Linear Differential Equations with Constant Coefficients

A differential equation of the type

$$a_0\, d^n y/dx^n + a_1\, d^{n-1}y/dx^{n-1} + \ldots\ldots\ldots\ a_{n-1}\, dy/dx + a_n\, y = 0 \qquad (1)$$

is a linear, *homogeneous* differential equation of order n with $a_0, \ldots\ldots a_n$ constants.
We seek a solution of the type $y = e^{mx}$. This leads to $y^{(n)} = m^n e^{mx}$, n = 1,2,3….
Putting in the differential equation (1), we obtain

$$(a_0\, m^n + a_1 m^{n-1} + \ldots\ldots\ldots\ldots + a_{n-1} m + a_n)\, e^{mx} = 0.$$

This gives the algebraic equation of degree n in m known as *auxiliary equation*

$$a_0\, m^n + a_1 m^{n-1} + \ldots\ldots\ldots\ldots + a_{n-1} m + a_n = 0.$$

This can have all real distinct roots, real roots with some repeated roots, some or all roots complex which appear in conjugate pairs.

Case 1

If the auxiliary equation has real distinct roots
$m_1, m_2, m_3, \ldots\ldots$, then the DE (1) has linearly independent solutions

$$e^{m_1 x},\ e^{m_2 x},\ e^{m_3 x}$$

The general solution will be of the form

$$y(x) = C_1\, e^{m_1 x} + C_2\, e^{m_2 x} + C_3\, e^{m_4 x} + \ldots$$

Case 2

If the auxiliary equation have some repeated roots, say
$m = m_1 = m_2$, then the corresponding terms in the general solution will be of the form

$$C_1\, e^{mx} + C_2\, x\, e^{mx}$$

Case 3

If the auxiliary equation has the pair $m = \alpha \pm \beta$ as roots, the corresponding terms in the general solution will be

$$e^{\alpha x}\,[C_1 \cos(\beta x) + C_2 \sin(\beta x)]$$

Example (1)

$y'' + 5y' - 6y = 0$

The auxiliary equation is

$m^2 + 5m - 6 = 0.$

This has solutions $m = 1, -6$. The general solution is given by

$y(x) = C_1 e^{x} + C_2 e^{-6x}$.

Example (2)

$y'' + 10y' + 25y = 0$

The auxiliary equation is

$m^2 + 10m + 25 = 0.$

This has solutions $m = -5, -5$ (repeated roots)
The corresponding term in the general solution are

$Y(x) = C_1 e^{-5x} + C_2\, x\, e^{-5x}$.

Example (3)

$y''' - y = 0$

The auxiliary equation is

$m^3 - 1 = 0 \rightarrow (m - 1)(m^2 + m + 1) = 0$

This has roots

$m = 1, -\frac{1}{2} \pm \sqrt{(3/2)}\, i$

The corresponding terms in the general solution will be

$C_1 e^{x} + e^{-1/2\, x} [C_2 \cos(\sqrt{(3/2)}\, x) + C_3 \sin(\sqrt{(3/2)}\, x)]$.

Cauchy Euler Equation

The equation of the following type is known as the Cauchy Euler equation (homogeneous form)

$x^{n} y^{(n)} + a_1 x^{n-1} y^{(n-1)} + a_2 x^{n-2} y^{(n-2)} + \ldots + a_n y = 0.$

We can reduce this equation into DE with constant coefficients using the following change of independent variable

$x = e^{t}$, or $t = \ln x$.
Using chain rule

$dy/dx = (dy/dt)(dt/dx)$.
As $dt/dx = 1/x$,
$dy/dx = (1/x)(dy/dt)$.
Applying the chain rule again

$$d^2y/dx^2 = (-1/x^2)(dy/dt) + (1/x)(1/x)(d^2y/dt^2)$$
$$= (-1/x^2)(dy/dt) + (1/x)^2\, d^2y/dt^2)$$

Example (1)

Find a general solution to

$3x^2 y'' + 11 x y' - 3y = 0.$

Using above change of variables and values of y' and y'',

$3(d^2y/dt^2 - dy/dt) + 11\, dy/dt - 3y = 0$
which reduces to
$3\, d^2y/dt^2 + 8\, dy/dt - 3y = 0$
The roots of the auxiliary equation are
$m = 1/3$, and $m = -3$.
The general solution is
$y(t) = C_1 e^{-(1/3)t} + C_2 e^{-3t}$,

using $t = \ln x$ (or $x = e^t$)
$y(x) = C_1 x^{-1/3} + C_2 x^{-3}$.

Example (2)
Solve $x y^{//} - y^{/} = 0$.

This can be written in the Cauchy Euler form by multiplying throughout by x
$x^2 y^{//} - x y^{/} = 0$.
The above change of variable now gives

$d^2y/dt^2 – dy/dt – dy/dt = 0$.
Or, $d^2y/dt^2 – 2\, dy/dt = 0$.

The roots of the auxiliary equation are m = 0, 2.
The general solution in terms of t is

$y(t) = C_1 + C_2 e^{2t}$,

which in terms of x is

$y(x) = C_1 + C x^2$.