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Invariant solutions of certain nonlinear evolution type equations with small parameters

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Abstract

The Fisher equation, which arises in the study of reaction diffusion waves in biology, does not display a high level of symmetry properties. Consequently, only travelling wave solutions are obtainable using the method of invariants. This has a direct bearing on studying perturbed forms of the equation which may arise from considering, e.g., damping or dissipative factors. We show, here, how one can get around this limitation by appending some unknown function to the perturbation and obtain interesting practical results using invariants. The ideas have significant consequences for equations which do not admit large class of symmetry properties. The method used in this analysis is then extended to other classes of evolution type equations that involve perturbations, for, e.g., the KdV type equations.

Keywords: Perturbations of nonlinear evolution equations; Fisher; KdV; Modified KdV equations

1. Introduction

It can easily be shown that the Fisher equation:

$$u_t = u_{xx} + \lambda u(1-u) \tag{1.1}$$

which models the behaviour of reaction–diffusion waves in biology only admits point symmetries involving time and space translations (see [1], [2] for a mathematical treatment of the equation with specific reference to 'invariant theory'). Thus, invariant solutions are obtainable in terms of travelling waves only. Nevertheless, some interesting situations do arise here (see the above references). Interestingly, the equation displays Painleve properties for certain values of λ (see [1]) which is related to Lagrangian properties of the reduced travelling wave form [2]. The drawback with an equation admitting translation symmetries only is the restriction one is subjected to for analysis of perturbed forms of the equation using the now established method of 'approximate symmetries' (Baikov et al. [3]). For example, perturbations of the equation in damping or

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dissipative terms of the first-order, viz., u_t and u_x, is not possible. Also, we note that (1.1) does not admit first-order conservation laws. Some studies of the equation have also been done in [4].

Here, we suggest a way one can do an analysis, using the approximate symmetry method. Essentially, we study versions of the perturbed form including first-order dissipative terms:

$$u_t - u_{xx} - \lambda u(1-u) + \epsilon f(x,t)u_x = 0 \tag{1.2}$$

and show, using group methods, how one needs to classify the function $f(x,t)$ which results in imposing certain 'approximate symmetries'.

We then consider perturbations of the KdV and modified KdV equation:

$$u_t + \alpha(1+\beta u)uu_x + \gamma u_{xxx} = 0, \quad \alpha, \gamma > 0. \tag{1.3}$$

We summarize the essentials of the approximate symmetry method.

Consider an rth-order system of perturbed partial differential equations of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$ with ϵ a small parameter:

$$E^\beta(x, u, u_{(1)}, \ldots, u_{(r)}; \epsilon) = \mathrm{O}(\epsilon^{k+1}), \quad \beta = 1, \ldots, \tilde{m}, \tag{1.4}$$

where $u_{(1)}, u_{(2)}, \ldots, u_{(r)}$ denote the collections of all first, second, …, rth-order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots$, respectively, with the total differentiation operator with respect to x^i given by:

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, \ldots, n, \tag{1.5}$$

where the summation convention is used whenever appropriate.

The symmetry generator:

$$\mathscr{X} = X_0 + \epsilon X_1 + \cdots + \epsilon^k X_k \tag{1.6}$$

is called a kth-order approximate symmetry generator of (1.4) if

$$\mathscr{X}(E^\beta)\big|_{(1.4)} = \mathrm{O}(\epsilon^{k+1}) \tag{1.7}$$

holds, where

$$X_b = \xi_b^i \frac{\partial}{\partial x^i} + \eta_b^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_{b,i}^\alpha \frac{\partial}{\partial u_j^\alpha} + \zeta_{b,i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1}^\alpha i_2} + \cdots, \quad b = 0, \ldots, k$$

are Lie–Bäcklund symmetry operators and ξ_b^i, η_b^α are differential functions, and the additional coefficients are uniquely determined by the prolongation formulae:

$$\begin{aligned} \zeta_{b,i}^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \\ \zeta_{b,i_1 i_2 \cdots i_s}^\alpha &= D_{i_s}(\zeta_{i_1 i_2 \cdots i_{s-1}}^\alpha) - u_{j i_1 \cdots i_{s-1}}^\alpha D_{i_s}(\xi^j), \quad s > 1. \end{aligned} \tag{1.8}$$

In (1.7) and hereafter $|_{(1.4)}$ means "evaluated on the solutions of Eq. (1.4)".

For equations of first-order in the perturbed variable ϵ (see [3]), if X_0 is a generator of Lie (point) symmetry of a differential equation:

$$E_0 = 0, \tag{1.9}$$

then an *approximate symmetry*, $X = X_0 + \epsilon X_1$, of the perturbed differential equation

$$E_0 + \epsilon E_1 = 0 \tag{1.10}$$

is obtained by solving for X_1 in

$$X_1(E_0)|_{E_0=0} + H = 0, \tag{1.11}$$

where

$$H = \frac{1}{\epsilon} X_0(E_0 + \epsilon E_1)|_{E_0+\epsilon E_1=0} \tag{1.12}$$

(E_1 is the perturbation and H is referred to as an auxilliary function).

2. Perturbations of the Fisher equation

A point symmetry generator admitted by (1.1) is:

$$X_0 = \frac{\partial}{\partial t} + k\frac{\partial}{\partial x}. \tag{2.1}$$

When $k = 0$, the invariant solution is a steady one and $k \neq 0$ provide a travelling wave solution where k is the speed of the wave. If $X_1 = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial} + \phi\frac{\partial}{\partial u}$ in (1.11), by (1.12):

$$H = (kf_x + f_t)u_x, \tag{2.2}$$

Eq. (1.11) is

$$\phi^t - \phi^{xx} - \lambda\phi(1-2u) + (kf_x + f_t)u_x = 0, \tag{2.3}$$

from which the analysis reveals:

$$\begin{aligned}
&\tau = a(t), \quad \xi = b(x,t), \quad b = \tfrac{1}{2}a_t x + c(t),\\
&\phi = d(t,x)u + e(t,x),\\
&-\tfrac{1}{2}a_{tt}x - a_t - 2d_x + kf_x + f_t = 0,\\
&d_t u + e_t + d\lambda(u-u^2) - a_t\lambda(u-u^2) - (d_{xx}u + e_{xx}) - \lambda(du+e) + 2\lambda u(du+e) = 0.
\end{aligned} \tag{2.4}$$

The calculations reveal that $f(x,t)$ is of particular forms. We proceed further with a specific choice of f and determine approximate point symmetries of (1.2). We choose $f = \frac{-3}{2}\lambda x e^{\lambda t}$ with ($k = 0$, viz., the steady state case) so that (1.2) becomes:

$$u_t - u_{xx} - \lambda u(1-u) - \left(\frac{3}{2}\epsilon\lambda x e^{\lambda t}\right)u_x = 0. \tag{2.5}$$

We get:

$$\begin{aligned}
&\tau = 1 - 3\epsilon e^{\lambda t},\\
&\xi = \tfrac{-3}{2}\epsilon\lambda x e^{\lambda t},\\
&\phi = 3\lambda\epsilon e^{\lambda t}(u-1),
\end{aligned} \tag{2.6}$$

so that the approximate symmetry is:

$$G = \frac{-3}{2}\epsilon\lambda x e^{\lambda t}\frac{\partial}{\partial x} + (1 - 3\epsilon e^{\lambda t})\frac{\partial}{\partial t} + 3\epsilon\lambda e^{\lambda t}(u-1)\frac{\partial}{\partial u}. \tag{2.7}$$

The invariants of G are given by:

$$\frac{dx}{\frac{-3}{2}\epsilon\lambda x e^{\lambda t}} = \frac{dt}{1 - 3\epsilon e^{\lambda t}} = \frac{du}{3\epsilon\lambda(u-1)e^{\lambda t}}, \tag{2.8}$$

which are y and v, viz.,

$$y = \frac{x^2}{1 - 3\epsilon e^{\lambda t}} \tag{2.9}$$

and

$$v = (u-1)(1 - 3\epsilon e^{\lambda t}). \tag{2.10}$$

After substitutions and simplifications we get:

$$\frac{3\lambda\epsilon x^2 e^{\lambda t}v'}{1 - 3\epsilon e^{\lambda t}} + 3\epsilon\lambda e^{\lambda t}v - \frac{4x^2 v''}{1 - 3\epsilon e^{\lambda t}} - 2v' + \lambda v((1 - 3\epsilon e^{\lambda t}) + v) - 3\epsilon\lambda x^2 e^{\lambda t}v' = 0. \tag{2.11}$$

Therefore,

$$3\epsilon\lambda y e^{\lambda t}v' + 3\epsilon\lambda e^{\lambda t}v - 4yv'' - 2v + \lambda v((1 - 3\epsilon e^{\lambda t}) + v) - 3\epsilon\lambda x^2 e^{\lambda t}v' = 0, \tag{2.12}$$

which implies,

$$-4yv'' - 2v' + \lambda v(v+1) + 9\lambda\epsilon^2 y(\mathrm{e}^{\lambda t})^2 = 0. \tag{2.13}$$

Thus,

$$-4yv'' - 2v' + \lambda v(v+1) = 0. \tag{2.14}$$

Now solving the equation,

$$-4yv'' = 2v' - \lambda v(v+1), \tag{2.15}$$

with the Lie symmetry,

$$y^{\frac{1}{2}}\frac{\partial}{\partial y}, \tag{2.16}$$

having invariants $\alpha = v$ and $\beta = y^{\frac{1}{2}}v'$. That is,

$$\beta\frac{\mathrm{d}\beta}{\mathrm{d}\alpha} = \frac{\lambda}{4}(\alpha^2 + \alpha), \tag{2.17}$$

leading to $\beta = \frac{\lambda^{\frac{1}{2}}}{2}\alpha\left(1 + \frac{2}{3}\alpha\right)^{\frac{1}{2}}$. Substituting α and β we get

$$y^{\frac{1}{2}}v' = \frac{\lambda^{\frac{1}{2}}}{2}v\left(1 + \frac{2}{3}v\right)^{\frac{1}{2}}, \tag{2.18}$$

i.e.,

$$y^{\frac{1}{2}}\frac{\mathrm{d}v}{\mathrm{d}y} = \frac{\lambda^{\frac{1}{2}}}{2}v\left(1 + \frac{2}{3}v\right)^{\frac{1}{2}}. \tag{2.19}$$

We get two solutions

$$\ln\left[\frac{1 - \left(1 + \frac{2}{3}v\right)^{\frac{1}{2}}}{1 + \left(1 + \frac{2}{3}v\right)^{\frac{1}{2}}}\right] = \lambda^{\frac{1}{2}}y^{\frac{1}{2}} + c \tag{2.20}$$

and

$$\ln\frac{v}{1 + \left(1 + \frac{2}{3}v\right)^2} = \lambda^{\frac{1}{2}}y^{\frac{1}{2}} + c. \tag{2.21}$$

Replacing v and y in the latter leads to

$$\ln\frac{(u-1)(1 - 3\epsilon\mathrm{e}^{\lambda t})}{\left(1 + \left(1 + \frac{2}{3}(u-1)(1 - 3\epsilon\mathrm{e}^{\lambda t})\right)^{\frac{1}{2}}\right)^2} = \frac{\lambda^{\frac{1}{2}}x}{(1 - 3\epsilon\mathrm{e}^{\lambda t})} + c. \tag{2.22}$$

The perturbed wave solution is displayed below for (i) $0 \leqslant t \leqslant 5$, $-20 \leqslant x \leqslant 20$, $\lambda = 4$, $\epsilon = 0.1$ and (ii) $0 \leqslant t \leqslant 2$, $-100 \leqslant x \leqslant 20$, $\lambda = 4$, $\epsilon = 0.1$, respectively. (See Fig. 1 and Fig. 2.)

3. Perturbations of the KdV equation

We now consider perturbations of some cases of the KdV type equations:

$$u_t + \alpha(1 + \beta u)uu_x + \gamma u_{xxx} = 0, \quad \alpha, \gamma > 0, \tag{3.1}$$

which includes modified KdV equation and the Gardner equation:

$$u_t - 6uu_x + u_{xxx} = 12\delta u^2 u_x. \tag{3.2}$$

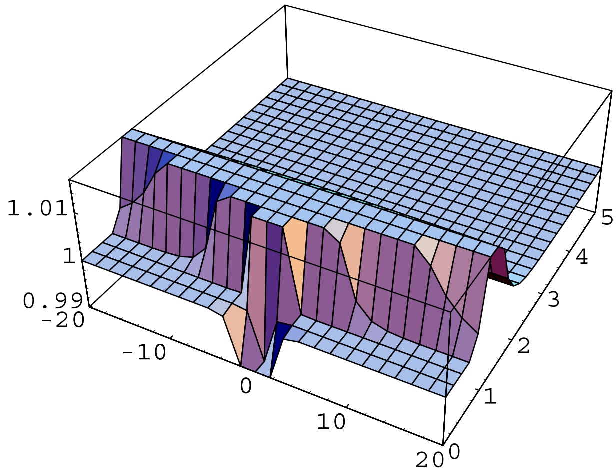

Fig. 1. Case (i): $0 \leqslant t \leqslant 5$, $-20 \leqslant x \leqslant 20$, $\lambda = 4$ and $\epsilon = 0.1$.

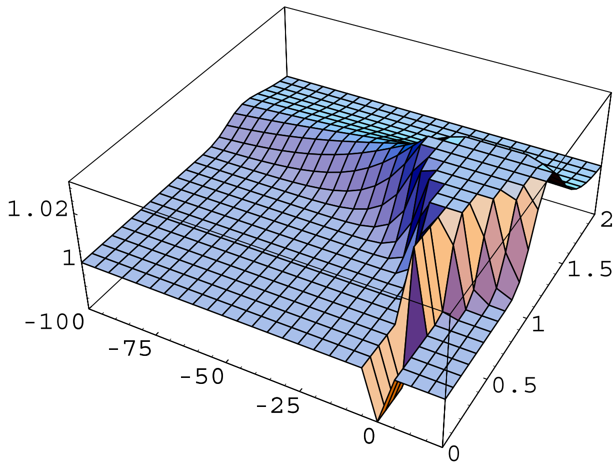

Fig. 2. Case (ii): $0 \leqslant t \leqslant 2$, $-100 \leqslant x \leqslant 20$, $\lambda = 4$ and $\epsilon = 0.1$.

Example 1. Firstly, as the case:

$$u_t - 6uu_x + u_{xxx} = 0 \tag{3.3}$$

is of particular physical interest (for e.g., it is widely used in the modelling of shallow water phenomena), we look at the analysis of a perturbed form:

$$u_t - 6uu_x + u_{xxx} + \epsilon u = 0. \tag{3.4}$$

As far as equation (3.3) is concerned, it is interesting to note that the dilation invariance under symmetry $x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}$ leads to the ode:

$$27y^3v''' + 18y^2v'' + 3yvv' + y(24 - y)v' - 2v^2 - 24v - 18y^2 = 0, \tag{3.5}$$

where $y = x^3/t$ and $v = x^2u$ and invariance under $X_0 = t\frac{\partial}{\partial x} - 1/6\frac{\partial}{\partial u}$ leads to the solution $u = 1/(6t)(k - x)$, k constant (see Fig. 3 for $0.1 \leqslant t \leqslant 100$, $-100 \leqslant x \leqslant 100$, $k = 1$ and $\epsilon = 0.1$).

A nontrivial approximate symmetry of (3.4) is $X = t - (\epsilon/2)t^2\frac{\partial}{\partial x} + (-1/6 + (\epsilon/6)t)\frac{\partial}{\partial u}$ so that with $y = t$ the new dependent variable $v(y)$ is obtainable from:

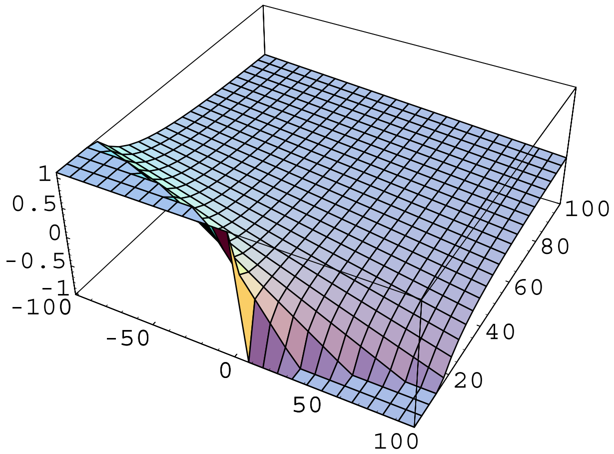

Fig. 3. Unperturbed solution; $0.1 \leqslant t \leqslant 100$, $-100 \leqslant x \leqslant 100$, $k = 1$ and $\epsilon = 0.1$.

$$v' + v\left(\epsilon + 2\frac{1 - \epsilon y}{2y - \epsilon y^2}\right) = 0, \tag{3.6}$$

so that $u(x,t) = x\frac{1-\epsilon y}{2y-\epsilon y^2} + k\frac{e^{-\epsilon t}}{t(\epsilon t-2)}$ (see graph in the following Fig. 4 perturbed solution, $0.1 \leqslant t \leqslant 100$, $-100 \leqslant x \leqslant 100$, $k = 1$ and $\epsilon = 0.5$).

Example 2. The calculations regarding the perturbation of the combined KdV-modified KdV equation, viz.,

$$u_t + \alpha(1 + \beta u)uu_x + \gamma u_{xxx} = \epsilon f(t, x, u, u_{(1)}, \ldots, u_{(r)}), \tag{3.7}$$

does not reveal anything significant. Also, some analysis of the Gardner equation (3.2) has been dealt with elsewhere, for e.g., [5]. Here, we consider the possibility of regarding the uu_x term as being a perturbation in the modified KdV equation. Then the reverse is studied, i.e., $u * 2u_x$ is regarded as a perturbation of the KdV equation, viz., supposing the term 12δ in the Gardner equation is 'small'.

(a) The first of these cases is:

$$u_t + 6u^2u_x + u_{xxx} + \epsilon uu_x = 0, \tag{3.8}$$

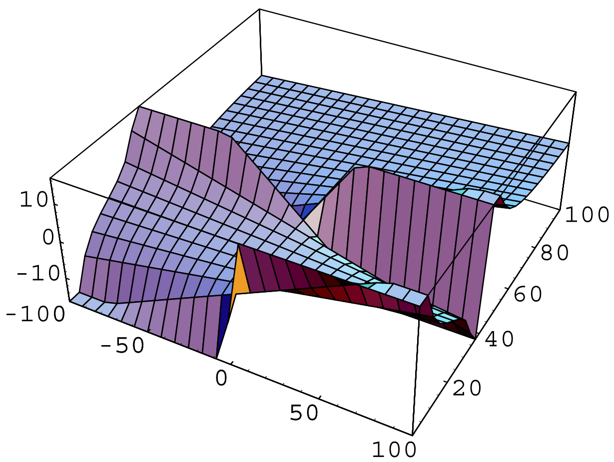

Fig. 4. Case (i): Perturbed solution; $0.1 \leqslant t \leqslant 100$, $-100 \leqslant x \leqslant 100$, $k = 1$ and $\epsilon = 0.1$.

where the unperturbed equation is a modified KdV equation $u_t + 6u^2u_x + u_{xxx} = 0$ with a Lie symmetry generator (scaling) $X_0 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}$. The auxilliary function $H = uu_x$ and the approximate part of the required symmetry, $X_1 = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u}$, is given by:

$$\phi^t + 12\phi uu_x + 6u^2\phi^x + \phi^{xxx} + uu_x = 0. \tag{3.9}$$

The calculations yield an approximate symmetry:

$$X = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \left(u + \frac{\epsilon}{12}\right)\frac{\partial}{\partial u}. \tag{3.10}$$

The invariants are $y = x^3/t$ and $v = (u + \epsilon/12)x$. The reduced equation is an ode in $v = v(y)$:

$$27y^3v''' + 27y^2v'' - y^2v' - 6v - 6v^3 + 6yv' + 18yv^2v' = 0, \tag{3.11}$$

whose analytical treatment is nontrivial.

(b) Now we consider the perturbed KdV equation supposing that the perturbation is $12\delta u^2u_x = \epsilon u^2u_x$, i.e.,

$$u_t - 6uu_x + u_{xxx} + \epsilon u^2u_x = 0. \tag{3.12}$$

A point symmetry generator admitted by the unperturbed equation $u_t - 6uu_x + u_{xxx} = 0$ is $X_0 = t\frac{\partial}{\partial x} - \frac{1}{6}\frac{\partial}{\partial u}$. Here, $H = -\frac{1}{3}uu_x$. Then $X_1 = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u}$ is obtained from:

$$\phi^t - 6\phi u_x - 6u\phi^x + \phi^{xxx} - \frac{1}{3}uu_x = 0, \tag{3.13}$$

giving an approximate generator $X = \left(t - \frac{1}{36}\epsilon x\right)\frac{\partial}{\partial x} - \frac{1}{12}\epsilon t\frac{\partial}{\partial t} - \frac{1}{6}\frac{\partial}{\partial u}$. The invariants of X are $y = t^{-\frac{1}{3}}x + \frac{18}{\epsilon}t^{\frac{2}{3}}$ and $v = u - \frac{2}{\epsilon}\ln t$ leading to

$$-\frac{1}{3}v'\left(xt^{-\frac{1}{3}} + \frac{18}{\epsilon}t^{\frac{2}{3}} - \frac{54}{\epsilon}t^{\frac{2}{3}}\right) + \frac{2}{\epsilon} - 6vv't^{\frac{2}{3}} - \frac{12}{\epsilon}v't^{\frac{2}{3}}\ln t + v''' + \epsilon v'v^2t^{\frac{2}{3}} + \frac{4}{\epsilon}vv't^{\frac{2}{3}}\ln t + \frac{4}{\epsilon^2}v't^{\frac{2}{3}}(\ln t)^2 = 0, \tag{3.14}$$

from which simplifications lead to

$$v''' + \frac{2}{\epsilon} - \frac{1}{3}yv' + t^{\frac{2}{3}}v'\left[\frac{16}{\epsilon} - 6\left(v + \frac{2}{\epsilon}\ln t\right) + \epsilon\left(v + \frac{2}{\epsilon}\ln t\right)^2\right] = 0. \tag{3.15}$$

Here, however, the cancellation of the 'original variables' cannot be achieved. This may be explained by the nature of the perturbation term δu^2u_x not being 'small enough' in comparison to other terms in the equation. Thus, a further analysis is not pursued.

4. Conclusion

We have considered perturbations of well known evolution type equations that arise in mathematical physics. The analysis is carried out using the now well known approximate symmetry method. Interesting results are obtained and in some cases we have presented three-dimensional plots of the invariant solutions. It may be worth pursuing a study of these perturbed forms using conservation laws and associated symmetries in the manner described in [6] and [7].

References

[1] H.I. Abdel-Gawad, A method for finding the invariants and exact solutions of coupled non-linear differential equations with applications to dynamical systems, Int. J. Non-linear Mech. 38 (2003) 429–440.

[2] A.H. Kara, C.M. Khalique, Nonlinear evolution type equations and their exact solutions using inverse variational methods, J. Phys. A. 38 (2005) 4629–4636.

[3] V.A. Baikov, R.K. Gazizov, N.H. Ibragimov, in: N.H. Ibragimov (Ed.), CRC Handbook of Lie Group Analysis of Differential Equations, vol. 3, CRC Press, Boca Raton, Florida, 1996.

[4] B.Y. Guo, X. Chen, Analytic solutions of the Fisher equation, J. Phys. A 24 (3) (1991) 645–650.

[5] A.H. Bokhari, A.H. Kara, F.D. Zaman, Soliton and other exact solutions of the combined KdV-modified KdV and KdV-generalized KdV equations, Il Nuovo Cimento B 120 (2005) 393–396.

[6] A.H. Kara, F.M. Mahomed, G. Unal, Approximate symmetries and conservation laws with applications, Int. J. Theoret. Phys. 38 (9) (1999) 2389–2399.

[7] A.H. Davison, A.H. Kara, Potential symmetry generators and associated conservation laws of perturbed nonlinear equations, Appl. Math. Comput. 156 (1) (2004) 271–285.