Introduction to Differential Equations and Linear Algebra

Ordinary Differential Equation (ODE)

An equation that contains one or several (ordinary) derivatives of one or more dependent variables with respect to a single independent variable is called an ODE.

Some ODE's

(1): dy/dx=1; (2): dy/dx+dw/dx = x;

(3): $d^2y/dx^2+dy/dx+y=0$; (4): $d^2y/dx^2+(dy/dx)+y=0$;

(5): ydx + (x-1) dy=0. (6): $dy/dx+x(y)^2=1$ (nonlin)

(7): $(dy/dx)^2=xy$ (nonlin) (8): ydy/dx=1 (nonlin)

(9): dy/dx=coty (nonlin) (10): $y^{//}+p(x)y^{/}+q(x)y=r(x)$ (Linear ODE)

Order of an ODE

The order of the highest derivative in an ODE is called the order of the ODE.

Where & how do the ODE's arise

The ODE's arise in many engineering problems and those situations that represent various natural phenomenons:

ODE	Situation represented by the ODE
$dy(t)/dt=k\,y(t)$ $\Rightarrow y(t)=c\,e^{kt}$	Population growth rate (dy/dt) proportional to population present
$d^2y(t)/dt^2=kg$ $\Rightarrow y(t)=(k/2)gt^2+v_o t+y_o$	Acceleration of a falling object is proportional to "g"
$dT/dt=k(T_{surr}-T_{body})$ $\Rightarrow dT/dt>0$ if $T_{surr}>T_{body}$ and $dT/dt<0$ if $T_{surr}>T_{body}$	Newton's law of cooling
$dV(t)/dt=-k\sqrt{y}$	Time rate of change of volume proportional to the depth of water

Do All ODE's Have Solutions? NO

Example 1: $(dy/dx)^2 + y^2 = -1$ has no solution.

Example 2: $(dy/dx)^2 + (y)^2 = 0$ has a *trivial* solution only. *(y=0 is called trivial solution)*

Order of ODE's:

Order of the highest derivative that appears in an ODE is called the order of the

ODE.

Example 1: $dy/dx = 1$ 1st order ODE

Example 2: $d^2y/dx^2 + dy/dx + y = 0$ 2nd ordr ODE

Example 3: $x^2y'''y' + 2e^x y'' = (x^2+2)y^2$ 3rd order ODE ($y' = dy/dx$)

Representation of ODE's

- ☻ $F(x, y, y')=0$, 1^{st} order ODE
- ☻ $F(x,y,y',y'')=0$ 2^{nd} order ODE
- ☻ $F(x,y,y',\ldots,y(n))=0$ n^{th} order ODE

Solution of ODE's

$Y = u(x)$ is called a solution of the above ODE's if $F(x, u, u')$, $F(u, u', u'')$, and …$F(x, u, u', \ldots, u^{(n)})$ are all identically equal to zero on some interval (a, b). This means that all of the ODE's become identities when $y, y', \ldots, y^{(n)}$ are replaced respectively by $u, u', \ldots, u^{(n)}$.

Initial Value Problem

A problem $dy/dx = f(x, y)$ is called an initial value problem if it is subject to a condition that its solution $y(x_o) = y_o$ on an interval containing x_o. Such a condition gives rise to unique solution of the ODE as opposed to a problem with no such condition. By unique means that this condition determines the constant that appears in the solution of the ODE by a unique value.

Example: Find the constant "C" in $y(t)= Ce^{kt}$ that is a solution of the ODE $dy(t)/dt=ky$ subject to $y(0)=2$.

Solution:

Step 1: $dY(t)/dt= Cke^{kt} = k\,(Ce^{kt})= ky$. Thus the given $y(t)$ is a solution of the ODE.

Step 2: $y(0) = C = 2$. Thus the solution of the ODE is $y(t) = 2\,e^{kt}$.

Integrals as General and Particular Solutions

1^{st} order ODE $dy(x)/dx = f(x)$:

Integrate both sides of the above equation over "x" to get

$$y(x) = \int f(x)dx + C \qquad (1)$$

Eq. (1) is called the General Solution of the equations $dy/dx = f(x)$ with C integration constant.

Particular Solution of a 1^{st} Order ODE (We understand it by an example)

 Solve the initial value problem $dy/dx=2x+3$ subject to $y(1)=2$.

Solution: Integrate both sides of the above equation to get:

$$Y(x) = x^2 + 3x + C$$

The above solution is a general solution. Now we substitute the initial condition y(1)=2 to find that:

$$2 = (1)^2 + 3(1) + c \Rightarrow c = -2.$$

This condition gives particular solution of the above equation:

$$Y(x) = x^2 + 3x - 2$$

Velocity and Acceleration:

 The motion (curve) of a particle along a straight line is represented by $x = f(t)$.

 At any time " t " the velocity of the particle is defined by: $V = dx/dt = f'(t)$.

 The acceleration of the particle is represented by: $a = dV/dt = d^2x/dt^2$

 Newton's second law implies from here: $F = ma \Rightarrow F = m\, d^2x/dt^2$

The above equation being a 2nd order ODE there are two constants involved and are determined by initial position $X(0)=X_0$ and initial velocity $V(0)=V_0$ in the following way:

Write $dV/dt = a$ (a being constant) and integrate with respect to " t " to get

$$V = dx/dt = \int a\,dt + A = at + A. \qquad (1)$$

- Require that $V(0)=V_0$ to get $V_0=A$. Thus equation (1) becomes:

$$V = dx/dt = at + V_0 \qquad (2)$$

- Integrate equation (2) over " t " once again to get:

$$x(t) = \frac{1}{2}at^2 + V_0 t + B \qquad (3)$$

- Require that $X(0)=X_0$ to get $B=X_0$. Thus equation (3) takes the form:

$$x(t) = \frac{1}{2}at^2 + V_0 t + X_0 \qquad (4)$$

1.4: Separable Equations and their Applications

1st order Separable ODE dy(x)/dx = F(x, y):

The above equation is called separable if we can write: F(x, y) = f(x)/g(y). Thus the ODE becomes:

$$dy/dx = f(x)/g(y) \Rightarrow g(y)dy = f(x)dx \qquad (1)$$

Integrating (1) on both sides we get:

$$H(y) = K(x) + C, \quad where \quad H(y) = \int g(y)dy \quad and \quad K(x) = \int f(x)dx \qquad (2)$$

Illustration by Example: Solve $dy/dx = -6xy$ subject to y(0)=7.

Solution:

- The ODE is separable and becomes: $dy/y = -6xdx$.
- Its general solution is: $\ln y = -3x^2 + \ln C \quad or \quad y = Ce^{-3x^2}$
- Initial condition gives: $7 = C$
- Thus the initial value solution becomes: $y = 7e^{-3x^2}$

Example: Find general solution of the $dy/dx+2xy^2=0$.

Rewrite the differential equations as

$dy/dx = -2xy^2$

Separate variables

$$\frac{dy}{y^2} = -2xdx$$

Integrate both sides

$$\int \frac{dy}{y^2} = \int -2xdx$$

which gives

$y^{-1} = (-x^2+c)$

or $y = 1/(c-x^2)$

First Order Linear Differential Equation

Consider the ODE $dy/dx + P(x)y = Q(x)$ where $P(x) \& Q(x)$ are continuous on the interval of solution of the ODE. Then the integrating factor, $\rho(x)$, is defined by:

$$\rho(x) = e^{\int P(x)dx}.$$

Multiply by this integrating factor on both sides and write the result as:

$$\frac{d}{dx}(e^{\int P(x)dx} y) = Q(x)e^{\int P(x)dx}$$

Integrating the above equation on both sides gives:

$$y(x) = e^{-\int P(x)dx}[\{Q(x)e^{\int P(x)dx}\}dx + c]$$

 Example: Solve the ODE $\frac{dy}{dx} - y = \frac{11}{8}e^{-\frac{x}{3}}$ subject to y(0)=-1.

 Solution.

Step 1. The IF for the above ODE is: $\rho(x) = e^{\int -dx} = e^{-x}$

Step 2. The ODE becomes: $\frac{d}{dx}(e^{-x}y) = \frac{11}{8}e^{-\frac{4x}{3}}$

Step 3. Integrating above gives: $y(x) = -\frac{33}{32}e^{-\frac{x}{3}} + e^{x}c$

Step 4. Use the initial condition y(0)=-1 to get:

$$-1 = -\frac{33}{32} + c \Rightarrow c = \frac{1}{32}$$

Step 5. The solution of the ODE becomes:

$$y(x) = (-33e^{-\frac{x}{3}} + e^{x})/32$$