Determination of Near Surface Velocity Profile by Linearized Inversion of Scattered Surface Wavefield

F. D. Zaman and Abdullah Al-Ramadhan

August, 2004

King Fahd University of Petroleum & Minerals
Department of Mathematical Sciences

Overview

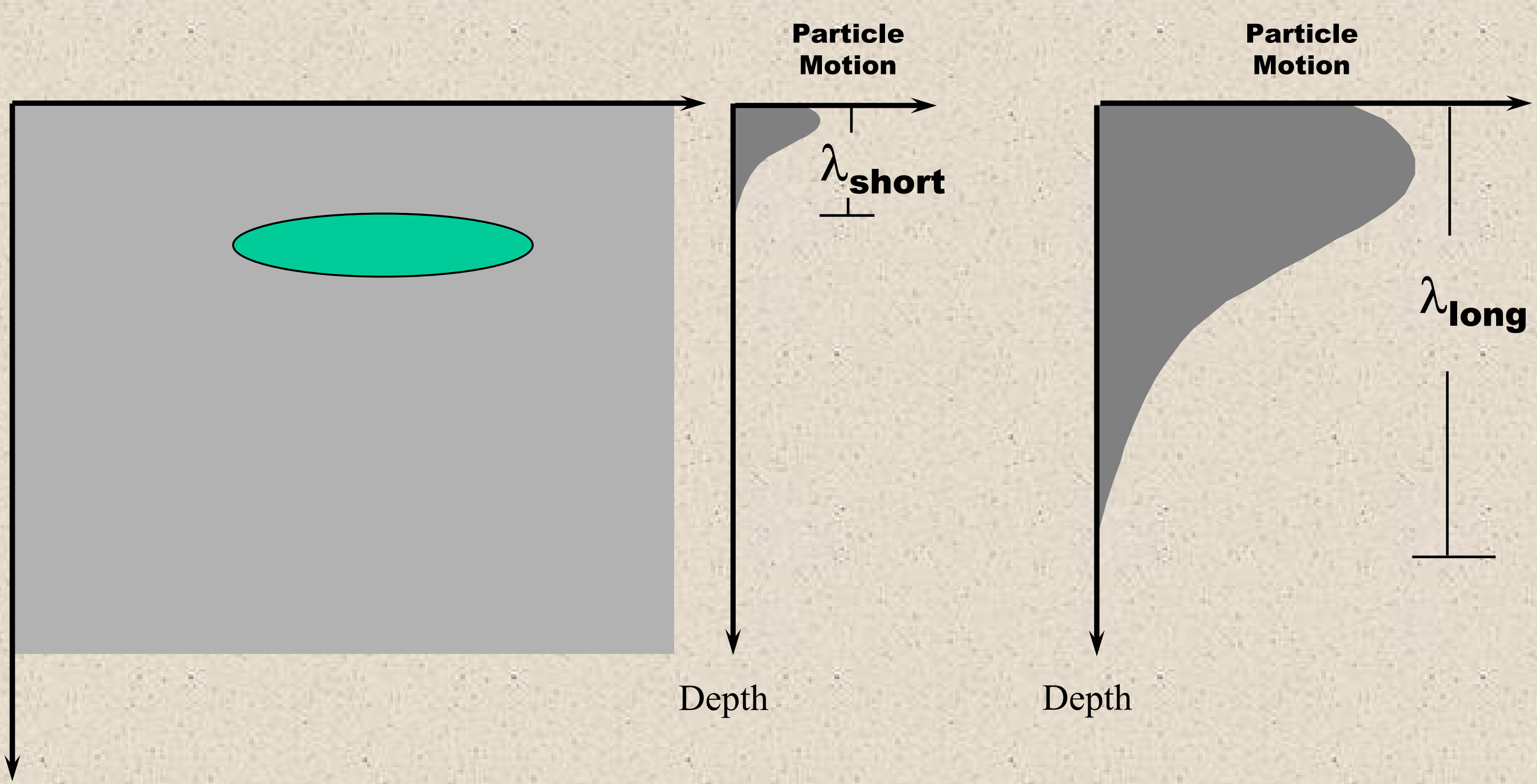
- Motivation
- Surface Wave Method
- Linearizing the Scattered Wavefield
- The Green Function Method
- The Green Function for Love Wave
- The Green Function for Rayleigh Waves
- Linearized Inversion
- Linearized Inversion Algorithm
- Conclusions and Recommendations
- Acknowledgments

Motivation

- An accurate description of the **velocity profile** of the near surface earth is essential for the definition and interpretation of the shallow geology.
- Detection of the **heterogeneity** and **lithology** of the near surface layer.
- Detection of the depth of the **heterogeneity**.

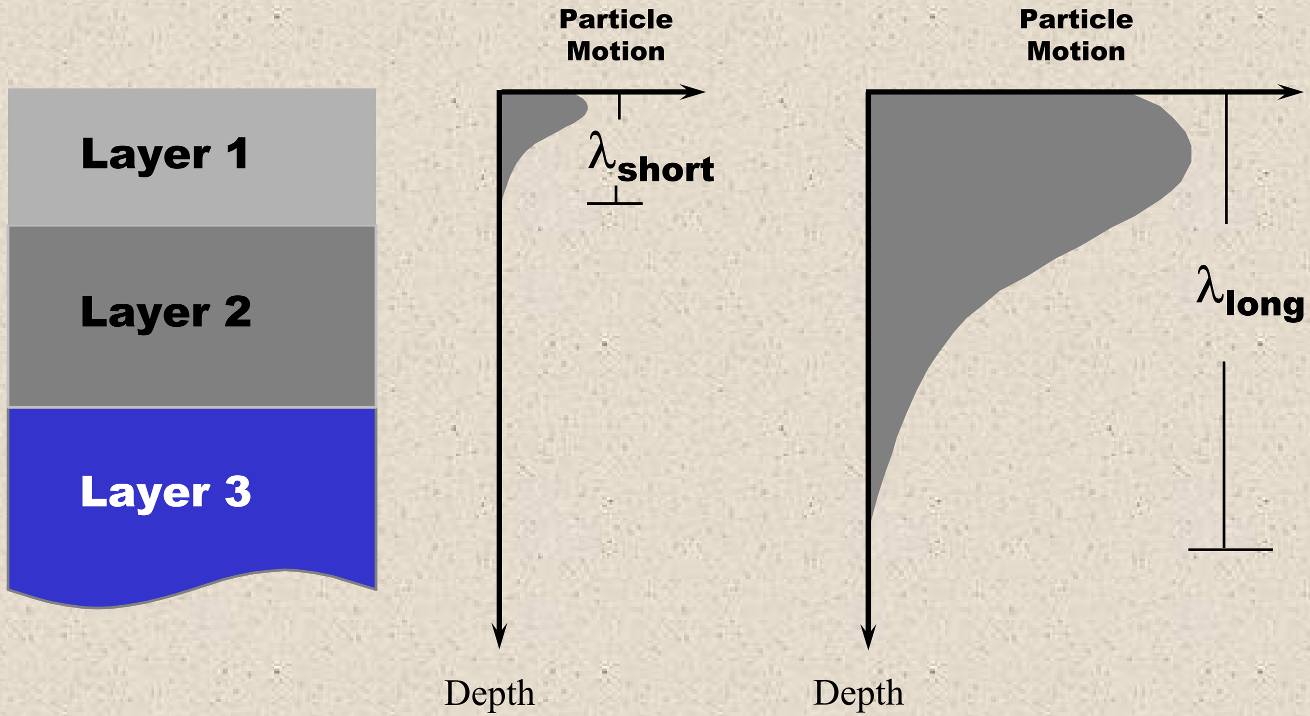
Surface Wave Method

- Use the dispersion of Surface waves to detect heterogeneity and variations in velocity profile

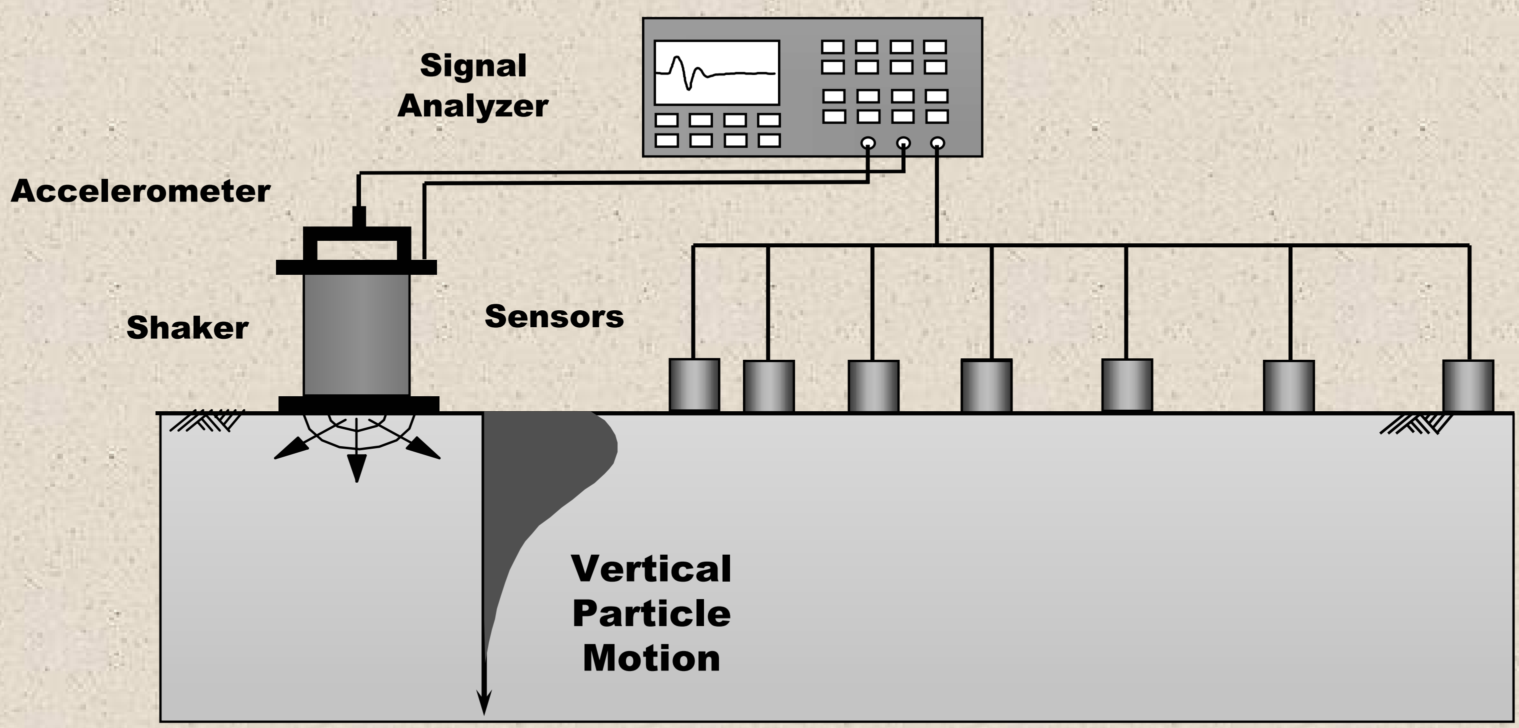


Depth Detection by Surface Wave

- Use the dispersion of Rayleigh surface waves to infer the depth of the heterogeneity



Geometry of the Problem



We start the investigation of the near surface earth by exciting a band-limited point source located at $\mathbf{x}_s$ and at time $t = 0$. The point source will generate a surface wavefield $u(\mathbf{x}_s, \omega)$, which will be measured on the surface by a number of receivers located at positions $\mathbf{x}_g$.

The Governing Equation

The propagation of the wavefield is governed by the Helmholtz equation:

$$Lu(\mathbf{x}_s,\omega) = [\nabla^2 + \frac{\omega^2}{\beta^2}]u(\mathbf{x}_s,\omega) = -F(\omega)\delta(\mathbf{x}-\mathbf{x}_s) \text{ and } z \geq 0,$$

where $F(\omega)$ represents the Fourier transform of the source and can be described as a continuous function with a compact support, $\delta(\mathbf{x}-\mathbf{x}_s)$ is the Dirac delta function, ω is the angular frequency, and β is the shear velocity of the medium. Furthermore, the incident wavefield and all the other scattered wavefields produced by the heterogeneity in the near surface earth satisfy the Sommerfeld radiation conditions. That is, ru is bounded and:

$$r[\frac{du}{dr} - \frac{i\omega}{\beta}] \to 0 \text{ as } r \to \infty,\ r = |\mathbf{x}|.$$

Linearizing the Scattered Wavefield

The medium's shear speed β can be decomposed into a known background speed $\beta_0(\mathbf{x})$ and a perturbation $\eta(\mathbf{x})$ which represents the heterogeneity in the near surface earth as

$$\frac{1}{\beta^2} = \frac{1}{\beta_0^2[1+\varepsilon\eta]}.$$

Expanding the above equation as a power series, one obtains:

$$\frac{1}{\beta_0^2[1+\varepsilon\eta]} = \frac{1}{\beta_0^2}\{1 - (\varepsilon\eta) + (\varepsilon\eta)^2 - (\varepsilon\eta)^3 + \ldots..\}.$$

Neglecting $O(\varepsilon^2)$ and higher orders, we arrive at the following:

$$\frac{1}{\beta_0^2[1+\varepsilon\eta]} = \frac{1}{\beta_0^2}[1 - \varepsilon\eta(\mathbf{x})],$$

Linearizing the Scattered Wavefield

By substituting last equation into the Helmholtz equation, we get the following equation:

$$[\nabla^2 + \frac{\omega^2}{\beta_0^2}]u(\mathbf{x}_s,\omega) - \varepsilon\eta\frac{\omega^2}{\beta_0^2}u(\mathbf{x}_s,\omega) = -F(\omega)\delta(\mathbf{x}-\mathbf{x}_s).$$

Denoting $[\nabla^2 + \frac{\omega^2}{\beta_0^2}]$ by L_0 and $\eta\frac{\omega^2}{\beta_0^2}$ by L_1 we obtain:

$$L_0 u(\mathbf{x}_s,\omega) - \varepsilon L_1 u(\mathbf{x}_s,\omega) = -F(\omega)\delta(\mathbf{x}-\mathbf{x}_s).$$

Linearizing the Scattered Wavefield

The perturbation procedure is then applied to the wavefield $u(\mathbf{x}_s, \omega)$ by decomposing the wavefield into an incident wavefield, $u_0(\mathbf{x}_s, \omega)$ representing the response of the near surface earth in the absence of the heterogeneity $\eta(\mathbf{x})$ and $u_m(\mathbf{x}_s, \omega)$ represent the modulation of $u_0(\mathbf{x}_s, \omega)$ in response to the presence of the heterogeneity $\eta(\mathbf{x})$ where m denotes the degree of scattering.

Then, the wavefield can be expressed mathematically as a perturbation series:

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots$$

By substituting and by collecting terms of equal powers of ε, we obtain:

$$\boxed{\begin{array}{c} L_0 u_0 = -F(\omega)\delta(\mathbf{x} - \mathbf{x}_s) \\ \hline \square \\ \hline L_0 u_m = L_1 u_{m-1} \qquad m \geq 1 \end{array}}$$

Linearizing the Scattered Wavefield

By introducing the adjoint operator and the adjoint Green function at the receiver location $\mathbf{x}$, we get:

$$L_0^* G^* (\mathbf{x}_s, \mathbf{x}, \omega) = \delta(\mathbf{x}_s - \mathbf{x})$$

where L_0^* represents the adjoint operator and $G^*(\mathbf{x}_s, \mathbf{x}, \omega)$ represents the adjoint Green function of the direct operator L_0 and of the direct Green function $G(\mathbf{x}, \mathbf{x}_s, \omega)$, respectively.

We state the Green theorem as:

$$\int_D \{G^* L_0 u_0 - u_0 L_0^* G^*\} dV = \int_{\partial D} \{G^* (\hat{n} \bullet \nabla) u_0 - u_0 (\hat{n} \bullet \nabla) G^*\} dS,$$

where D is the domain of the problem, $\hat{n}$ is a unit vector directed outward and normal to the boundary surface ∂D.

Linearizing the Scattered Wavefield

For the problem in hand, the adjoint operator is the same as the direct operator. Thus,

$G(\mathbf{x}, \mathbf{x}_s, \omega) = G(\mathbf{x}_s, \mathbf{x}, \omega)$. Consequently, the solution is given by:

$u_0 = -GF(\omega)$
$\square$
$u_m = GL_1 u_{m-1} \qquad m \geq 1$

And we obtain:

$$u = u_0 - \varepsilon G L_1 u_0 + \varepsilon^2 G L_1 G L_1 u_0 + \ldots$$

where u represents the total wavefield consisting of the unperturbed field, the single scattered field, the double scattered field and the higher orders of scattered fields and is known as the Neumann scattering series.

Linearizing the Scattered Wavefield

Retaining the first two terms, one obtains the Born approximation:

$$u = u_0 - \varepsilon G L_1 u_0 .$$

The single scattered wavefield u_1 is given by $-GL_1u_0$ and it depends linearly on the perturbation of the medium. Therefore, the process of linearizing the scattered wavefield is complete.

Linearizing the Scattered Wavefield

So far we have been confining ourselves to the scattered wavefield at $\mathbf{x}_s$, but our goal is to obtain an integral equation for $u_1(\mathbf{x}_g, \mathbf{x}_s, \omega)$:

$$L_0 u_1 = -L_1 u_0 = -\frac{\omega^2}{\beta_0^2}\eta(\mathbf{x})G(\mathbf{x}_s, \mathbf{x}, \omega).$$

Then the integral equation for the single scattered wavefield:

$$u_1(\mathbf{x}_g, \mathbf{x}_s, \omega) = -\int_D \frac{\omega^2}{\beta_0^2} G(\mathbf{x}_s, \mathbf{x}, \omega) F(\omega) \eta(\mathbf{x}) G(\mathbf{x}, \mathbf{x}_g, \omega) dV.$$

The integral relates the observed single scattered wavefield $u_1(\mathbf{x}_g, \mathbf{x}_s, \omega)$ to the heterogeneity $\eta(\mathbf{x})$. It also represents the forward modeling equation for the scattered wavefield.

The Green Function Method

Let us now address the Green function technique to find an inverse of a linear differential operator. Assume that we want to find the solution of the following equation:

$$Lu = -f, \;\; a < x < b; \;\; B_1(u) = 0 \;\; \& \;\; B_2(u) = 0$$

Where L is a linear differentiable operator of order n.

The Green function of the above equation is given by:

$$Lg(x, x') = -\delta(x - x'),$$

with the following properties:

$Lg(x, x') = 0$ for $x \neq x'$
$\frac{d^k g(x, x')}{dx^k}$ continuous at $x = x'$ for $k = 0, 1, 2, \ldots, n-2$
$\frac{d^{n-1} g(x, x')}{dx^{n-1}} \Big\vert_{x=x'-\varepsilon}^{x=x'+\varepsilon} = 1$ the jump condition.

The adjoint Green function $g^*(x, x')$ is defined as:

$$\boxed{L^* g^*(x', x) = -\delta(x' - x),}$$

The Green Function for Love Wave

The Love wave differential operator will be used to derive in three steps the Green function for the Love wave in the half-space overlain by a layer.

First, we will obtain the Green function $G(z, z_0)$ for an infinite medium by using the Green function method.

$$G(z, z_0) = \left\{ \boxed{\begin{array}{ll} L(G) = 0; & -\infty \le z < z_0 \\ \hline L(G) = 0; & z_0 < z < +\infty \end{array}} \right\}$$

$G(z, z_0)$ is continuous at $z = z_0$

$G(z, z_0)$ statisfies the jump condition, $\frac{dG}{dz}\Big|_{z_0^-} - \frac{dG}{dz}\Big|_{z_0^+} = 1,$

with $G \to 0$ as $z \to \pm\infty$ and $L \equiv \frac{d^2}{dz^2} - (\kappa^2 - \frac{\omega^2}{\beta^2}),$

The Green Function for Love Wave

$$G(z,z_0) = \left\{ \boxed{\begin{array}{ll} A(z_0)\exp(v_\beta z), & -\infty < z < z_0 \\ \hline B(z_0)\exp(-v_\beta z), & z_0 < z < +\infty \end{array}} \right\}.$$

Applying the continuity condition at $z = z_0$, we get:

$$G(z,z_0) = \left\{ \boxed{\begin{array}{ll} C\exp[v_\beta(z - z_0)]; & -\infty < z < z_0 \\ \hline C\exp[v_\beta(z_0 - z)]; & z_0 < z < +\infty \end{array}} \right\}.$$

The jump condition then gives $-Cv_\beta - Cv_\beta = 1$, that is $C = -\frac{1}{2v_\beta}$

Substituting for C to get:

$$\boxed{G(z,z_0) = -\frac{1}{2v_\beta}\exp[-v_\beta|z - z_0|],}$$

The Green Function for Love Wave

Second, we derive the Green function for a half-space by using the method of images. We will consider that we have sources $\delta(z-z_0)$ at $z = z_0$ and a negative image source $-\delta(z-z_0)$ at $z = -z_0$, thus we have:

$$\frac{d^2}{dz^2}G(z-z_0) - v_\beta^2 G(z-z_0) = \delta(z-z_0) - \delta(z+z_0).$$

Applying the principle of superposition, we get:

$$G(z,z_0) = -\frac{1}{2v_\beta}\exp[-v_\beta|z-z_0|] + \frac{1}{2v_\beta}\exp[-v_\beta|z+z_0|],$$

which represents the Green function for a half-space bounded by zero and infinity.

However, the half-space in our model is overlain by a layer of thickness H and extends downwards to infinity. Therefore, we will put the sources $\delta(z-z_0)$ at $z = 0$ and a negative image source $-\delta(z-z_0)$ at $z = 2H$, thus the required Green function will have the following form:

$$G(z,z_0) = -\frac{1}{2v_\beta}\{\exp[-v_\beta|z-z_0|] - \exp[-v_\beta|z+z_0-2H|]\}.$$

The Green Function for Love Wave

The Green function of the layer has the following form:

$$G(z,z_0) = \left\{ \begin{array}{|l|} \hline -\frac{1}{2\nu_\beta}\exp[\nu_\beta(z-z_0)] + A\exp(\nu_\beta z) + B\exp(-\nu_\beta z);\ \ 0 \le z < z_0 \\ \hline -\frac{1}{2\nu_\beta}\exp[-\nu_\beta(z-z_0)] + A\exp(\nu_\beta z) + B\exp(-\nu_\beta z);\ \ z_0 < z \le H \\ \hline \end{array} \right\},$$

differentiating with respect to z, we get:

$$\frac{d}{dz}G = \left\{ \begin{array}{|l|} \hline -\frac{1}{2}\exp[\nu_\beta(z-z_0)] + A\nu_\beta\exp(\nu_\beta z) - B\nu_\beta\exp(-\nu_\beta z) \\ \hline \frac{1}{2}\exp[-\nu_\beta(z-z_0)] + A\nu_\beta\exp(\nu_\beta z) - B\nu_\beta\exp(-\nu_\beta z) \\ \hline \end{array} \right\},$$

and applying the boundary condition, we obtain the required Green function for the layer

$$\begin{array}{|r|} \hline G(z,z_0) = -\dfrac{1}{2\nu_\beta}\{\exp[-\nu_\beta|z-z_0|] + \\ \hline \exp(\nu_\beta z)[\dfrac{\exp[-\nu_\beta(H-z_0)] + \exp[-\nu_\beta(H+z_0)}{\exp(\nu_\beta H) - \exp(-\nu_\beta H)}] + \\ \hline \exp(-\nu_\beta z)[\dfrac{\exp[\nu_\beta(H-z_0)] + \exp[-\nu_\beta(H-z_0)}{\exp(\nu_\beta H) - \exp(-\nu_\beta H)}]\}. \\ \hline \end{array}$$

The Differential Operator for Rayleigh Waves

The Rayleigh wave differential operator is given as a two coupled equations:

$$(\lambda+\mu)\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_3}{\partial x_3}\right)+\mu\left(\frac{\partial^2 u_1}{\partial x_1^2}+\frac{\partial^2 u_1}{\partial x_3^2}\right)=\rho\frac{\partial^2 u_1}{\partial t^2}$$

$$(\lambda+\mu)\frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_3}{\partial x_3}\right)+\mu\left(\frac{\partial^2 u_3}{\partial x_3^2}+\frac{\partial^2 u_3}{\partial x_1^2}\right)=\rho\frac{\partial^2 u_3}{\partial t^2}.$$

The displacement fields of the Rayleigh wave are given by the equations:

$u_1 = f_1(z)\exp[i(\omega t-\kappa x)]$
$u_2 = 0$
$u_3 = f_3(z)\exp[i(\omega t-\kappa x)].$

The Differential Operator for Rayleigh Waves

Substituting all the derivativs and simplifying, we finally obtain:

$$\frac{d^4 f_1}{dz^4} + \frac{\alpha^2}{\beta^2} v_\alpha^2 \frac{d^2 f_1}{dz^2} = \rho\alpha^2\sigma\{\rho\beta^2\sigma\frac{d^2 f_1}{dz^2} - \frac{\beta^2}{\alpha^2} v_\beta^2 [\frac{1}{\rho\beta^2\sigma}(\frac{d^2 f_1}{dz^2} + \frac{\beta^2}{\alpha^2} v_\beta^2 f_1)]\}$$

$$\frac{d^4 f_3}{dz^4} + \frac{\beta^2}{\alpha^2} v_\beta^2 \frac{d^2 f_3}{dz^2} = \rho\beta^2\sigma\{\rho\alpha^2\sigma\frac{d^2 f_3}{dz^2} - \frac{\alpha^2}{\beta^2} v_\alpha^2 [\frac{1}{\rho\alpha^2\sigma}(\frac{d^2 f_3}{dz^2} + \frac{\alpha^2}{\beta^2} v_\alpha^2 f_3)]\},$$

where $\alpha = \sqrt{\frac{\lambda+2\mu}{\rho}}$,$\beta = \sqrt{\frac{\mu}{\rho}}$, $v_\alpha^2 = \frac{\omega^2}{\alpha^2} - k^2$, $v_\beta^2 = \frac{\omega^2}{\beta^2} - k^2$, $\sigma = \frac{ik(\lambda+\mu)}{\mu(\lambda+2\mu)}$,

and after some simplifications, we end up with the following two decoupled equations:

$$\frac{d^4 f_1}{dz^4} + \{\frac{\beta^2}{\alpha^2} v_\beta^2 + \frac{\alpha^2}{\beta^2} v_\alpha^2 - \rho^2\alpha^2\beta^2\sigma^2\}\frac{d^2 f_1}{dz^2} + v_\alpha^2 v_\beta^2 f_1 = 0$$

$$\frac{d^4 f_3}{dz^4} + \{\frac{\beta^2}{\alpha^2} v_\beta^2 + \frac{\alpha^2}{\beta^2} v_\alpha^2 - \rho^2\alpha^2\beta^2\sigma^2\}\frac{d^2 f_3}{dz^2} + v_\alpha^2 v_\beta^2 f_3 = 0,$$

The Differential Operator for Rayleigh Waves

The Rayleigh wave operator L is defined as:

$$L \equiv \frac{d^4}{dz^4} + \frac{d^2}{dz^2}\left\{\frac{\beta^2}{\alpha^2}\nu_\beta^2 + \frac{\alpha^2}{\beta^2}\nu_\alpha^2 - \rho^2\alpha^2\beta^2\sigma^2\right\} + \nu_\alpha^2\nu_\beta^2,$$

which can be simplified to::

$$L \equiv \frac{d^4}{dz^4} + \left\{\nu_\alpha^2 + \nu_\beta^2\right\}\frac{d^2}{dz^2} + \nu_\alpha^2\nu_\beta^2.$$

The Green Function for Rayleigh Waves

There will be two Green functions for the Rayleigh waves operator L corresponding to P and SV surface waves, namely G_1 and G_3. The Green functions $\{G_j(z,z_0)\ , j = 1,3\}$ must satisfy the following conditions:

$$\left\{\begin{array}{|ll|}\hline L(G_j) = 0 & 0 \leq z < z_0 \\ \hline L(G_j) = 0 & z_0 < z < \infty \\ \hline \end{array}\right\},$$

$B_1(G_1) : \frac{d^2G_1}{dz^2} - \{2k^2 \frac{(v_\beta^2 - v_\alpha^2)}{(v_\beta^2 + k^2)} - v_\beta^2\}G_1 = 0,$ at $z = 0$
$B_2(G_1) : \frac{d^3G_1}{dz^3} - \{2k^2 \frac{(v_\beta^2 - v_\alpha^2)}{(v_\beta^2 + k^2)} + v_\alpha^2 - 2v_\beta^2\} \frac{dG_1}{dz} = 0,$ at $z = 0,$

$B_1(G_3) : \frac{d^2G_3}{dz^2} - \{2k^2 \frac{(v_\alpha^2 - v_\beta^2)}{(v_\beta^2 + k^2)} - v_\alpha^2\}G_3 = 0,$ at $z = 0$
$B_2(G_3) : \frac{d^3G_3}{dz^3} - \{\frac{k^2v_\beta^2 + 2v_\alpha^4 - v_\beta^4}{(v_\beta^2 - k^2 - 2v_\alpha^2)}\} \frac{dG_3}{dz} = 0,$ at $z = 0,$

The Green Function for Rayleigh Waves

$$G_j(z,z_0)\ ;\ \frac{\partial G_j}{\partial z}\ ;\ \frac{\partial^2 G_j}{\partial z^2} \quad \text{are continuous at} \quad z = z_0,$$

$$\frac{\partial^3}{\partial z^3}G_j(z_0^{+0},z_0) - \frac{\partial^3}{\partial z^3}G_j(z_0^{-0},z_0) = 1,$$

$$G_j \longrightarrow 0 \ \ as \ \ z \longrightarrow \infty, \ \ j = 1,3.$$

$$\frac{d^4 G_j}{dz^4} + \{v_\alpha^2 + v_\beta^2\}\frac{d^2 G_j}{dz^2} + v_\alpha^2 v_\beta^2 G_j = 0,$$

which has the characteristic equation:

$$r^4 + \{v_\alpha^2 + v_\beta^2\}r^2 + (v_\alpha^2 v_\beta^2)r = 0.$$

The characteristic equation has the four roots $\pm i v_\alpha$ and $\pm i v_\beta$. Corresponding to those roots, we get four linearly independent solutions: $\exp\{iv_\alpha z\}, \exp\{-iv_\alpha z\}, \exp\{iv_\beta z\}, \exp\{-iv_\beta z\}$.

The Green Function for Rayleigh Waves

The general solution for the Green functions can be expressed as:

$$G_j(z, z_0) = \begin{cases} A_j(z_0)\exp\{i\nu_\alpha z\} + B_j(z_0)\exp\{-i\nu_\alpha z\} + \\ C_j(z_0)\exp\{i\nu_\beta z\} + D_j(z_0)\exp\{-i\nu_\beta z\}, & z < z_0 \\ \square \\ E_j(z_0)\exp\{i\nu_\alpha z\} + F_j(z_0)\exp\{i\nu_\beta z\}, & z > z_0 \end{cases},$$

with $\mathrm{Im}(\nu_\alpha) > 0$ and $\mathrm{Im}(\nu_\beta) > 0$.

The Green Function for Rayleigh Waves

After applying the boundary conditions, we obtain the Green function for the P surface wave:

$$G_1(z, z_0) = \frac{i}{2\chi_1(v_\alpha^2 - v_\beta^2)}\{\chi_3[\frac{\exp\{i(v_\beta z_0 + v_\alpha z)\}}{v_\beta} - \frac{v_\beta \exp\{i(v_\alpha z_0 + v_\beta z)\}}{k^2}] -$$

$$\chi_2[\frac{\exp\{iv_\alpha(z_0 + z)\}}{v_\alpha} + \frac{\exp\{iv_\beta(z_0 + z)\})}{v_\beta}]\} \; +$$

$$\frac{i}{2(v_\alpha^2 - v_\beta^2)}\{[\frac{\exp\{iv_\alpha(z - z_0)\}}{v_\alpha} - \frac{\exp\{iv_\beta(z - z_0)\}}{v_\beta}]H(z - z_0)$$

$$+ [\frac{\exp\{iv_\alpha(z_0 - z)\}}{v_\alpha} - \frac{\exp\{iv_\beta(z_0 - z)\}}{v_\beta}]H(z_0 - z)\}$$

The Green Function for Rayleigh Waves

Similarly, the Green function for the SV surface wave is:

$$G_3(z,z_0) = \frac{i}{2\chi_1(v_\alpha^2 - v_\beta^2)}\{\chi_3[\frac{v_\alpha \exp\{i(v_\beta z_0 + v_\alpha z)\}}{k^2} - \frac{\exp\{i(v_\alpha z_0 + v_\beta z)\}}{v_\alpha}] +$$

$$\chi_2[\frac{\exp\{iv_\alpha(z_0 + z)\}}{v_\alpha} + \frac{\exp\{iv_\beta(z_0 + z)\})}{v_\beta}]\} \quad +$$

$$\frac{i}{2(v_\alpha^2 - v_\beta^2)}\{[\frac{\exp\{iv_\alpha(z - z_0)\}}{v_\alpha} - \frac{\exp\{iv_\beta(z - z_0)\}}{v_\beta}]H(z - z_0)$$

$$+ [\frac{\exp\{iv_\alpha(z_0 - z)\}}{v_\alpha} - \frac{\exp\{iv_\beta(z_0 - z)\}}{v_\beta}]H(z_0 - z)\},$$

where $H(z - z_0)$ is the Heaviside unit step function. The last two equations represent the Green functions for SV waves.

The Linearized Inversion for Scattered Surface Waves

We now turn our attention to the inverse problem: Given $u_1(\mathbf{x}_g, \mathbf{x}_s, \omega)$ for all $\mathbf{x}_s \in \partial D_1$ and all $\mathbf{x}_g \in \partial D_2$, where ∂D_j, are arbitrarily fixed open subsets in ∂D and all $\omega \in (0, \omega_0)$, find $\eta(x,z)$.

Tthe linearized forward formula can be expressed as:

$$u_1(x_g, x_s, z, \omega) = -\frac{\omega^2}{\beta_0^2} F(\omega) \int_D G^*(x_s, x, z, \omega)\eta(x,z)G^*(x, x_g, z, \omega) \exp[i2(\kappa x - \kappa z)]dxdz,$$

wher $G(x_s,x,z,\omega) = S(x_s,\omega)G^*(x_s,x,z,\omega)\exp[i(\kappa x - \kappa z)]$ and $G(x,x_g,z,\omega) = R(x_g,\omega)G^*(x,x_g,z,\omega)\exp]i(\kappa x - \kappa z)]$.

Further $\kappa = \frac{\omega}{\beta_0}$.Thus we have:

$$u_1(x_g, x_s, z, \omega) = -\frac{\omega^2}{\beta_0^2} F(\omega) \int_D G^*(x_s, x, z, \omega)\eta(x,z)G^*(x, x_g, z, \omega) \exp[\frac{i2\omega}{\beta_0}(x - z)]dxdz,$$

which represents a fourier-like transform which can be inverted using the inverse Fourier transform and we get:

$$\eta((x,z) = -\frac{1}{\pi\beta_0^2} \int_{-\infty}^{+\infty} \omega^2 F(\omega) G^+(x_s, \omega)\, u_1(x_g, x_s, z, \omega) G^+(x_g, \omega) \exp[-\frac{i2\omega}{\beta_0}(x - z)]d\omega$$

The Linearized Inversion for Scattered Surface Waves, the Algorithm

The forward integral formula is digitized by dividing the volume into regular cells to get:

$$u_1(x_i, x_j, z_n, \omega_k) = \sum_{n=1}^{N} \sum_m \frac{\omega_k^2}{\beta_{0m}^2} F(\omega_k) G_i^+(x_i, z_n, \omega_k) \eta(x_m, z_n) G_j^+(x_j, z_n, \omega_k) \exp[\frac{i2\omega_k}{\beta_{0_m}}(x_m - z_n)]$$

where i represents the source number, j represents the receiver number, k represents the frequency number, and m represents the midpoint cell number between the source i and the receiver j and can be written as $m = i + j$. Thus, $u_1(x_i, x_j, z_n, \omega_k)$ represents single scattered wavefield generated at the source i and recorded at the receiver j with a frequency component k. $\eta(x_m, z_n)$ represents the velocity perturbation within midpoint cell m and layer n. To simplify the problem, we define:

$$A_{m,n}(x_i, x_j, z_n, \omega_k) = \frac{\omega_k^2}{\beta_{0m}^2} F(\omega_k) G_i^+(x_i, z_n, \omega_k) G_j^+(x_j, z_n, \omega_k) \exp[\frac{i2\omega}{\beta_0}(x_m - z_n)]$$

where $A_{m,n}(x_i, x_j, z_n, \omega_k)$ represents all the single scattered wavefield corresponding to the midpoint cell m and layer n.

The Linearized Inversion for Scattered Surface Waves, the Algorithm

Then the discretized forward formula can be written as:

$$u_1(x_i, x_j, z_n, \omega_k) = \sum \sum A_{m,n}(x_i, x_j, z, \omega_k)\eta(x_m, z_n).$$

The last expression can be further simplified as:

$$d = A\eta.$$

Multiplying both sides of the last equation by $(A^*A)^{-1}A$ where A^* is the complex conjugate transpose of A to get:

$$\eta = (A^*A)^{-1}Ad.$$

The Linearized Inversion for Scattered Surface Waves, the Algorithm

Therefore, the inverse formula for the perturbation is given by:

$$\eta(x_{i+j}, z_n) = \sum_{\substack{sources \\ receivers \\ frequencies \\ layers}} \bar{d}_{ij}(\omega_k),$$

suggesting that the summation is done over the sources, receivers, frequencies and layers. Hence, the last formula will give us the perturbations which will allow us to update the velocity of the studied medium. Consequently, we have arrived at our goal "determination of the near surface velocity profile by linearized inversion of scattered surface wavefield".

Thank you