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# Investigation of the Initial Inverse Problem in the Heat Equation

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We investigate the inverse problem in the heat equation involving the recovery of the initial temperature from measurements of the final temperature. This problem is extremely ill-posed and it is believed that only information in the first few modes can be recovered by classical methods. We will consider this problem with a regularizing parameter which approximates and regularizes the heat conduction model.

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## 1 Introduction

The classical direct problem in heat conduction is to determine the temperature distribution of a body as the time progresses. The task of determining the initial temperature distribution from the final distribution is distinct from the direct problem and is identified as the initial inverse heat conduction problem. This type of inverse problem is extremely ill-posed, e.g., Engle [\[1\]](#). There is an alternative approach which consists of a reformulation of the classical heat equation by a hyperbolic heat equation (see Weber [\[2\]](#), Elden [\[3\]](#), and Masood et al. [\[4\]](#)).

We will present an alternative approach which approximates and regularizes the initial inverse heat conduction solution. The need to consider the alternative formulation has some physical advantages. In many applications, one encounters a situation where the usual parabolic heat equation does not serve as a realistic model. Since the speed of propagation of the thermal signal is finite, e.g., for short-pulse laser applications, the hyperbolic differential equation correctly models the problem; see Vedavarz et al. [\[5\]](#) and Gratzke et al. [\[6\]](#) among others. The initial inverse problem in the hyperbolic heat equation is stable and well posed. Moreover, numerical methods for hyperbolic problems are efficient and accurate. We will utilize the small value of the parameter and apply the WKBJ (Wentzel, Kramers, Brillouin, and Jeffreys) method to solve the initial inverse problem, see Bender and Orszag [\[7\]](#).

## 2 Initial Inverse Problem in the Heat Equation

Supposing we have a metal bar, which for the sake of convenience we take to extend over the interval  $0 \leq x \leq \pi$ , whose temperature at the point  $x$  and at time  $t$  is given by the function  $u(x, t)$ . Then, for an appropriate choice of units,  $u(x, t)$  satisfies the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0 \quad (1)$$

with homogeneous Dirichlet boundary conditions

$$u(0, t) = u(\pi, t) = 0 \quad (2)$$

We assume the final temperature distribution of the bar at time  $t = T$

$$f(x) = u(x, T) \quad (3)$$

and we want to recover the initial temperature profile of the bar

$$g(x) = u(x, 0) \quad (4)$$

The condition (2) can be replaced by an insulated boundary, i.e.,  $u_x(0,t) = u_x(\pi,t) = 0$ , since it is important in some applications, see for example Beck et al. [8] and Al-Khalidy [9].

We assume by the separation of the variables, the solution of the direct problem of the form

$$u(x, t) = \sum_{n=1}^{\infty} v_n(t) \phi_n(x) \quad (5)$$

The eigenfunctions of  $d^2/dx^2$  given by  $\phi_n(x) = \sqrt{2/\pi} \sin(nx)$  form a complete orthonormal system in  $L^2[0, \pi]$ .

Thus,  $g(x) \in L^2[0, \pi]$  can be expanded as

$$g(x) = \sum_{n=1}^{\infty} c_n \phi_n(x), \quad x \in [0, \pi] \quad (6)$$

So, we can write the solution of the direct problem (1) in the form

$$u(x, t) = \sum_{n=1}^{\infty} c_n \exp[-n^2 t] \phi_n(x) \quad (7)$$

Now by applying (3), we can write

$$f(x) = \int_0^{\pi} K(x, \zeta) g(\zeta) d\zeta \quad (8)$$

which is an integral equation of the first kind. The singular system of the integral Eq. (8) is given by

$$\{\exp[-n^2 T], \phi_n(x), \phi_n(x)\} \quad (9)$$

Now by application of Picard's theorem the inverse problem can be solved if

$$\sum_{n=1}^{\infty} \exp[2n^2 T] |f_n|^2 < \infty \quad (10)$$

where

$$f_n = \int_0^\pi \phi_n(\zeta) f(\zeta) d\zeta \quad (11)$$

are the classical Fourier coefficients of  $f$ . Now again using Picard's theorem, we can recover the initial profile by the following expression

$$g(x) = \sum_{n=1}^{\infty} \exp[n^2 T] f_n \phi_n(x) \quad (12)$$

Picard's theorem demonstrates the ill-posed nature of the problem considered. If we perturb the data by setting  $f_\delta = f + \delta \phi_n$  we obtain a perturbed solution  $g_\delta = g + \delta \phi_n \exp[n^2 T]$ . Hence, the ratio  $\|g_\delta - g\| / \|f_\delta - f\| = \exp[n^2 T]$  can be made arbitrarily large due to the fact that the singular values  $\exp[-n^2 T]$  decay exponentially. The influence of errors in the data  $f$  is obviously affected by the rate of this decay. So in regularizing, we will confine ourselves to lower modes by only retaining the first few terms in the series (12). This technique of truncating the series is known as truncated singular value decomposition (TSVD), see Hansen [10].

### 3 The Hyperbolic Model

The method we apply is similar to the quasi-inverse method of Lions [11]. The Lions' method is based on replacing the problem (1,2,3,4) by a problem for equation of higher order with a small parameter. There are several methods for solving ill-posed problems. The quasi-solution method to solve the equation of the first kind was introduced by Ivanov [12]. The essence of this method is to change the notion of solution of an ill-posed problem so that, for certain conditions, the problem of its determination will be well-posed. Tikhonov's regularization method is widely used for solving linear and nonlinear operator equations of the first kind, see Tikhonov and Arsenin [13]. Iterative methods are applied to solve different problems and particularly these methods can also be applied to solve operator equations of the first kind. Moulthanovsky [14] applied such an iterative method to solve an initial inverse heat transfer problem. The projective methods for solving various ill-posed problems are based on the representation of the approximate solution as a finite linear combination of a certain functional system, see, e.g., Vasin and Ageev [15].

The methods mentioned in the previous paragraph may be applied for solving the extensive class of inverse problems.

These methods do not take into account the specific character of concrete inverse problems. The Lion's method and the method we present in this paper take into account peculiarities of the inverse problem. There is an alternative approach to the inverse heat conduction problem [2][3], which consists of introducing a small damping parameter with the term  $\partial^2 u / \partial t^2$ . So, let us consider the following hyperbolic heat equation

$$\epsilon \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad \epsilon > 0, \quad 0 < x < \pi, \quad (13)$$

together with conditions (2,3,4) and one additional condition

$$\frac{\partial u}{\partial t}(x, 0) = 0. \quad (14)$$

Following the same procedure as that for the parabolic heat equation and assuming the solution of the form (5), for  $\epsilon \rightarrow 0^+$  we get the following ordinary differential equation

$$\epsilon \frac{d^2 a_n(t)}{dt^2} + \frac{da_n(t)}{dt} + n^2 a_n(t) = 0, \quad \epsilon > 0, \quad t > 0, \quad (15)$$

subject to

$$a_n(0) = c_n, \quad \text{and} \quad \frac{da_n(0)}{dt} = 0. \quad (16)$$

This is a singular perturbation problem, so we seek the WKBJ solution to this problem [7]. The WKBJ solution to (15) is

$$a_n(t) = \left( \frac{\epsilon n^2 - 1}{2\epsilon n^2 - 1} \right) c_n \exp[-n^2 t] + \left( \frac{\epsilon n^2 c_n}{2\epsilon n^2 - 1} \right) \exp\left[n^2 t - \frac{t}{\epsilon}\right] \quad (17)$$

Using Picard's theorem the solution exists if

$$\sum_{n=1}^{\infty} \frac{|f_n|^2}{\left\{ \left( \frac{\epsilon n^2 - 1}{2\epsilon n^2 - 1} \right) \exp[-n^2 T] + \left( \frac{\epsilon n^2}{2\epsilon n^2 - 1} \right) \exp\left[n^2 T - \frac{T}{\epsilon}\right] \right\}^2} < \infty \quad (18)$$

and the solution is given by

$$g(x) = \sum_{n=1}^{\infty} \frac{f_n \phi_n(x)}{\left\{ \left( \frac{\epsilon n^2 - 1}{2\epsilon n^2 - 1} \right) \exp[-n^2 T] + \left( \frac{\epsilon n^2}{2\epsilon n^2 - 1} \right) \exp\left[n^2 T - \frac{T}{\epsilon}\right] \right\}} \quad (19)$$

Supposing  $\epsilon \rightarrow 0^+$  in the expression (19), we get the solution of the heat conduction problem (12). So, the solution of the heat equation can also be treated as the limiting case of the hyperbolic heat equation.

**Example.** Let us consider the initial temperature distribution of the form  $g(x) = \sqrt{2/\pi} \sin(mx)$ , where  $m$  is some fixed integer, then the final data for (19) and (12) can be given by

$$f_m = \left( \frac{\epsilon m^2 - 1}{2\epsilon m^2 - 1} \right) \exp[-m^2 T] + \left( \frac{\epsilon m^2}{2\epsilon m^2 - 1} \right) \exp \left[ m^2 T - \frac{T}{\epsilon} \right] \quad (20)$$

and

$$f_m = \exp[-m^2 T]. \quad (21)$$

The expression (20) represents the final data for the hyperbolic model corresponding to the assumed initial profile  $g(x)$  in the absence of the noise. Using (20) in the expression (19), the initial profile  $g(x)$  can be recovered exactly. The expression (21) represents the final data for the parabolic model corresponding to the assumed initial profile  $g(x)$  in the absence of the noise and the initial profile can be recovered exactly by using it in the expression (12).

Now we analyze the models by adding white Gaussian noise to the data (21). The reason to use (21) as final data is that the exact measured data would be of this form. In Figs. 1,2,3, we use the noisy data (white Gaussian noise+(21)) in both parabolic heat conduction and hyperbolic heat models and see the mean behavior of 100 independent realizations.

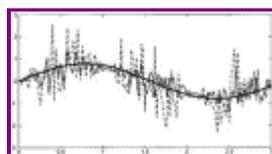


Figure 1.

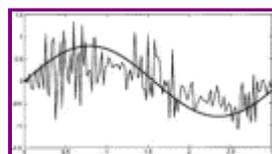


Figure 2.

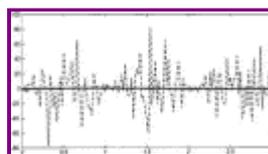


Figure 3.

We have considered the second mode, that is,  $m = 2$  in Figs. 1,2,3 as well as retaining the first three terms ( $N = 3$ ) in series (12) and (19). In Fig. 1, the hyperbolic heat model behaves better than the parabolic heat conduction model for SNR=50 dB. We have increased the level of noise in Figs. 2,3 to SNR=20 dB. The inherent instability of the parabolic heat conduction model is clear from Fig. 3 by observing the range of the vertical axis.

## Conclusions

The inverse solution of the heat conduction model is characterized by discontinuous dependence on the data. It is shown that in case of noisy data, the hyperbolic model approximates the exact initial profile better than the parabolic heat conduction model. Further, in the case of noisy data, the information about the initial profile cannot be recovered for higher modes by the parabolic heat conduction model but by the hyperbolic model some useful information may be recovered. However for higher modes the information recovered by the hyperbolic model is better than the parabolic model but it may not be good enough for a particular application.

## Acknowledgment

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## Nomenclature

$|\cdot|$  = absolute value

$\|\cdot\|$  = norm

$$c_n = \int_0^{\pi} g(x) \phi_n(x) dx$$

$$K(x, \zeta) = \sum_{n=1}^{\infty} \exp[-n^2 T] \phi_n(\zeta) \phi_n(x)$$

$$u_x = \partial u / \partial x$$

SNR = signal-to-noise ratio

WKBJ = Wentzel, Kramers, Brillouin, and Jeffreys

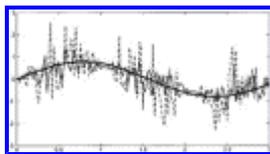
$\phi_n(x)$  = eigenfunctions

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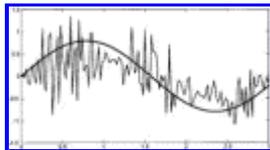
## FIGURES



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Fig. 1 The case of noisy data with  $\text{SNR}=50$  dB,  $N=3$ ,  $m=2$ ,  $T=1$ , and  $\epsilon=0.04$ . The noisy data used in the heat conduction solution (12) is represented by the dotted line and in the damped wave solution (19) by the thin solid line and the noiseless temperature by the thick solid line. [First citation in article](#)

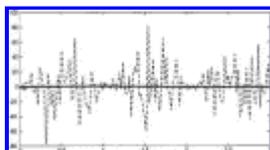
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[Full figure](#) (7 kB)

Fig. 2 Response of the damped model (19) in the case of noisy data with  $\text{SNR}=20$  dB,  $N=3$ ,  $m=2$ ,  $T=1$ ,  $\epsilon=0.07$  [First citation in article](#)

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Fig. 3 Response of the classical heat model (12) in the case of noisy data with  $\text{SNR}=20$  dB,  $N=3$ ,  $m=2$ ,  $T=1$  [First citation in article](#)

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