

The linear theory of elasticity is based upon the assumption that the strain tensor $u_{ij}$ depends linearly on the displacement vector $u_i$ as

$$u_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{1.1}$$

The elastic energy $E$ in this case is given by

$$E = \int \left( \frac{\lambda}{2} u_{ii}^2 + \mu u_{ij}^2 \right) \mathrm{d}\mathbf{r} \tag{1.2}$$

where $\mathbf{r}$ is the position vector of the point $x_i$ and $\lambda$ and $\mu$ are Lame's coefficients. If the displacement in the elastic medium is not small, the nonlinear strain tensor takes the form

$$u_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}). \tag{1.3}$$

Moreover, the elastic energy up to third-order takes the form

$$E = \int \left( \frac{\lambda}{2} u_{ii}^2 + \mu u_{ij}^2 + \frac{1}{3} A u_{ij} u_{jk} u_{ki} + B u_{ij}^2 u_{kk} + \frac{1}{3} C u_{ii}^3 \right) \mathrm{d}\mathbf{r} \tag{1.4}$$

where $A$, $B$ and $C$ are constants. We consider the one-dimensional motion in which $u_1 = u$ and the nonlinear strain (1.3) has only one component given by

$$u_{11} = u_x + \frac{1}{2}u_x^2. \tag{1.5}$$

Equation (1.4) reduces to

$$E = \int \rho\left(\alpha u_x^2 + \frac{1}{6}\beta u_x^3\right) \mathrm{d}\mathbf{r} \tag{1.6}$$

where $\alpha = \frac{\lambda+2\mu}{\rho}$, and $\beta = \frac{3(\lambda+2\mu)+2(A+3B+C)}{\rho}$, $\rho$ is the density of the elastic medium. The equation of nonlinear wave motion can then be written as

$$u_{tt} - (\alpha + \beta u_x)u_{xx} = 0, \quad \alpha, \beta > 0, \tag{1.7}$$

where, usually, $\alpha > \beta$. As noted in [1], Equation (1.7) corresponds to the continuum limit of the Fermi-Pasta-Ulam equation. Therein, iter alia, Apostol has presented an asymptotic series expansion for the case $\beta/\alpha$ small – we discuss a symmetry analysis.

The analysis here involves a symmetry analysis by which exact (similarity) solutions are obtained for a large class of (1.7). Section 2 in the paper will take the following form. In 2.1, we present solutions based on the assumption that $\alpha$ and $\beta$ are constants (as in [1]). Then, in 2.2 a number of variable versions of $\alpha$ and $\beta$ are considered with an emphasis on these being functions of $x$. In Section 2.3, we present the details of an *approximate symmetry* analysis on versions of (1.7) which include small perturbations that arise in practise.

In short, a point symmetry analysis involves finding the point dependent one parameter Lie groups of transformations that keep invariant the equation in question. These will take the form

$$X = \xi(x, t, u)\partial_x + \tau(x, t, u)\partial_t + \phi(x, t, u)\partial_u. \tag{1.8}$$

The invariants of $X$ provide the new variables and mechanism for reducing the differential in question. For a detailed analysis on the subject, see [2–4]. It will not be the goal of this paper to classify completely the variable velocity case as this is an enormous task. Instead, we aim to provide some exact solutions for certain practical situations and reductions which may lend themselves to numerical analysis, if need be. We also comment, below, the role of classifying 'optimal' systems.

## 2. Exact solutions

In this section, we do a similarity analysis for the various cases mentioned. In some cases, we present the solutions graphically. When the reduced equation is an ordinary differential equation (ode) that cannot be solved analytically, it may be studied via some numerical scheme. The Equation (1.7) generates the six dimensional Lie algebra of point symmetries with basis

$$\begin{aligned}
&\partial_t, \quad \partial_x, \quad \partial_u \\
&x\partial_x + t\partial_t + u\partial_u, \quad t\partial_u \\
&\frac{-\beta}{2\alpha}\partial_t + \left(x + \frac{\beta}{\alpha}u\right)\partial_u
\end{aligned} \tag{2.1}$$

### 2.1. Similarity solutions of the basic equation

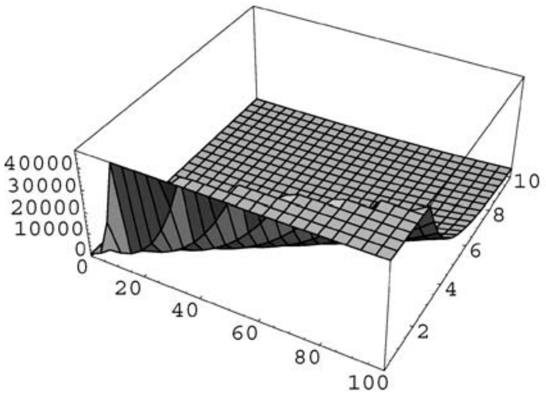
One way to proceed hereon is to classify the optimal system of one-dimensional subalgebras from (2.1) and then construct invariant solutions (see [4]). This need to be done as this procedure provides a systematic way of finding invariant solutions. The calculations are relatively complicated dealing with the classification of the orbits of the adjoint representations. A way dealing with this procedure is described in [4]. Here, however, we are interested in determining the interesting exact solutions constructed from the algebra in (2.1) and to present them graphically. In the group theory method, it is understood that whilst one of the solutions may satisfy a given set of boundary/initial conditions, another may not. The choice of the exact solution is, thus, determined by the specific model. Alternatively, one may build in the conditions from the outset of the calculations as described in, for e.g., [3].

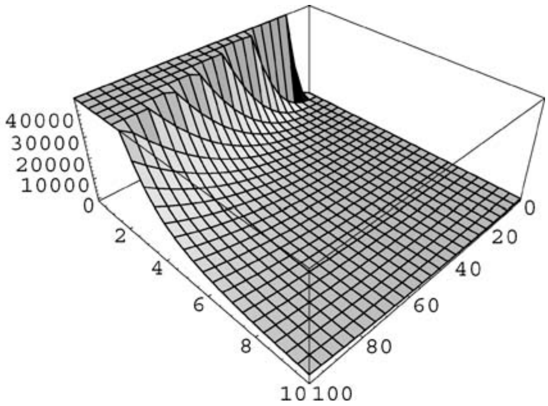
The travelling wave solution obtained by taking the symmetry generator $\partial_t + c\partial_x$ gives a simple solution of the type $u(x, t) = Ax + Bct$. The more interesting similarity solutions are obtainable from other point symmetry generators.

$X = x\partial_x + t\partial_t + u\partial_u$ gives invariants $y = \frac{y}{x}$ and $v = \frac{u}{t}$ $(v = v(y))$ which when substituted into (1.7) gives yields the exact solution

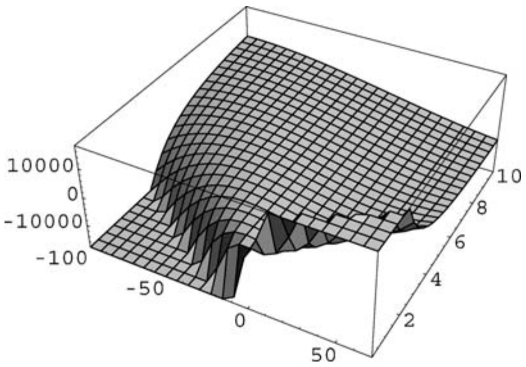
$$u(x, t) = \frac{t}{\beta}\left(\frac{1}{3}\frac{x^3}{t^3} - \alpha\frac{x}{t}\right) + kt, \tag{2.2}$$

$k$ constant. The figures below correspond to $\alpha = \beta = 1$



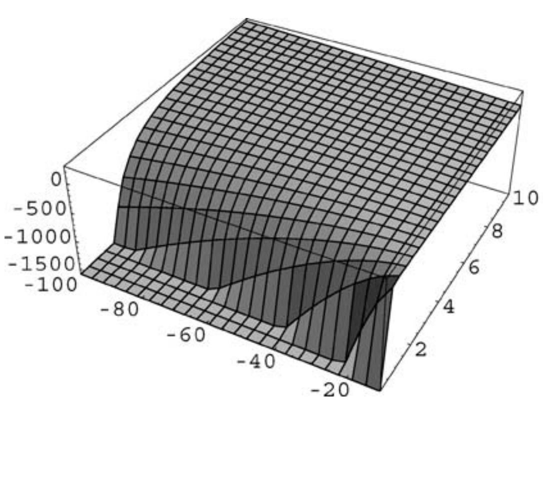

and $\alpha = 1$, $\beta = .01$ yields

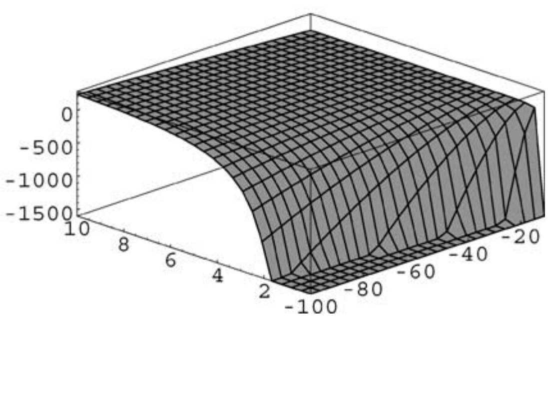



Equation (1.7), via its symmetry generator $\frac{-\beta}{2\alpha}\partial_t + (x + \frac{\beta}{\alpha}u)\partial_u$ leads to invariants $y = x$ and $v = t^2(x + \frac{\beta}{\alpha}u)$ which reduces (1.7) to the ode $6v - v'v'' = 0$ which has solution

$$u(x, t) = \frac{\alpha}{\beta}\left[-x + \frac{1}{18t^2}(k - 23\sqrt{3}(-x - c)^{3/2})\right], \tag{2.3}$$

$k$ and $c$ are constants. The following is a representation for $\alpha = \beta = k = 1$ and $c = 10$.




2.2. Variable velocities of the longitudinal and transversal plane waves

We present, here, solutions obtained for certain classes of variable values of the velocities $\alpha$ and $\beta$. A proper treatment of this problem is to doing a complete group classification of the resultant algebras that arise which in turn will give rise to specific forms of the velocities. As mentioned in the introduction, this is an enormous task and not the purpose here. Instead, we choose beforehand the kind of velocities that arise in practice and determine the Lie point symmetries and corresponding exact solutions.

(i) If $\alpha$ and/or $\beta$ are dependent on $x$, Equation (1.7) looses the translation in $x$ symmetry and, hence, the travelling wave solution is also absent. As a first case, suppose we append linear terms to $\alpha$ and $\beta$ in (1.7), viz.,

$$u_{tt} - (\alpha + \gamma x + (\beta + \delta x)u_x)u_{xx} = 0 \tag{2.4}$$

The algebra of point symmetries are

$$X_1 = t\partial_u, \quad X_2 = \partial_u, \quad X_3 = \partial_t,$$
$$X_4 = \frac{\beta + \delta x}{\delta}\partial_x + t\partial_t - \frac{\gamma x}{\delta}\partial_u$$

whose nonzero commutators are $[X_1, X_4] = X_1$ and $[X_1, X_3] = -X_2$.
The symmetry $X_4$ leads to invariants $y = (\beta + \delta x)t^{-1}$ and $v = u + \frac{\gamma x}{\delta} - \frac{\gamma\beta}{\delta^2}\ln(\beta + \delta x)$ which when substituted into (2.4) leads to the ode

$$-\delta^3 y^3 v' v'' - \delta^2 \alpha y^2 v'' + v' + \gamma\beta\delta y^2 v' + \alpha\beta\gamma = 0. \tag{2.5}$$

It is clear that further analysis will require a sophisticated method. For example, one may consider values for $\delta^n$ being small enough for some choice of $n$ to regard (2.5) as a perturbation of an exact equation. Also, one may resort to some numerical technique.
The combined generator $X_1 + \frac{1}{a}X_3$ yields $y = x$ and $v = u - at$ which leads to $v'' = 0$ or $v' = -\frac{\alpha+\gamma y}{\beta+\delta y}$ so that an exact solution of (2.4) is

$$u(x,t) = at - \frac{\gamma x}{\delta} - \frac{-\gamma\beta + \alpha\delta}{\delta^2} \times \ln(\beta + \delta x) + k, \tag{2.6}$$

(ii) The specific case of the above

$$u_{tt} - (\gamma x + \delta x u_x)u_{xx} = 0 \tag{2.7}$$

admits, inter alia, the dilation symmetry $X = 2x\partial_x + t\partial_t + 2u\partial_u$ and $Y = \frac{\gamma x}{\delta}\partial_x + (x + \frac{2\gamma u}{\delta})\partial_u$. The first has invariants $y = \frac{x}{t^2}$ and $v = \frac{u}{t^2}$ leading to the ode

$$-\gamma y v' v'' + 4y^2 v'' - \delta y v'' + 2v = 0$$

and $Y$ has invariants $y = t$ and $v = u - at$ leading to the ode

$$v'' - 4\gamma v^2 = 0 \tag{2.8}$$

whose solution is given by

$$y = \int \frac{\mathrm{d}v}{\sqrt{4\gamma v^3 + k}}, k$$

constant. The integral is obtainable in terms of special (elliptic) functions. For $k = 0$, we get a solution for (2.8) to be

$$t = \frac{x}{\sqrt{\gamma u + \delta x}} + k^*.$$

We note, however, that in these cases with $\alpha = \beta = 0$ in (2.4) we have a physically uninteresting situation.

(iii) As an example of $\alpha(x) > \beta(x)$, we choose the equation

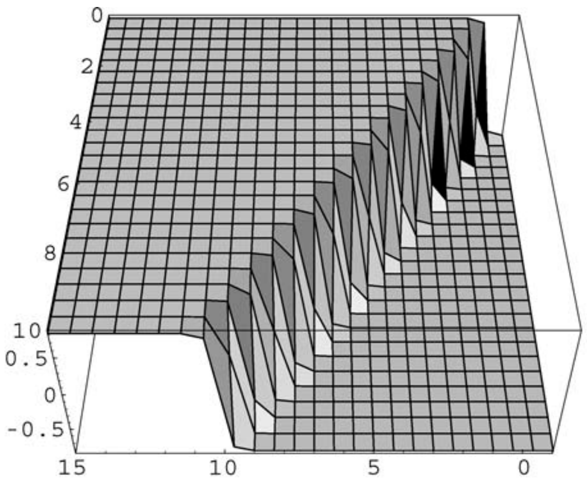
$$u_{tt} - (e^{2x} + x^2 u_x)u_{xx} = 0. \tag{2.9}$$

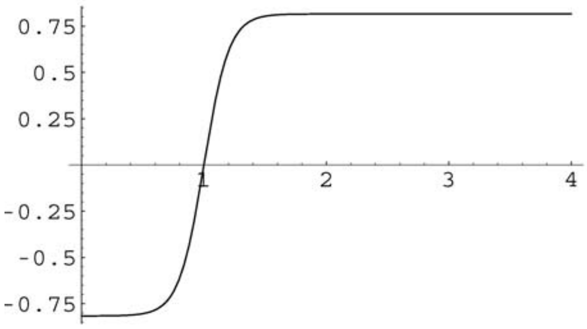
One of its symmetry generators is $X = A\partial_t + (Bt + C)\partial_u$, $A \neq 0$, $B$ and $C$ constants, produces invariants $y = x$ and $v = u - \frac{1}{A}(\frac{1}{2}Bt^2 + Ct)$ reducing (2.9) to the ode $(e^{2y} + y^2 v)v'' = \frac{B}{A}$ which is easily reducible to first-order.

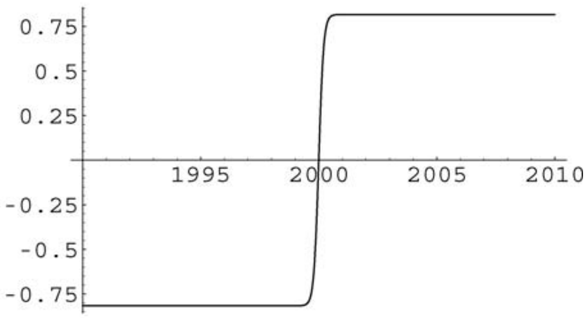
(iv) As a point of interest, suppose $\alpha$ is a quadratic function of the displacement $u^2$ and $\beta = 1$, viz., $u_{tt} - (u^2 + u_x)u_{xx} = 0$, we have an interesting travelling wave type solution arising from the point symmetry generator $\partial_t + c\partial_x$. Firstly, the new variables $y = x - ct$ and $v = u$ casts (1.7) as the ode $6v^2 + v' - 4 = 0$ whose solution leads to

$$u(x,t) = \frac{\sqrt{\frac{2}{3}}\left(-1 + e^{2\sqrt{6}(2(x-t))}\right)}{1 + e^{2\sqrt{6}(2(x-t))}}. \tag{2.10}$$

The 3-$d$ graph is presented below followed by 2-d graphs fixed at $t = 1$ and $t = 2000$, respectively.




### 2.3. A discussion on perturbation cases

One may pursue the physically relevant situation of including small perturbations in (1.7) (like a damping term) or interpreting some of the physical terms in the equation to signify perturbations. Here too one may appeal to the symmetry based method. The essentials of the approximate symmetry method can be summarized as follows. For equations of first-order in the perturbed variable $\epsilon$ (see [5], an interesting account is given in [6]), if $X^0$ is a generator of Lie (point) symmetry of a differential equation $E_0 = 0$, then an *approximate symmetry*, $X = X^0 + \epsilon X^1$, of the perturbed differential equation $E_0 + \epsilon E_1 = 0$ is obtained by solving for $X^1$ in

$$X^1(E_0)|_{E_0=0} + H = 0, \tag{2.11}$$

where $H = \frac{1}{\epsilon} X^0(E_0 + \epsilon E_1)|_{E_0+\epsilon E_1=0}$ ($E_1$ is the perturbation and $H$ is referred to as an auxilliary function). In [7], an approximate symmetry analysis of a different class of perturbed wave equation is given. Here, without details, we comment on the kind of situations that arise.

(i) In

$$u_{tt} - (\alpha + \beta u_x)u_{xx} + \epsilon u_t = 0, \tag{2.12}$$

(1.7) is appended with a small damping term $u_t$. The only nontrivial approximate symmetry is $X = 10x\partial_x + (10t + \epsilon t^2)\partial_t + (10u - 4\epsilon t(u + x))\partial_u$ which has invariants given by $y = \frac{(10+\epsilon t)x}{t}$ and $v$ where $u = v(10^5 t + 5.10^4 \epsilon t^2) - \epsilon(4.10^7)tx$. One may then proceed as above.

(ii) If we set $\epsilon = \beta/\alpha$ following from $\alpha \gg \beta$ (see [1]) we get (1.7) becoming the perturbed equation

$$u_{tt} - \alpha(1 + \epsilon u_x)u_{xx} = 0. \tag{2.13}$$

A possibly convenient way of dealing with the equation is to consider a conserved form $D_t(u_t) - \alpha D_x(u_x + \frac{1}{2}\epsilon u_x^2) = 0$ which leads to the potential form

$$v_x = u_t, \quad v_t = \alpha\left(u_x + \frac{1}{2}\epsilon u_x^2\right). \tag{2.14}$$

Thus, one may pursue the approximate symmetries of the potential system instead. Firstly, it can be shown that the Lie point symmetry generators of the unperturbed system $v_x = u_t$, $v_t = \alpha u_x$ is

$$X_1 = \alpha u \partial_x + v \partial_t, \quad X_2 = A(x,t)\partial_x + C(x,t)\partial_t$$
$$X_3 = \phi(x,t,u,v)\partial_u + \zeta(x,t,u,v)\partial_v, \tag{2.15}$$

where $\alpha C_x = A_t$, $C_t = A_x$ and $\zeta_u = \alpha\phi_v$, $\zeta_v = \phi_u$, $\zeta_x = \phi_t$, $\zeta_t = \alpha\phi_x$. It is interesting to note that there are a number potential (nonlocal) symmetries of $u_{tt} - \alpha u_{xx} = 0$, the most obvious one being $X_1$. It turns out that $X_1$ does not lead to any approximate symmetries of (2.13) and $X_2 + X_3$ leads to $X_2 + X_3 + \epsilon X^1$, where $X^1$ is of type

$$X_1^1 = \alpha u \partial_x + v \partial_t, \qquad X_2^1 = F(x,t)\partial_x + G(x,t)\partial_t,$$
$$X_3^1 = \Phi(x,t,u,v)\partial_u + \Psi(x,t,u,v)\partial_v, \tag{2.16}$$

where $\alpha G_x = F_t$, $G_t = F_x$ and $\Psi_u = \alpha\Phi_v$, $\Psi_v = \Phi_u$, $\Psi_x = \Phi_t$, $\Psi_t = \alpha\Phi_x$ and also $\phi_u = C_t$, $\phi_z = C_x$. This implies that the approximate symmetries are unstable. Whilst the calculations reveal some interesting properties of the wave equation and this perturbation, it does not serve to construct solutions for (2.13). It seems, therefore, that an approximate symmetry study, as prsented here, of the equation is not feasible. However, one may pursue alternative approximate methods as discussed in detail in [8].

## 3. Conclusion

We have shown that a similarity analysis of a nonlinear wave equation in elasticity can be tackled using the group theoretical method. The symmetries, via their invariants, reduce the equations to odes which may be solved easily. When the reduced equation cannot be solved directly, one may consider a numerical scheme on the reduced ode. A variety of situations were considered.

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