Finite Element Computations of Pure-Streamfunction Equation of The Ladyzhenskaya Equations for Incompressible Fluid

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Abstract: - In this paper we consider a pure-streamfunction equation of Ladyzhenskaya equations. For certain values of the parameters of the equation, the studied equation becomes identical to the pure-streamfunction equation of the Navier-Stokes equations. A weak form, a finite element method approximation procedures and an iterative method for solving the discrete nonlinear problems are provided. Using the Bogner-Fox-Schmidt element, the steady 2-D incompressible flow in a driven cavity is solved using a grid mesh of 16X16. Streamfunction contours are also displayed showing the main features of the flow.

Keywords: - Navier-Stokes equations, Ladyzhenskaya model, subgridscale model, finite element, streamfunction formulation

1. Introduction

Understanding turbulent flow is central to many important problems including environmental and energy related applications (global change, mixing of fuel and oxidizer in engines and drag reduction), aerodynamics (maneuvering flight of jet aircraft) and biophysical applications (blood flow in the heart). However, in many situations it is still not clear which models are most appropriate, especially in the case of turbulent flows.

The Navier-Stokes equations are generally accepted as providing an accurate model for the incompressible motion of viscous fluids in practical situations. This research will consider one model introduced by Ladyzhenskaya [13,14,15]. The study of this model may be justified through a variety of physical and mathematical arguments. The following paragraphs, summarized from [5], address the reasons for choosing the Ladyzhenskaya model and the attractions of the streamfunction formulation.

The first reason for the study of the Ladyzhenskaya model is from a modeling stand point. The Stokes hypothesis which defines an ordinary fluid (water or air, for example) leads to a specific mathematical form of the nonlinear relation between the stress and the velocity fields, see [19] for details. If one requires that the relation between the stress and the velocity be linear, then one arrives at the Navier-Stokes equations. However, if one retains the Stokes hypotheses defining a fluid and then retains some of the nonlinear terms in the general constitutive relation which a Stoksian fluid must satisfy, then one arrives at the Ladyzhenskaya model considered here, see [14,15]. Thus, from a modeling stand point, the Navier-Stokes equations are a special case of the Ladyzhenskaya equations. This leads to the obvious conclusion that any flow which can be accurately described by solutions of the Navier-Stokes equations can be at least as accurately described by solutions of the Ladyzhenskaya equations.

The second reason for the study of the Ladyzhenskaya model comes from the field of turbulence modeling. For certain values of the parameter q, the Ladyzhenskaya equations considered here are identical to Smagorinsky model [21]. Thus, from a practical engineering point of view, the study of Ladyzhenskaya equations and of proprieties of their solution is of substantial interest.

The third reason for the study of the Ladyzhenskaya model comes from a mathematical stand point. Ladyzhenskaya has shown [13,14,15] that solutions of the equations in the non-stationary case and in three space dominions, are globally unique in time. The analogous result for the Navier-Stokes equations has not been proved and is believed not to be true. The condition derived in [5] which guarantees the uniqueness of the solutions of the stationary Ladyzhenskaya model is, in some sense, less pessimistic than the analogous condition for the Navier-Stokes model. Also the analogous condition for the stationary Ladyzhenskaya model [5,9] generally guarantees uniqueness for the higher values of the Reynolds number than that predicted for the Navier-Stokes model. So indeed from a mathematical point of view, this is a good motivation for this study.

This research also takes the advantage of using the streamfunction formulation of a Ladyzhenskaya model. The first attraction of the streamfunction formulation is that the incompressibility constrain is automatically satisfied. The second attraction is that the pressure is not present in the weak form, and there is only one scalar unknown to solve for.

2. Model Equation

The model we work with is as follows, consider the motion of stationary ideal incompressible viscous fluids in a bounded domain Ω in R^2 with Lipshitz boundary $\partial\Omega$, and let u denote the velocity field, p the pressure, ϕ the streamfunction, and f the body force per unit mass. Then the model is given by [SL]:

$$\partial_{xx}(A(\psi)\ \psi_{xx}) + 2\partial_{xy}(A(\psi)\ \psi_{xy}) + \partial_{yy}(A(\psi)\ \psi_{yy}) - \psi_y\Delta\psi_x + \psi_x\Delta\psi_y = f_{2,x} - f_{1,y} \quad \text{in} \quad \Omega, \tag{1}$$

$$\psi = \frac{\partial\psi}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \tag{2}$$

Where $A(\psi)$ is defined by

$$A(\psi) = \varepsilon_0 + \varepsilon_1 \|\vec{\Delta}\psi\|^{q-2}, \tag{3}$$

and

$$\vec{\Delta}\psi = \text{grad (grad } \psi) = [\psi_{xx}, \psi_{xy}, \psi_{yx}, \psi_{yy}]^T,$$

and

$$\|\vec{\Delta}\psi\| = [\psi_{xx}^2 + 2\psi_{xy}^2 + \psi_{yy}^2]^{1/2}, \tag{4}$$

with $\varepsilon_0 (= 1/\text{Re})$, ε_1 where Re is the Reynolds number and $q-2>0$. We also consider the Ladyzhenskaya equation [L]

$$-\sum\partial_k(\hat{A}(u)\partial_k u_j) + \sum u_k\partial_k u_j + \partial_j p = f_j \text{ in } \Omega, \tag{5}$$

$$div \quad u = 0 \text{ in } \Omega, \tag{6}$$

$$u = 0 \text{ on } \partial\Omega, \tag{7}$$

where in (5), $j=1,2$ and $\hat{A}(u)$ is defined by

$$\hat{A}(u) = \varepsilon_0 + \varepsilon_1 |\nabla u|^{q-2} \quad with \quad q-2>0, \tag{8}$$

and

$$|\nabla u| = [\sum_{i,j=1}^{2}(\partial_j u_j)^2]^{\frac{1}{2}}. \tag{9}$$

The Ladyzhenskaya equations, (5-7), have been proposed in [13,14,15]. Finite element error analysis of this model was carried out in Du and Gunzburger [4,5] under a global uniqueness (small data) condition. Layton gave in [16] an error analysis for high Re number also he provides a formula for choosing q and ε_1 so that one can construct a higher-order method which is just as stable as a first-order upwind methods. Iterative method for solving the discrete nonlinear problems (5-7) is given in [4]. If we set $\varepsilon_1 = 0$, equation [SL] reduces to the streamfunction form of the Navier-Stokes equations [SNS]

$$\varepsilon_0\Delta^2\psi - \psi_y\Delta\psi_x + \psi_x\Delta\psi_y = \text{curl}\vec{f} \text{ in } \Omega, \tag{10}$$

$$\psi = 0 \text{ on } \partial\Omega, \tag{11}$$

$$\frac{\partial\psi}{\partial\hat{n}} = 0 \text{ on } \partial\Omega. \tag{12}$$

The standard weak formulation of [SNS] first appeared in 1979 in [10]. Finite element analysis for [SNS] can be found in [10,11,12]. Cayco and Nicolaides [3] studied a general analysis of convergence for the weak form of (10-12). Fairag in [6,7,8] studied two-level finite element analysis of (10-12) and some computational aspects.

3. Notations, Function Spaces, and Variational Formulation

We first need to define some function spaces and associated norms. More details concerning these spaces can be found in [1]. Let Ω be a bounded, simply connected, polygonal domain in R^2. $L^2(\Omega)$ is the Hilbert space of Lebesgue square integrable functions with norm $\|\bullet\|_0$ and $L_0^2(\Omega)$ is the subspace of $L^2(\Omega)$ consisting of functions with zero mean. Let $H^m(\Omega)$ be the usual Sobolev space consisting of functions which together with their distributional derivatives up through order m are in $L^2(\Omega)$. Denote the norm on $H^m(\Omega)$ by $\|\bullet\|_m$. Let $H_0^m(\Omega)$ be the completion of $C_0^\infty(\Omega)$ under the $\|\bullet\|_m$ norm. We equip $H_0^m(\Omega)$ with the seminorm $|\bullet|_m$, which is a norm equivalent to $\|\bullet\|_m$. Also, the dual of space $H_0^m(\Omega)$ is denoted by $H^{-m}(\Omega)$, with norm $\|\bullet\|_{-m}$. Let $[H^m(\Omega)]^2$ be the space $H^m(\Omega)\times H^m(\Omega)$ and $[H_0^m(\Omega)]^2$ be the space $H_0^m(\Omega)\times H_0^m(\Omega)$ equipped with the following norm

$$\|\vec{u}\|_m = (\|u_1\|_m^2 + \|u_2\|_m^2)^{1/2} \text{ and}$$

$$|\vec{u}|_m = (|u_1|_m^2 + |u_2|_m^2)^{1/2} \text{ where } \vec{u} = (u_1, u_2)^T.$$

For each $\phi \in H^1(\Omega)$, define

$$\text{curl}\phi = (\phi_y, -\phi_x)^T.$$

For each $\vec{u} = (u_1, u_2)^T \in [H^1(\Omega)]^2$, define

$$\text{curl}\vec{u} = (\partial u_2 / \partial x) - (\partial u_1 / \partial y) . \qquad (13)$$

We define the space $L^p(\Omega) := \left\{\varphi: \|\varphi\|_{L^p(\Omega)} < \infty\right\}$ where $\|\varphi\|_{L^p(\Omega)} := (\int_\Omega |\varphi|^p \, d\Omega)^{1/p}$. We also define the Sobolev space

$W^{m,p}(\Omega) := \{\varphi \in L^p(\Omega) : \partial^\alpha \varphi \in L^p(\Omega) \;\; \forall \; |\alpha| \le m\}$

which is a Banach space for the norm $\|\varphi\|_{m,p,\Omega} := (\sum_{|\alpha| \le m} \int_\Omega |\partial^\alpha \varphi(x)|^p \, d\Omega)^{\frac{1}{p}}$, p<∞. We define for $q > 0$, $W_0^{2,q}(\Omega) :=$ completion of $C_0^\infty(\Omega)$ in the $W^{2,q}$-norm.

We now present the weak formulation for problem [SL] which can be obtained through the standard procedure, e.g. multiplying the original equation by test functions and integrating by parts [WSL]:

Find $\psi \in W_0^{2,q}(\Omega)$ such that , for all $\phi \in W_0^{2,q}(\Omega)$,

$$a(\psi, \phi) + \hat{a}(\psi, \psi, \phi) + b(\psi; \psi, \phi) = (\vec{f}, curl\phi), \qquad (14)$$

where

$$a(\psi, \phi) = \varepsilon_0 \int_\Omega \Delta\psi \, \Delta\phi \, dx \; dy, \qquad (15)$$

$$\hat{a}(\psi, \xi, \phi) = \varepsilon_1 \int_\Omega \| \vec{\Delta}\psi \|^{q-2} \, \vec{\Delta}\xi \, \vec{\Delta}\phi \, dx \; dy, \qquad (16)$$

$$b(\psi; \xi, \phi) = \int_\Omega \Delta\psi(\xi_y \; \phi_x - \xi_x \; \phi_y) \, dx \; dy, \qquad (17)$$

$$(\vec{f}, curl\phi) = \int_\Omega \vec{f}.curl\phi \; dx \, dy, \qquad (18)$$

If we set $\varepsilon_1 = 0$, equation (14) reduces to the standard weak form of the streamfunction of the Navier-Stokes equations. Existence and uniqueness and finite element error analysis of the weak form [WSL] were carried out in [9]. Equation (14) has a unique solution under a certain condition depending on its parameters. This uniqueness property has been proved in [9].

Exact solutions obtained through theoretical analysis are very limited. Thus, solving the problem by numerical methods becomes very important. In this paper, we focus our attention toward finite element approximations of the model problem [WSL] described above.

4. Discretization:

For simplicity, we assume that Ω is a polygonal domain. Let Ω^h be a regular finite element triangulation where h is a discretization parameter that tends to zero. We define a finite-dimensional space X^h such that $X^h \subset W_0^{2,q}(\Omega)$. Then, we approximate [WSL] by the following discrete problem:

Find $\psi^h \in X^h$ such that , for all $\phi^h \in X^h$,

$$a(\psi^h, \phi^h) + \hat{a}(\psi^h, \psi^h, \phi^h) + b(\psi^h; \psi^h, \phi^h) = (\vec{f}, curl\,\phi^h). \qquad (19)$$

Existence and uniqueness of the solution to (19) can be found in [9]. The inclusion $X^h \subset W_0^{2,q}(\Omega)$ requires the use of finite-element functions that are continuously differentiable over Ω. Argyis Triangle, Clough-Tocher Triangle, Bogner-Fox-Schmidt Rectangle, Bicubic Spline Rectangle (see [8]) are examples of finite-element spaces for the streamfunction formulation of the Ladyzhenskaya model. We will impose boundary conditions by setting all the degrees of freedom at the boundary nodes to be zero and the normal derivative equal to zero at all vertices and nodes on the boundary.

In our computations, we use the Bogner-Fox-Schmidt Rectangle. In this element, the functions are bicubic polynomials within each rectangle. The degrees of freedom are chosen to be the function value, the first derivatives, and the mixed second derivative at the vertices. We set the function and the normal derivative values equal to zero at all vertices on the boundary.

In [9], we established the error bound given in the following theorem. This theorem and its proof can also be found in [9].

Theorem: Let $X^h \subset W_0^{2,q}(\Omega)$ be a finite element space. Let ψ be the solution to (14) and ψ^h be solution to (19). Then for h sufficiently small, ψ^h satisfies

$$\left\|\psi - \psi^h\right\|_{H_0^2(\Omega)} \le C\,(\text{Re}) \inf_{\psi^h \in X^h} \left\|\psi - \psi^h\right\|_{H_0^2(\Omega)},$$

where $C\,(\text{Re})$ is a positive constant and depends on the Reynolds number.

As an example, if the Bogner-Fox-Schmidt Rectangles are used, then there exist a positive constant C such that $\left|\psi - \psi^h\right|_2 \le Ch^2$. For each of the elements mentioned above, Table (1) shows the error estimates.

Element	Estimate
Argyis Triangle	$\left\|\psi - \psi^h\right\|_2 \le O(h^4)$
Clough-Tocher Triangle	$\left\|\psi - \psi^h\right\|_2 \le O(h^2)$
Bogner-Fox-Schmidt Rectangle	$\left\|\psi - \psi^h\right\|_2 \le O(h^2)$
Bicubic Spline Rectangle	$\left\|\psi - \psi^h\right\|_2 \le O(h^2)$

Table(1): Accuracy of Finite Elements for the Straemfunction Formulation.

5. Iterative method and Algorithm

We have formulated the discrete approximation problems for our model in previous sections. In addition, many finite-element spaces applicable for the approximation procedures have been discussed. Now, the discrete formulation (19) may be converted into a system of nonlinear algebraic equations by explicitly choosing bases for X^h. In this section, we will describe an iterative method for the nonlinear system resulting from the discretization.

To construct an iterative method, we first start with an initial guess, then use an approximate nonlinear system as an iterative scheme to produce a sequence of solutions that is expected to converge to the exact solution of the original system.

In this method, we linearize the added nonlinear term and then solve the nonlinear system of equations. Let $\psi^{(0)} \in X^h$ be given; then we define the sequence $\psi^{(n)} \in X^h$ for $n = 1, 2, 3, \ldots$, to be the solution of the following nonlinear discrete system:

Find $\psi^{(n)} \in X^h$ such that , for all $\phi \in X^h$,

$$\begin{aligned} &a(\psi^{(n)}, \phi) + \hat{a}(\psi^{(n-1)}, \psi^{(n)}, \phi) \\ &+ b(\psi^{(n)}; \psi^{(n)}, \phi) = (\vec{f}, curl\,\phi). \end{aligned} \tag{20}$$

This method requires small modification on the stiffness matrix which results from the approximation procedure of the Navier-Stokes equations. The resulting system from this method is nonlinear system and if we use Newton's method to solve this nonlinear system, then the resulting matrix from each iteration is nonsymmetric whose symmetric part is positive definite. Moreover, the resulting matrix is sparse. The suggested linear solver for such system is any Conjugate Gradient alike method. Some of them are Generalized Minimal Residual (GMRES), Bi Conjugate Gradient (BiCG), Conjugate Gradient Square Method (CGS) and Bi Conjugate Gradient Stabilized (Bi-CGSTAB), (see Templates [18]). It is known that the (n+1)-th error is proportional to the square of the n-th error. So that the convergence is very rapid once the errors are small. So we need to pick a good initial guess to solve the nonlinear system of equations. Choosing $\psi^{(n-1)}$ to be the initial guess in solving the nonlinear system (20) is considered to be a good choice because as n increases $\psi^{(n)}$ going closer and closer to the exact solution. For the first iteration we will choose $\psi^{(0)} = 0$ this will lead us to solve the streamfunction equation of the Navier-Stokes equations. Therefore, the approximate solution for [SNS] will be the initial guess for the second iteration in (20). One requirement which may be imposed upon any iterative method is that it is norm-reducing in sense that

$$\| residual^{(n+1)} \| \le \| residual^{(n)} \|, \; k = 0, 1, \cdots,$$

holds in some norm. The Newton method does not necessarily satisfy this requirement. One simple modification of Newton method is adding the following IF statement

If ($\| residual^{(n+1)} \| \ \ge \ \| residual^{(n)} \|$) then

$\psi^{(n+1)} = \psi^{(n)} - \frac{1}{2}(updte)$

else

$\psi^{(n+1)} = \psi^{(n)} - (update)$

end if

Discussion of this modification and other techniques for solving nonlinear system of equations can be found in [17].

6. Computational Experiments

We consider the driven cavity problem in the two-dimensional box $[0,1]\times[0,1]$ with no-slip boundary conditions, i.e, $u_1 = u_2 = 0$ in all boundaries except $y = 1$, where $u_1 = 1$. This problem have been studied and addressed by many researchers including Ghia, Ghia, Shin [20], and Betts-Haroutunian [2]. The numerical computational in this section was obtained using a TOSHIBA Satellite Pro with Intel Mobile CPU 1.7GHz running Windows XP. Bogner-Fox-Schmit elements are used with 16×16 grid points. We pick two values of the Reynolds number. We used the iterative method (20) and applied Newton's method to solve the nonlinear system. We choose the Bi Conjugate Gradient Stabilized method (Bi-CGSTAB), (see Templates [18]), to solve the linear system resulting from each Newton's iterate. The stopping criteria for the problem is

$\| \psi^{(n+1)} - \psi^{(n)} \| \ \le TOL$ and $\| residual \| \ \le TOL$

with $TOL = 10^{-5}$

where the above two norms are in the discrete $L_2 - norm$. We choose $\varepsilon_1 = 1.0E-20$ and $q = 4$. We compute an approximate solution for Re =1, 20, 40, 60, 80 and 100. Figures (1-6) display streamfunction contours. Our computations show that we get a stable approximation of the solution for the Ladyzhenskaya model. As seen on Figures (1-6), the top right corner (where the moving wall moves towards the stationary wall) shows that the streamfunction contours are very smooth even with high Reynolds numbers. It is also seen that the number of vortices in the bottom right and left corners increases as the Reynolds number increases. The biggest vortex in the bottom left corner gets bigger as the Reynolds number increases.

7. Summary

The streamfunction equation of the Ladyzhenskaya equations and its weak formulation are presented. An algorithm for solving the discrete form is given with some applicable finite elements. This algorithm is used to solve a driven cavity problem in a square. Detailed solutions are presented.

Acknowledgment: This work was supported by KFUPM under grant INT-278. The author wishes to express his gratitude to KFUPM for providing the financial support. The author is also grateful to UNITEN, where this work has been completed, while on research attachment at Department of Engineering Sciences and Mathematics, UNITEN.

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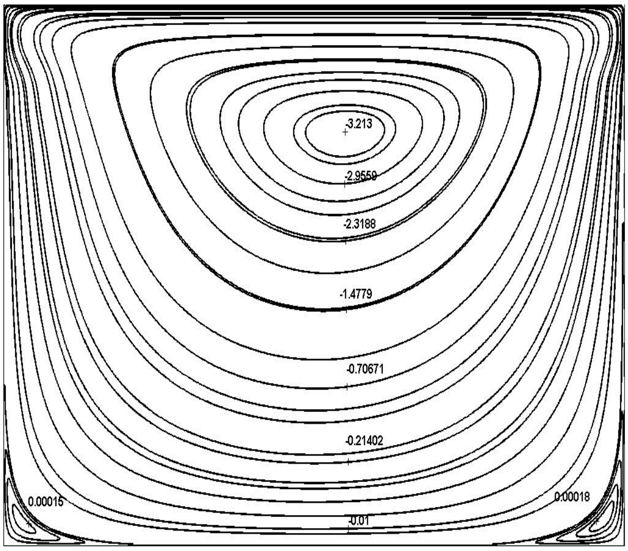
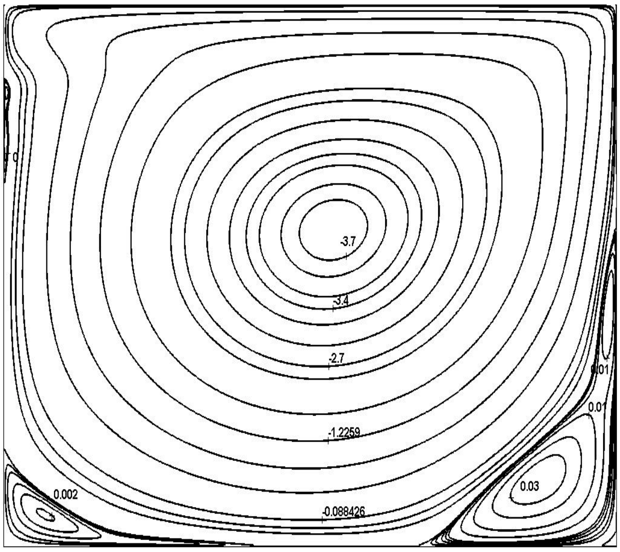
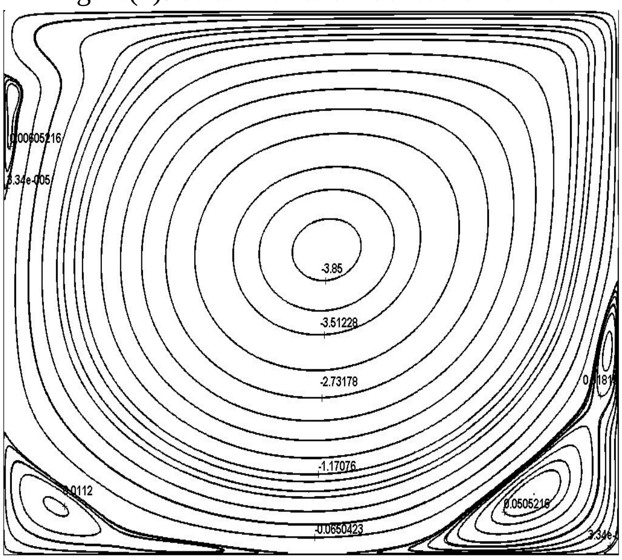


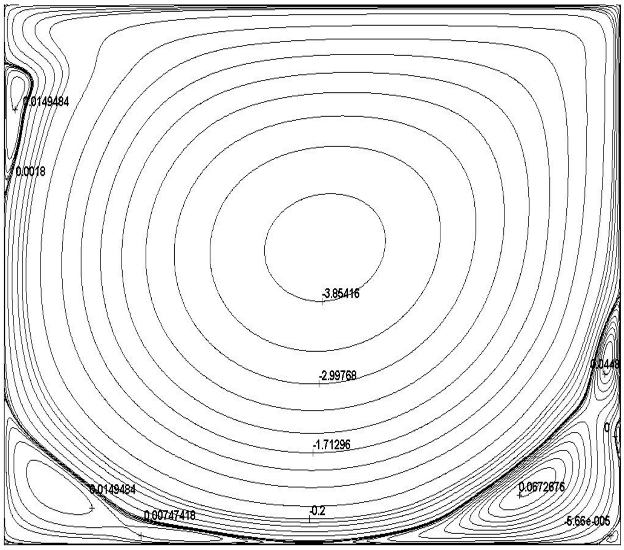
Figure(1): Streamfunction Contours Re=1



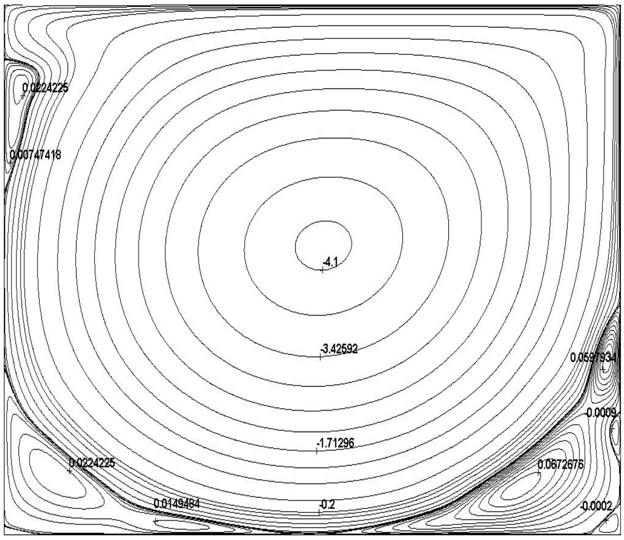
Figure(2): Streamfunction Contours Re=20



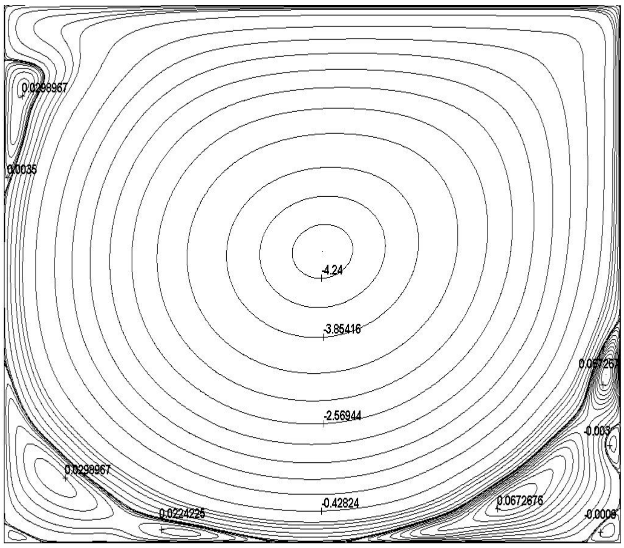
Figure(3): Streamfunction Contours Re=40



Figure(4): Streamfunction Contours Re=60



Figure(5): Streamfunction Contours Re=80



Figure(6): Streamfunction Contours Re=100