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Investigation of the direct and inverse solutions for torsional waves propagating in a cylinder

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Abstract

We investigate the direct and inverse problem associated with the torsional waves propagating in a cylinder. We analyse the usual wave equation as well as the damped wave equation and consider the problem of recovering the initial profile from the observations of the final profile. This inverse problem arises when experimental measurements are taken at any given time, and it is desired to calculate the initial profile. An integral representation for the problem is found, from which a formula for initial disturbance is derived using Picard's criterion and the singular system of the associated operators.

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1. Introduction

The classical direct problem in wave propagation is to determine the wave distribution in a medium as the time progresses. The task of determining the initial distribution from the final distribution is distinct from the direct problem and is identified as the initial inverse problem [2,4,5,7]. In many physical applications, one encounters the situation where the usual wave equation does not serve as a realistic model. For instance, the wave equation does not model correctly if the medium of propagation offers resistance, so a damping term which is proportional to velocity is introduced in the wave equation. We introduce a damping term in the wave equation [6] and study its effect on the inversion. The damping may be caused due to impurities in the medium, distributed boundary frictions or small viscous effects.

In the second section, the problem of torsional waves propagating in a cylinder is formulated. The direct and the inverse problems without damping are discussed in the third section. In the fourth section a damping term is introduced in the governing equation and its effects on the direct and the inverse problem is studied. Also an example is presented to check the validity of the inverse solutions. Finally, in the last section the results are summarized.

2. Formulation of the problem

The displacement components in the cylindrical polar coordinates are u_r, u_θ, u_z and the components of stress are σ_{rr}, $\sigma_{r\theta}$, $\sigma_{\theta\theta}$, etc. The torsional waves propagating in a cylinder involve only u_θ while other components are zero. Moreover, U_θ is independent of θ. The governing equation for torsional waves propagating in a cylinder is given by [1]

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} = \frac{\partial^2 u_\theta}{\partial z^2} + \frac{1}{c^2}\frac{\partial^2 u_\theta}{\partial t^2}, \quad r, z \in R, \tag{1}$$

where $c = \sqrt{\mu/\rho}$ is the shear velocity and is the region

$$R = \{(r,z)/r \in [0,a], \quad z \in [0,b]\}. \tag{2}$$

The field $u_\theta(r, z, t)$ satisfies the following boundary conditions

$$u_\theta(r,0,t) = 0, \quad u_\theta(r,b,t) = h, \tag{3}$$

$$\frac{\partial u_\theta(r,z,0)}{\partial t} = 0. \tag{4}$$

The surface of the cylinder is kept stress free which gives

$$\sigma_{r\theta} = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = 0, \quad \text{at } r = a. \tag{5}$$

Also we assume the displacement at $r = 0$ is finite.

We consider the inverse problem of finding initial disturbance $g(r, z)$ from the information of final profile $f(r, z)$ for some future time $t = T$, so that

$$g(r,z) = u_\theta(r,z,0), \tag{6}$$

$$f(r,z) = u_\theta(r,z,T). \tag{7}$$

3. The direct and inverse problem without damping

We assume that the solution of the direct problem (1) is of the form

$$u_\theta(r,z,t) = \sum_{n=1}^{\infty} u_{n,m}(t)\phi_{n,m}(r,z). \tag{8}$$

The solution of the corresponding eigenvalue problem is given by

$$\phi_{n,m}(r,z) = \frac{2}{a\sqrt{b}J_0'(a\lambda_n)}\sin(\mu_m z)J_0(\lambda_n r), \tag{9}$$

where $'$ denotes derivative with respect to r, $\mu_m = \frac{m\pi}{b}$, and λ_n are solutions of

$$\lambda_n J_0'(\lambda_n a) + \frac{J_0(\lambda_n a)}{a} = 0. \tag{10}$$

The eigen functions $\phi_{n,m}(r, z)$ form a complete orthonormal system in $H_r[R]$. Thus $g(r, z) \in H_r[R]$ can be expanded as

$$g(r,z) = \sum_{n,m=1}^{\infty} c_{n,m}\phi_{n,m}(r,z), \quad r,z \in R, \tag{11}$$

where

$$c_{n,m} = \int_0^a \int_0^b \tau g(\tau,\eta)\phi_{n,m}(\zeta,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau. \tag{12}$$

Substituting (8) in (1), (6) and (4), and using orthonormality property of the eigen functions leads to the following ordinary differential equation

$$\frac{1}{c^2}\frac{\mathrm{d}^2 v_{n,m}(t)}{\mathrm{d}t^2} + [\mu_m^2 + \lambda_n^2]v_{n,m}(t) = 0, \quad t > 0, \tag{13}$$

$$v_{n,m}(0) = c_{n,m}, \tag{14}$$

$$\frac{\mathrm{d}v_{n,m}(0)}{\mathrm{d}t} = 0. \tag{15}$$

This initial value problem can easily be solved and so (8) takes the form

$$u_\theta(r,z,t) = \sum_{n=1}^{\infty} c_{n,m} \cos\left(\sqrt{\mu_m^2 + \lambda_n^2}ct\right)\phi_{n,m}(r,z). \tag{16}$$

Now we use condition (7) to write the final profile in the form

$$\begin{aligned} f(r,z) &= \sum\nolimits_{n,m=1}^{\infty} c_{n,m} \cos\left(\sqrt{\mu_m^2 + \lambda_n^2}cT\right)\phi_{n,m}(r,z) \\ &= \int_0^a \int_0^b \tau g(\tau,\eta) K(r,z,\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau, \end{aligned} \tag{17}$$

where

$$K(r,z,\tau,\eta) = \sum_{n,m=1}^{\infty} \cos\left(\sqrt{\mu_m^2 + \lambda_n^2}cT\right)\phi_{n,m}(\tau,\eta)\phi_{n,m}(r,z). \tag{18}$$

Thus the inverse problem is reduced to solving the integral equation of the first kind. The singular system for the integral operator in (17) is given by

$$\left[\cos\left(\sqrt{\mu_m^2 + \lambda_n^2}cT\right); \phi_{n,m}(r,z), \phi_{n,m}(r,z)\right]. \tag{19}$$

It now follows from Picard's theorem [2,3] that our inverse problem is solvable if and only if

$$\sum_{n,m=1}^{\infty} \frac{|f_{n,m}|^2}{\left[\cos\left(\sqrt{\mu_m^2 + \lambda_n^2}cT\right)\right]^2} < \infty, \tag{20}$$

where

$$f_{n,m} = \int_0^a \int_0^b f(\tau,\eta)\phi_{n,m}(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau, \tag{21}$$

are classical Fourier coefficients of f. In this case solution [2] is given by

$$g(r,z) = \sum_{n,m=1}^{\infty} \frac{f_{n,m}\phi_{n,m}(r,z)}{\cos\left(\sqrt{\mu_m^2 + \lambda_n^2}cT\right)}. \tag{22}$$

Eq. (22) represents the inverse solution in case there is no damping. The initial profile $g(r, z)$ can be calculated from the information of the final data $f(r, z)$, which is used to calculate the Fourier coefficients appearing in the expression (22).

4. The direct and inverse problem in the presence of the damping

The governing equation for torsional waves with a damping term $\delta\frac{\partial u_\theta}{\partial t}$, propagating in a cylinder is given by

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{\partial^2 u_\theta}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u_\theta}{\partial t^2} + \delta\frac{\partial u_\theta}{\partial t}, \quad r, z \in R, \ \delta, t \geqslant 0, \tag{23}$$

together with initial and boundary conditions (3)–(7). Eq. (23) has a dissipation or damping term which is proportional to velocity with constant of proportionality being δ. Following the same procedure as in the previous section, instead of Eq. (13) in this case we have the following ordinary differential equation

$$\frac{1}{c^2}\frac{\mathrm{d}^2 v_{n,m}(t)}{\mathrm{d}t^2} + \delta\frac{\mathrm{d}v_{n,m}(t)}{\mathrm{d}t} + [\mu_m^2 + \lambda_n^2]v_{n,m}(t) = 0, \quad t \geqslant 0, \tag{24}$$

together with conditions (14) and (15). Eqs. (24), (14) and (15) can be solved easily to yield the solution

$$v_{n,m}(t) = \exp\left(-\frac{c^2\delta}{2}t\right)\left\{c_{n,m}\cos(\sigma t) + \frac{\delta c_{n,m}}{2k_n}\sin(\sigma t)\right\}, \ 4c^2(\mu_m^2+\lambda_n^2) > \delta^2 c^4, \tag{25}$$

$$u_{n,m}(t) = \exp\left(-\frac{c^2\delta}{2}t\right)\left\{c_{n,m}\cosh(\sigma t) + \frac{\delta c_{n,m}}{2k_n}\sinh(\sigma t)\right\}, \ 4c^2(\mu_m^2+\lambda_n^2) < \delta^2 c^4, \tag{26}$$

$$v_{n,m}(t) = c_{n,m}\exp\left(-\frac{c^2\delta}{2}t\right), \ 4c^2(\mu_m^2+\lambda_n^2) = \delta^2 c^4, \tag{27}$$

where

$$\sigma = \frac{\sqrt{|4c^2(\mu_m^2+\lambda_n^2) - \delta^2 c^4|}}{2}. \tag{28}$$

Therefore Eq. (8) can be written as

$$u_\theta(r,z,t) = \sum_{n,m=1}^{\infty}\exp\left(-\frac{c^2\delta}{2}t\right)\left\{c_{n,m}\cos(\sigma t) + \frac{\delta c_{n,m}}{2k_n}\sin(\sigma t)\right\}\phi_{n,m} \times (r,z), \quad 4c^2(\mu_m^2+\lambda_n^2) > \delta^2 c^4, \tag{29}$$

$$u_\theta(r,z,t) = \sum_{n,m=1}^{\infty}\exp\left(-\frac{c^2\delta}{2}t\right)\left\{c_{n,m}\cosh(\sigma t) + \frac{\delta c_{n,m}}{2k_n}\sinh(\sigma t)\right\} \times \phi_{n,m}(r,z), \quad 4c^2(\mu_m^2+\lambda_n^2) < \delta^2 c^4, \tag{30}$$

$$u_\theta(r,z,t) = \sum_{n,m=1}^{\infty} c_{n,m}\exp\left(-\frac{c^2\delta}{2}t\right) \times \phi_{n,m}(r,z), \quad 4c^2(\mu_m^2+\lambda_n^2) = \delta^2 c^4. \tag{31}$$

These solutions show that higher modes are damped in an oscillatory manner and lower modes are damped monotonically. The oscillatory damping is a serious structural drawback because fatigue stresses are caused due to such oscillations. Now we use condition (7) to write the final profile in the form

$$f(r,z) = \int_0^a \int_0^b \tau g(\tau,\eta) K(r,z,\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau \tag{32}$$

with

$$K(r,z,\tau,\eta) = \left[\exp\left(-\frac{c^2\delta}{2}T\right)\sum_{n,m=1}^{\infty}\phi_{n,m}(\tau,\eta)\phi_{n,m}(r,z) \times\left\{\cos(\sigma T) + \frac{\delta}{2k_n}\sin(\sigma T)\right\}\right], \quad 4c^2(\mu_m^2+\lambda_n^2) > \delta^2 c^4, \tag{33}$$

$$K(r,z,\tau,\eta) = \left[\exp\left(-\frac{c^2\delta}{2}T\right)\sum_{n,m=1}^{\infty}\phi_{n,m}(\tau,\eta)\phi_{n,m}(r,z) \times\left\{\cosh(\sigma T) + \frac{\delta}{2k_n}\sinh(\sigma T)\right\}\right], \quad 4c^2(\mu_m^2+\lambda_n^2) < \delta^2 c^4, \tag{34}$$

$$K(r,z,\tau,\eta) = \exp\left(-\frac{c^2\delta}{2}T\right)\sum_{n,m=1}^{\infty}\phi_{n,m}(\tau,\eta)\phi_{n,m}(r,z), \quad 4c^2(\mu_m^2+\lambda_n^2) = \delta^2 c^4. \tag{35}$$

Thus the inverse problem is reduced to solving the Fredholm integral equation of the first kind. The singular systems for the integral operators in Eq. (32) are given by

$$\left\{\exp\left(-\frac{c^2\delta}{2}T\right)\left(\cos(\sigma T) + \frac{\delta}{2k_n}\sin(\sigma T)\right); \phi_{n,m}(r,z), \phi_{n,m}(r,z)\right\}, \quad 4c^2(\mu_m^2+\lambda_n^2) > \delta^2 c^4, \tag{36}$$

$$\left\{\exp\left(-\frac{c^2\delta}{2}T\right)\left(\cosh(\sigma T) + \frac{\delta}{2k_n}\sinh(\sigma T)\right); \phi_{n,m}(r,z), \phi_{n,m}(r,z)\right\}, \quad 4c^2(\mu_m^2+\lambda_n^2) < \delta^2 c^4, \tag{37}$$

$$\left\{\exp\left(-\frac{c^2\delta}{2}T\right); \phi_{n,m}(r,z), \phi_{n,m}(r,z)\right\}, \quad 4c^2(\mu_m^2+\lambda_n^2) = \delta^2 c^4. \tag{38}$$

Now by Picard's theorem, using singular systems (36)–(38), the inverse problem is solvable iff

$$\sum_{n,m=1}^{\infty} \frac{\exp(c^2\delta T)|f_{n,m}|^2}{\left(\cos(\sigma T) + \frac{\delta}{2k_n}\sin(\sigma T)\right)^2} < \infty, \quad 4c^2(\mu_m^2 + \lambda_n^2) > \delta^2 c^4, \tag{39}$$

$$\sum_{n,m=1}^{\infty} \frac{\exp(c^2\delta T)|f_{n,m}|^2}{\left(\cosh(\sigma T) + \frac{\delta}{2k_n}\sinh(\sigma T)\right)^2} < \infty, \quad 4c^2(\mu_m^2 + \lambda_n^2) < \delta^2 c^4, \tag{40}$$

$$\sum_{n,m=1}^{\infty} \exp(c^2\delta T)|f_{n,m}|^2 < \infty, \quad 4c^2(\mu_m^2 + \lambda_n^2) = \delta^2 c^4, \tag{41}$$

where $f_{n,m}$ are classical Fourier coefficients of $f(x)$ given by the expression (21). In this case, by Picard's theorem, the solutions are given by

$$g(r,z) = \sum_{n,m=1}^{\infty} \frac{\exp\left(\frac{c^2\delta T}{2}\right) f_{n,m}\phi_{n,m}(r,z)}{\left(\cos(\sigma T) + \frac{\delta}{2k_n}\sin(\sigma T)\right)}, \quad 4c^2(\mu_m^2 + \lambda_n^2) > \delta^2 c^4, \tag{42}$$

$$g(r,z) = \sum_{n,m=1}^{\infty} \frac{\exp\left(\frac{c^2\delta T}{2}\right) f_{n,m}\phi_{n,m}(r,z)}{\left(\cosh(\sigma T) + \frac{\delta}{2k_n}\sinh(\sigma T)\right)}, \quad 4c^2(\mu_m^2 + \lambda_n^2) < \delta^2 c^4, \tag{43}$$

$$g(r,z) = \sum_{n,m=1}^{\infty} \exp\left(\frac{c^2\delta T}{2}\right) f_{n,m}\phi_{n,m}(r,z), \quad 4c^2(\mu_m^2 + \lambda_n^2) = \delta^2 c^4. \tag{44}$$

From Eqs. (39)–(41) it is clear that $f_{n,m}$ should decay faster in case of oscillatory damping as compared to monotonic damping. In case of oscillatory damping, $f_{n,m}$ should be smooth enough to ensure that Eq. (39) is satisfied. This can be more easily achieved by restricting to lower modes only. The solution to the undamped wave equation can be obtained from the damped wave solution by taking $\delta = 0$ in the damped wave solution.

Example. Let us consider the initial distribution of the form

$$g(r,z) = \phi_{n,m}(r,z). \tag{45}$$

First we solve the direct problem (23), to find the final profile $f(r, z)$. The solution of the direct problem is

$$\begin{aligned} f(r,z) &= \exp\left(-\frac{c^2\delta}{2}T\right)\left\{c_{n,m}\cos(\sigma T) + \frac{\delta c_{n,m}}{2k_n}\sin(\sigma T)\right\}\phi_{n,m}(r,z), \\ &\qquad 4c^2(\mu_m^2 + \lambda_n^2) > \delta^2 c^4, \\ &= \exp\left(-\frac{c^2\delta}{2}T\right)\left\{c_{n,m}\cosh(\sigma T) + \frac{\delta c_{n,m}}{2k_n}\sinh(\sigma T)\right\}\phi_{n,m}(r,z), \\ &\qquad 4c^2(\mu_m^2 + \lambda_n^2) < \delta^2 c^4, \\ &= \exp\left(-\frac{c^2\delta}{2}T\right)\phi_{n,m}(r,z), \quad 4c^2(\mu_m^2 + \lambda_n^2) = \delta^2 c^4. \end{aligned} \tag{46}$$

Our aim is to use the final profile given by (46) to recover back the initial disturbance given by (45). From Eq. (21) the Fourier coefficients are given by

$$f_{n,m} = \exp\left(-\frac{c^2\delta}{2}T\right)\left\{c_{n,m}\cos(\sigma T) + \frac{\delta c_{n,m}}{2k_n}\sin(\sigma T)\right\},\ 4c^2(\mu_m^2 + \lambda_n^2) > \delta^2 c^4, \tag{47}$$

$$f_{n,m} = \exp\left(-\frac{c^2\delta}{2}T\right)\left\{c_{n,m}\cosh(\sigma T) + \frac{\delta c_{n,m}}{2k_n}\sinh(\sigma T)\right\},\ 4c^2(\mu_m^2 + \lambda_n^2) < \delta^2 c^4, \tag{48}$$

$$f_{n,m} = \exp\left(-\frac{c^2\delta}{2}T\right),\ 4c^2(\mu_m^2 + \lambda_n^2) = \delta^2 c^4. \tag{49}$$

We use the final data given by Eqs. (47)–(49) in Eqs. (42)–(44) respectively to recover the initial profile given by Eq. (45).

5. Conclusions

It has been shown here by classical means that damping of the medium play a role in the initial profile reconstruction. It is clear from the processing formulas (42)–(45) and (22) that neglecting the damping of the medium may give misleading results. Since every medium has damping so it is more realistic to use the damped wave equation instead of the undamped wave equation for the reconstruction of the initial profile from the information of the final profile.

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References

[1] J.D. Achenbach, Wave Propagation in Elastic Solids, North-Holland Publishing Company, 1975.
[2] Heinz W. Engl, Martin Hanke, Andreas Neubauer, Regularization of Inverse Problems, Kluwer, Dordrecht, 1996.
[3] Charles W. Groetsch, Elements of Applicable Functional Analysis, Dekker, New York, 1980.
[4] K. Masood, S. Messaoudi, F.D. Zaman, Initial inverse problem in heat equation with Bessel operator, Int. J. Heat Mass Transfer 45 (14) (2002) 2959–2965.
[5] G. Nakamura, S. Saitoh, A. Syarif, Representation of initial heat distribution by means of their heat distribution as a function of time, Inverse Problems 15 (1999) 1255–1261.
[6] Ivar Stakgold, Boundary Value Problems of Mathematical Physics, Vol. II, MacMillan, New York, 1968.
[7] C.F. Weber, Analysis and solution of ill-posed inverse heat conduction problem, Int. J. Heat Mass Transfer 24 (1981) 1783–1792.