Study of Acoustic Waves in Oceans

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The study of acoustic wave propagation in the ocean has attracted considerable attention in the past. This has been motivated by the need to understand naval detection and marine seismology. For this purpose normal mode analysis has been one of the earliest methods used. As the change in the ocean properties effect the acoustic wave propagation, it is focus of our attention to model and analyze more general situations. We present some interesting models of ocean which can take into account the change in density or shape of seabed in a more realistic way. Perturbation analysis is used to study these models.

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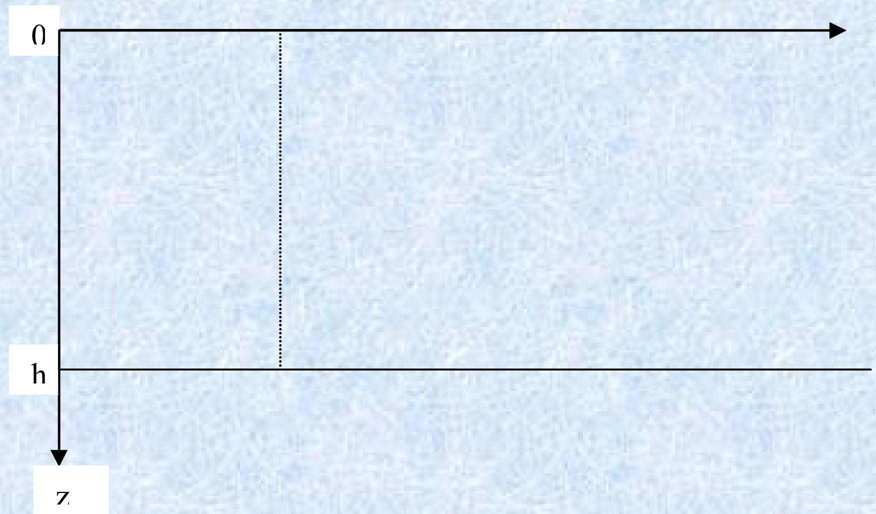


$$p_{rr}(r,z)+\frac{1}{r}p_r(r,z)+p_{zz}(r,z)+k^2p(r,z)=0$$

$p(r,0) = 0$ AND $p_z(r,h) = 0$.

We solve using separation of variables

$$p(r,z)=\Phi(r)\theta(z)$$

- The Idealized Depth Equation is the Sturm Liouville Problem

$$\phi^{//}(z) + k^2\phi(z) = \lambda\phi(z)$$

$$\phi(0)=0 \quad and \quad \phi^{/}(h)=0$$

The normalized eigenfunctions are

$$\phi_m(z) = \sqrt{\frac{h}{2}} \sin \frac{(2m-1)\pi z}{2h}$$

$$\lambda_m = k^2 - \left[\frac{(2m-1)}{2h}\right]^2$$

$$m = 1, 2, 3 \;.......... \quad .....$$

- An ocean with depth dependent properties leads to

$$\psi''(z) + k^2 n^2(z)\psi(z) = \lambda\psi(z)$$

with boundary conditions

$$\psi(0) = 0 \quad and \quad \psi'(h) = 0$$

The index of refraction

$$n^2(z) = 1 + \varepsilon s(z)$$

Contains a perturbation term $\varepsilon s(z)$

The case ε = 0 represents an idealized ocean

The Perturbation Method

- Find the eigenfunctions and the eigenvalues of the perturbed problem in terms of eigenfunctions and eigenvalues of the unperturbed or idealized problem and the perturbation term ϵ s(z).
- The eigenvalues of the unperturbed problem are *real* and eigenfunctions form a *complete orthonormal* set.

- The Procedure
- Expand in power series in e
- $$\lambda_m = \lambda_m^{(0)} + \varepsilon \lambda_m^{(1)} + \varepsilon^2 \lambda_m^{(2)} + ...$$
- and
$$\psi_m(z) = \psi_m(z)^{(0)} + \varepsilon \psi_m(z)^{(1)} + \varepsilon^2 \psi_m(z)^{(2)} + ...$$
- However, $\psi_m(z)^{(1)}$ can only be determined as Fourier series
- $$\psi_m(z)^{(1)} = \sum_1^{\infty} \alpha^{(1)}{}_{mk} \phi_k(z)$$

Perturbation Results

- The first correction terms in the perturbation series can be found as

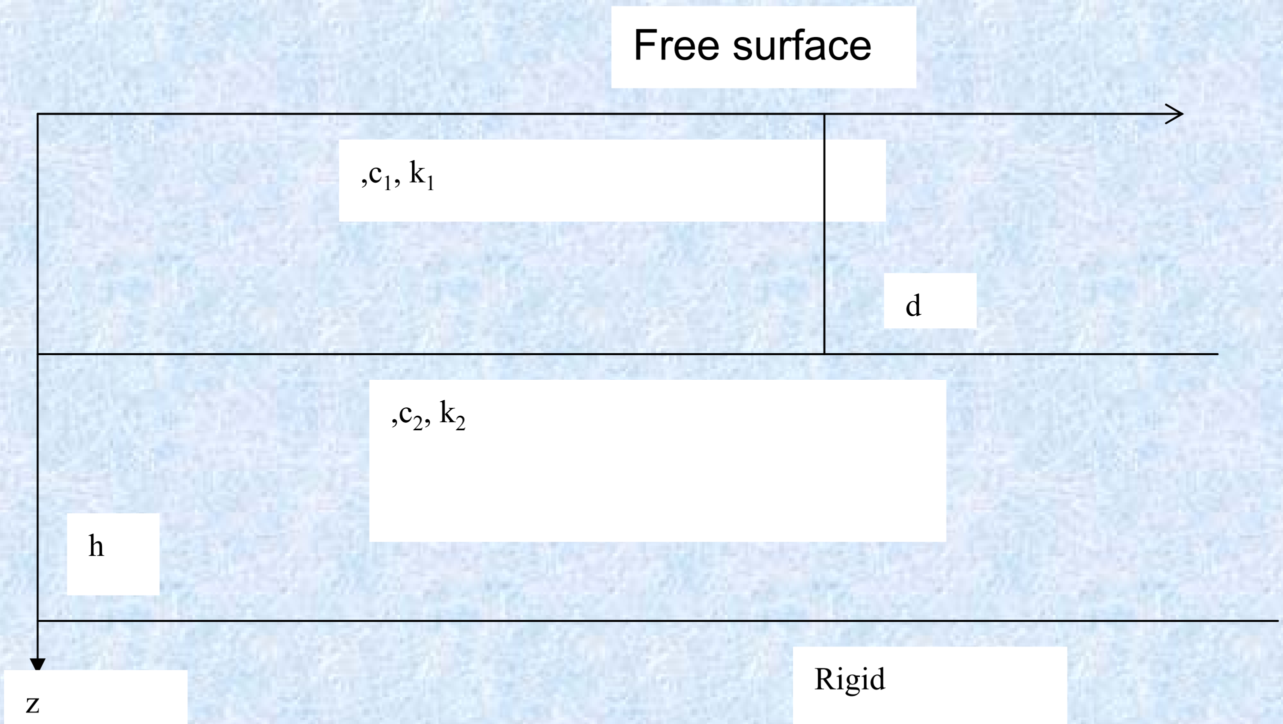
$$\lambda_m^{(1)} = \frac{2k^2}{h}\int_0^h s(z)\left[\sin\frac{(2m-1)\pi z}{2h}\right]^2 dz$$

The Fourier coefficien ts are

$$\alpha^{(1)}{}_{mn} = \frac{2k^2}{h}\int s(z)\sin\frac{(2m-1)\pi z}{2h}\sin\frac{(2n-1)\pi z}{2h}dz$$

$$\alpha^{(1)}{}_{mm} = 0$$

Layered Ocean Model



- Ocean properties are depth dependent
- The variations is piecewise constant
- Depth equation has piecewise constant coefficients
- Rigid seabed assumption gives nice normal mode theory

Zaman, Al-Muhiameed (Applied Acoustics, 2000)

- Layered model of ocean is used
- Seabed is assumed to be reflecting type
- Some energy is absorbed and some reflected back
- Neumann Boundary Conditions are satisfied
- The ocean properties are assumed to undergo small depth dependent change in the lower layer

Sturm Liouvilles Problem

- The depth equation in this case is

$$\phi_{zz}^{(i)}(z) + k_i^2 (n^{(i)})^2 \phi^{(i)}(z) = \lambda_m^{(i)} \phi^{(i)}(z),\ i = 1, 2$$

$$n^{(1)} = 1; \quad (n^{(2)})^2 = 1 + \varepsilon s(z)$$

$$\phi^{(1)}(0) = 0, \quad \phi^{(2)}(h) + \alpha\phi_z^{(2)}(h) = 0$$

$$\phi^{(1)}(d) = \phi^{(2)}(d), \quad \frac{1}{\rho^{(1)}}\phi_z^{(1)}(d) = \frac{1}{\rho^{(2)}}\phi_z^{(2)}(d)$$

$$\lambda_{1m}^{(2)} = k^2 \int_d^h s(z)\phi_{om}^{(2)}(z)\,dz$$

$$\lambda_{2m}^{(2)} = \int_d^h (-\lambda^{(2)}{}_{1m} + k_2{}^2 s(z))\,\phi_{0m}^{(2)}\phi_{0m}^{(2)}\,dz$$

Perturbed seabed Model (ICIAM 2003)

- Ocean consisting of two layers
- Depth dependent properties in one of the layers
- Presence of source emitting acoustic signals
- Seabed having a small undulation
- Due to perturbed boundary, variables can not be separated
- Instead of separation of variables, Fourier transform in x may be used
- Inhomogeneity due to source can be tackled using the Green Function

- The field equations in this case are

$$\frac{\partial^2 p^{(1)}}{\partial x^2} + \frac{\partial^2 p^{(1)}}{\partial z^2} + k^2{}_1(z)\, p^{(1)} = -2\pi\delta\,(z - z_s)$$

$$\frac{\partial^2 p^{(2)}}{\partial x^2} + \frac{\partial^2 p^{(2)}}{\partial z^2} + k^2{}_2(z)\, p^{(2)} = 0,$$

$$\textit{At perturbed seabed} \quad z = z_b$$

$$z_b = d_2 + \varepsilon h(x) \quad 0 \le x \le L$$

$$z_b = d_2 \quad \textit{otherwise}$$

$$\frac{\partial \;\; p^{(2)}(x, z_b)}{\partial z} - \varepsilon \frac{dh}{dx} \frac{\partial p^{(2)}(x, z_b)}{\partial x}$$

$$\textit{All other boundary conditions are same}$$

An Ocean with Random Properties

- In certain situations the property variation is not determinate
- The density and hence wave number is assumed to be a random function of depth
- The depth equation is setup as a random eigenvalue problem
- Perturbation procedure leads to a weakly correlated process
- Variance in the eigenvalue can be estimated in terms of the correlation function

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$$\psi_{zz}(z) + k^2(1 + \varepsilon\ s(z,\gamma)\psi(z) = \lambda\psi(z)$$

$$\psi(0) = 0, \quad \psi(h) = 0$$

Assume

$$\langle s(z,\gamma)\rangle = 0$$

Perturbati on procedure $\Rightarrow$

$$\lambda_m{}^{(1)}(\gamma) = k^2 \int_0^h s(z,\gamma)\,\phi_m{}^2(z)dz$$

- Weakly Correlated Process
- A random process is weakly correlated if it is given as

$$u = \int_a^b p(x,\gamma)\phi(x)\,dx$$

Where $\phi(x)$ is determinate and $p(x,\gamma)$ is a random process. If $\langle s(z,\gamma)\rangle = 0$ then the second moment (variance) is given by

$$\mathrm{Var(u)} = b\int_{\mathrm{a}}\int_a^b \phi(x_1)\phi(x_2)k(x_1,x_2,\gamma)\,dx_1 dx_2$$

where $k(x_1, x_2, \gamma)$

is the correlation function of the random function.

- We thus can find the following for the correction term

$$\left\langle (\lambda_m^{(1)}(\gamma))^2 \right\rangle = \frac{4k^4}{h^2} \int_0^h \int_0^h K_s(z_1, z_2) \sin^2\left(\frac{(2m-1)\pi}{2h} z_1 \right) \sin^2\left(\frac{(2m-1)\pi}{2h} z_2 \right) dz_1 dz_2$$

If we assume the correlation function in the form

$$K_s(z_1, z_2) = \sigma^2 \exp\{-\alpha |z_1 - z_2|\},$$

σ^2 is variance of the random function s(z, γ)

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$$\left\langle (\lambda_m^{(1)}(\gamma))^2 \right\rangle = \frac{2k^4}{h^2}\sigma^2 \left\{ 1 - \frac{\sin 2\theta_m h}{2\theta_m h} + \frac{3}{4\alpha^2}\cos 4\theta_m h - 1) \right\}$$

where

$$\theta_m = \frac{(2m-1)\pi}{2h}$$