Thm 10.1 (P.152): u_h and u be solution

$$\|u_h(t)-u(t)\| \le \|v_h - v\| + ch^2\left(\|v\|_2 + \int_0^t \|u_t\|_2 ds\right)$$

proof: $u_h - u = \underbrace{(u_h - R_h u)}_{\theta} + \underbrace{(R_h u - u)}_{\rho}$

$$\|\rho(t)\| = \|R_h u - u\| \le ch^2 \|u(t)\|_2$$
$$= ch^2 \|u(0) + \int_0^t u_t ds\|_2$$
$$\le ch^2 \left(\|v\|_2 + \int_0^t \|u_t\|_2 ds\right) \quad \text{—— (1)}$$

$$(u_t, \varphi) + a(u, \varphi) = (f, \varphi) \quad \forall \varphi \in H_0^1$$
$$(u_{h,t}, \chi) + a(u_h, \chi) = (f, \chi) \quad \forall \chi \in S_h$$

$$((u_h - u)_t, \chi) + a(u_h - u, \chi) = 0$$
$$(\theta_t + \rho_t, \chi) + a(\theta + \rho, \chi) = 0$$
$$(\theta_t, \chi) + a(\theta, \chi) = -(\rho_t, \chi) - a(\rho, \chi)$$
$$= -(R_h u_t, \chi) + (u_t, \chi) - a(R_h u, \chi) + a(u, \chi)$$
$$= (f, \chi) - (R_h u_t, \chi) - a(u, \chi)$$
$$= (u_t, \chi) - (R_h u_t, \chi)$$
$$= (u_t - R_h u_t, \chi)$$
$$= -(\rho_t, \chi)$$

$$\boxed{(\theta_t, \chi) + a(\theta, \chi) = -(\rho_t, \chi)} \quad \forall \chi \in S_h$$

$\boxed{\text{Three Problems}}$

Cont: $(u_t, \varphi) + a(u, \varphi) = (f, \varphi) \quad \forall \varphi \in H_0^1$
$u(0) = v$

Semidiscrete: $(u_{h,t}, \chi) + a(u_h, \chi) = (f, \chi) \quad \forall \chi \in S_h$
$u_h(0) = v_h$

Full-discrete: $(\bar{\partial}_t U^n, \chi) + a(U^n, \chi) = (f, \chi) \quad \forall \chi \in S_h$
$U^0 = v_h$

$\boxed{\text{Three Operators}}$ I_h, P_h, R_h

(1) $I_h u \in S_h \quad \forall u \in C(\bar{\Omega})$
interpolation operator

(2) $P_h u \in S_h \quad \forall u \in L_2$
orthogonal projection w.r.t L_2-norm
$$(P_h u - u, \chi) = 0 \quad \forall \chi \in S_h$$

(3) $R_h u \in S_h \quad \forall u \in H_0^1$
orthogonal projection w.r.t energy-norm
$$a(R_h u - u, \chi) = 0 \quad \forall \chi \in S_h$$
Ritz-projection, Elliptic-projection.

Remark1: FEM solution is the Ritz-projection of the exact solution (i.e) $u_h = R_h u$

Remark2: $\|u - R_h u\| \le ch^2 \|u\|_2$ $\quad |u - R_h u|_1 \le ch\|u\|_2 \quad \forall u \in H^2 \cap H_0^1$

Thm 10.2:

$$|u_h(t) - u(t)|_1 \leq |v_h - v|_1 + ch\left\{\|v\|_2 + \|u(t)\|_2 + \left(\int_0^t \|u_t\|_1^2 ds\right)^{1/2}\right\}$$

Proof: $u_h - u = \underbrace{u_h - R_h u}_{\theta} + \underbrace{R_h u - u}_{\rho}$

$$|\rho(t)|_1 = |R_h u(t) - u(t)|_1 \leq ch\|u(t)\|_2 \quad \text{— (1)}$$

we need to estimate $|\theta(t)|_1$

$$(\theta_t, \chi) + a(\theta, \chi) = -(\rho_t, \chi) \quad \forall \chi \in S_h$$

let $\chi = \theta_t$

$$(\theta_t, \theta_t) + a(\theta, \theta_t) = -(\rho_t, \theta_t)$$

Expand $\|\rho_t + \theta_t\|^2 \geq 0$

$$\|\theta_t\|^2 + \frac{1}{2}\frac{d}{dt}|\theta|_1^2 = -(\rho_t, \theta_t)$$

$$\leq \frac{1}{2}\left(\|\rho_t\|^2 + \|\theta_t\|^2\right)$$

$$\frac{d}{dt}|\theta|_1^2 \leq \|\rho_t\|^2$$

integrate

$$|\theta(t)|_1^2 \leq |\theta(0)|_1^2 + \int_0^t \|\rho_t\|^2 ds \quad \text{— (2)}$$

$$|\theta(0)|_1^2 = |u_h(0) - (R_h u)(0)|_1^2 = |v_h - R_h v|_1^2$$

$$\leq \left(|v_h - v|_1 + |v - R_h v|_1\right)^2$$

Apply stability estimate:

$$\|\theta(t)\| \leq \|\theta(0)\| + \int_0^t \|\rho_t\| ds$$

$$\|\theta(0)\| = \|u_h(0) - (R_h u)(0)\| = \|v_h - R_h v\|$$

$$= \|v_h - v + v - R_h v\| \leq \|v_h - v\| + \|v - R_h v\|$$

$$\leq \|v_h - v\| + ch^2\|v\|_2$$

$$\|\rho_t\| = \|(R_h u - u)_t\| = \|R_h u_t - u_t\| \leq ch^2\|u_t\|_2$$

$$\int_0^t \|\rho_t\| ds \leq ch^2 \int_0^t \|u_t\|_2 ds$$

$$\|\theta(t)\| \leq \|v_h - v\| + ch^2\left(\|v\|_2 + \int_0^t \|u_t\|_2 ds\right) \quad \text{— (2)}$$

(1) and (2) give:

$$\|u_h - u\| \leq \|v_h - v\| + ch^2\left(\|v\|_2 + \int_0^t \|u_t\|_2 ds\right)$$

② implies

$$|\theta(t)|_1^2 \leq \left[|\theta(0)|_1 + \left\{ \int_0^t \|\rho_t\|^2 ds \right\}^{1/2} \right]^2 \qquad a^2+b^2 \leq (|a|+|b|)^2$$

$$|\theta(t)|_1 \leq |\theta(0)|_1 + \left\{ \int_0^t \|\rho_t\|^2 ds \right\}^{1/2} \quad \text{—③}$$

$$|\theta(0)|_1 = |u_h(0) - (R_h u)(0)|_1 = |v_h - R_h v|_1$$

$$\leq |v_h - v|_1 + |v - R_h v|_1 \leq |v_h - v|_1 + ch\|v\|_2 \quad \hookrightarrow ④$$

$$\|\rho_t\| = \|R_h u_t - u_t\| \leq ch\|u_t\|_1$$

$$\int_0^t \|\rho_t\|^2 ds \leq ch^2 \int_0^t \|u_t\|_1^2 ds$$

$$\left\{ \int_0^t \|\rho_t\|^2 ds \right\}^{1/2} \leq ch \left\{ \int_0^t \|u_t\|_1^2 ds \right\}^{1/2} \quad \text{—⑤}$$

③, ④, ⑤ $\Rightarrow$

$$|\theta(t)|_1 \leq |v_h - v|_1 + ch\left\{ \|v\|_2 + \left(\int_0^t \|u_t\|_1^2 ds \right)^{1/2} \right\} \quad \hookrightarrow ⑥$$

① and ⑥ $\Rightarrow$

$$|u_h(t) - u(t)|_1 \leq |\theta(t)|_1 + |\rho(t)|_1$$

$$\leq |v_h - v|_1 + ch\left\{ \|v\|_2 + \|u(t)\|_2 + \left(\int_0^t \|u_t\|_1^2 ds \right)^{1/2} \right\}$$

$\blacksquare$